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Quantum Field Theory of Many-Particle Systems - Problem Set 6

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 17.05.2024, 13:30**.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Lehmann representation

4+3 Punkte

The aim of this task is to derive the Lehmann representation for the retarded correlation function

$$C^{+}_{\hat{X}_1\hat{X}_2}(t) = -i\theta(t)\langle[\hat{X}_1(t), \hat{X}_2(0)]_{\zeta_X}\rangle \ ,$$

and for the time ordered correlation function in imaginary time

$$C^{\tau}_{\hat{X}_1\hat{X}_2}(\tau) = -\langle\hat{T}_\tau\hat{X}_1(\tau)\hat{X}_2(0)\rangle \ .$$

(a) Express the expectation value in the definition of the two correlation functions as a trace over the statistical operator $\mathrm{e}^{-\beta(\hat{H}-\mu\hat{N})}$ using exact eigenstates $\{|\psi_n\rangle\}$ of the full Hamiltonian. Insert a resolution of unity between the operators $\hat{X}_1$ and $\hat{X}_2$ to express the time evolution of $\hat{X}_1$ in terms of the eigenvalues K_n of $\hat{K} = \hat{H} - \mu\hat{N}$.

(b) Calculate the Fourier transforms

$$C^{+}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{i\omega t-\eta|t|}C^{+}(t)$$

and

$$C^{\tau}(i\omega_n) = \int_{0}^{\beta} \mathrm{d}\tau\, \mathrm{e}^{i\omega_n\tau}C^{\tau}(\tau) \ ,$$

by using the representation derived in part (a). The imaginary time correlation function is (anti-)periodic with period β according to $C^\tau(\tau+\beta) = \zeta_X C^\tau(\tau)$ and can be Fourier transformed with respect to bosonic (fermionic) Matsubara frequencies.

2. Green functions in real space

2+3 Punkte

The time ordered Green function in momentum space for a system of non-interacting fermions was in problem set 4 shown to be given by

$$G(t,\boldsymbol{k}) = -i\theta(t)\theta(\xi(\boldsymbol{k}))\mathrm{e}^{-i\xi(\boldsymbol{k})t} + i\theta(-t)\theta(-\xi(\boldsymbol{k}))\mathrm{e}^{-i\xi(\boldsymbol{k})t} \ ,$$

at zero temperature. Here $\xi(\boldsymbol{k}) = \varepsilon(\boldsymbol{k}) - \mu$ is the dispersion of free fermions. By performing a Fourier transform calculate the corresponding real-space time-ordered Green function $G(t,\boldsymbol{x})$ of a system of non-interacting fermions in (a) $d = 1$ dimensions and (b) $d = 3$ dimensions. Comment on your results.
Hint: In order to perform the integrals it is helpful to expand the dispersion around the Fermi surface. That is write $\xi(\boldsymbol{k}) \approx \boldsymbol{v}_F \cdot (\boldsymbol{k} - \boldsymbol{k}_F)$.