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Quantum Field Theory of Many-Particle Systems - Problem Set 10

Wintersemester 2015/2016

Abgabe: The problem set will be discussed in the tutorial on **Tuesday, 5.1.2016, 11:00**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_WS1516.html

18. BCS as a mean-field theory *4+2+4 Punkte*

In this problem we will consider electrons interacting through the same type of interaction as in problem 17 of the last sheet. That is we consider the following Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} + \sum_{\mathbf{k_1},\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) c^\dagger_{\mathbf{k_1}\uparrow} c^\dagger_{-\mathbf{k_1}\downarrow} c_{-\mathbf{k_2}\downarrow} c_{\mathbf{k_2}\uparrow},$$

where

$$V(\mathbf{k_1},\mathbf{k_2}) = \begin{cases} -g, & \epsilon_F \leq \frac{\hbar^2 k_i^2}{2m} \leq \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases} .$$

a) Proceeding analoguously to the Hartree-Fock mean-field theory in problem 15 convince yourself that a mean-field decoupling of the Hamiltonian yields

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c^\dagger_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \sum_{\mathbf{k}} \Delta^*_{\mathbf{k}} c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow} + \sum_{\mathbf{k_1},\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) \langle c^\dagger_{\mathbf{k_1}\uparrow} c^\dagger_{-\mathbf{k_1}\downarrow}\rangle \langle c_{\mathbf{k_2}\downarrow} c_{-\mathbf{k_2}\uparrow}\rangle,$$

with

$$\Delta_{\mathbf{k_1}} = -\sum_{\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) \langle c_{-\mathbf{k_2}\downarrow} c_{\mathbf{k_2}\uparrow}\rangle. \tag{1}$$

b) Show that the Hamiltonian can be diagonalised by the following unitary transformation

$$\begin{pmatrix} \alpha^\dagger_{\mathbf{k}\uparrow} \\ \alpha_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\mathbf{k}\uparrow} \\ c_{-\mathbf{k}\downarrow} \end{pmatrix},$$

such that

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \alpha^\dagger_{\mathbf{k}\sigma} \alpha_{\mathbf{k}\sigma} + \text{constant}.$$

Derive an expression for $E_{\mathbf{k}}$.

c) By writing the original fermionic operators in terms of their transformed counterparts, use Equation (1) to derive a self-consistency equation for $\Delta_{\mathbf{k}}$.

19. Electron Tunneling between two metals *2+2+2+4+2+2 Punkte*

The most important vertifications of the BCS theory of superconductivity came from electron tunneling experiments, in which the energy gap as a function of temperature was measured and showed excellent agreement with the BCS theory. In this problem electron tunneling between two normal metals will be discussed. Please note that you can solve most parts of the problem independently of the other parts.

We consider two metallic regions, one on the left (L) and one on the right (R), which are seperated by an insulating barrier. This setup is described by the Hamiltonian:

$$H = H_R + H_L + H_T$$
$$H_T = \sum_{\mathbf{k},\mathbf{p}} \left(T_{\mathbf{k},\mathbf{p}} c_{\mathbf{k}}^\dagger c_{\mathbf{p}} + \text{h.c} \right).$$

The first term H_R is the Hamiltonian for particles on the right side of the tunneling junction. It contains all many-body interactions. For a normal metal, it is characterized by an energy function ξ_p and a density of states ρ_F. Similarly, H_L describes all the physics for particles on the left side of the junction. these two parts of the Hamiltonian are considered to be strictly independent. Not only do they commute $[H_L, H_R] = 0$, but they commute term by term. The Hamiltonian on the right can be expressed in terms of one set of operators $c_{\mathbf{k}}$, and the Hamiltonian on the left side by an other set $c_{\mathbf{p}}$. These two sets of fermionic operators are independent, i.e.

$$\{c_{\mathbf{k}}, c_{\mathbf{p}}^\dagger\} = 0.$$

As, in the end, we will be interested in tunnel voltages of the order of the Debye energy, it is a good approximation to assume that the tunnel matrix elements are independent of energy

$$T_{\mathbf{k},\mathbf{p}} = T_0.$$

a) The tunneling current through the insulating region is expressed as the rate of change of the number of particles on, for example, the left-hand side of the junction $N_L = \sum_{\mathbf{p}} c_{\mathbf{p}}^\dagger c_{\mathbf{p}}$. Show that this rate of change

$$\dot{N}_L = i[H, N_L] = i[H_T, N_L]$$

is given by

$$\dot{N}_L = i \sum_{\mathbf{k},\mathbf{p}} \left(T_{\mathbf{k},\mathbf{p}} c_{\mathbf{k}}^\dagger c_{\mathbf{p}} - T_{\mathbf{k},\mathbf{p}}^* c_{\mathbf{p}}^\dagger c_{\mathbf{k}} \right).$$

b) We consider $H_0 = H_L + H_R$ as the unperturbed Hamiltonian, and H_T as a perturbation. Then, we can use the theory of linear response to obtain the tunnel current $I(t) = -e\langle \dot{N}_L(t)\rangle$ as

$$I(t) = -ei\int_{-\infty}^{t} dt' \langle [\dot{N}_L(t), H_T(t')]\rangle.$$

Here, the time dependence of operators is due to H_0, and we assume that there exists a difference in chemical potential $V = \mu_L - \mu_R$. Show that the current due to tunneling of single electrons is given by (assume that expectation values of the type $\langle c_{\mathbf{k}} c_{\mathbf{k}'} c^\dagger_{\mathbf{p}} c^\dagger_{\mathbf{p}'}\rangle$ vanish)

$$I_{\text{single}} = e\int_{-\infty}^{\infty} dt' \Theta(t-t') \left\{ e^{-ieV(t'-t)} \langle [A(t), A^\dagger(t')]\rangle - e^{-ieV(t-t')} \langle [A^\dagger(t), A(t')]\rangle \right\}.$$

Here the operator A is defined as

$$A = \sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}}.$$

c) Introducing the retarded correlation function

$$C^+_{A,A^\dagger}(t) = -i\Theta(t)\langle [A(t), A^\dagger(0)]\rangle,$$

show that the single particle tunnel current can be expressed in terms of its Fourier transform

$$C^+_{A,A^\dagger}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} C^+_{A,A^\dagger}(t),$$

as

$$I_{\text{single}} = ie\left[C^+_{A,A^\dagger}(eV) - \left(C^+_{A,A^\dagger}(eV)\right)^*\right] = -2e\text{Im}[C^+_{A,A^\dagger}(-eV)].$$

d) Show that

$$C^\tau_{A,A^\dagger}(i\omega_n) = \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 \frac{n_F(\xi_k) - n_F(\xi_p)}{i\omega_n + \xi_k - \xi_p}.$$

e) Show, by analytically continuing $i\omega_n \to \omega + i\eta$ your result from d), that the tunnel current is

$$I_{\text{single}} = 4\pi e \sum_{\mathbf{k},\mathbf{p}} |T_{\mathbf{k},\mathbf{p}}|^2 \delta(eV + \xi_k - \xi_p)[n_F(\xi_k) - n_F(\xi_p)].$$

f) Now make the assumption $T_{\mathbf{k},\mathbf{p}} = T_0$. Introduce the total densities of states $N_{R,L} = L^d \rho_{R,L}(\epsilon_F)$ which are densities of states per unit volume multiplied by the respective system volumes, to convert the momentum sums into integrals, and show that the tunnel current is given by

$$I_{\text{single}} = 4\pi e^2 N_R N_L |T_0|^2 V.$$