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Quantum Field Theory of Many-Particle Systems - Problem Set 5

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 10.05.2024, 13:30**.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Gaussian integrals

1+2+2+1 Punkte

In this problem we will consider different generalisations of the standard Gaussian integral.

(a) To begin with show that

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2} = \sqrt{\frac{2\pi}{a}},$$

where $\mathrm{Re}\, a > 0$.

(b) Generalise the above integral by adding a so-called source term in the exponent. That is calculate

$$\int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}ax^2 + Jx}.$$

(c) Now let $\boldsymbol{x}$ and $\boldsymbol{J}$ be N-dimensional real vectors, and let A be an $N \times N$ matrix. Show that the multi-dimensional analogues of the integrals in parts a and b become

$$\int d\boldsymbol{x}\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}} = (2\pi)^{N/2} \mathrm{det} A^{-1/2},$$

and

$$\int d\boldsymbol{x}\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x} + \boldsymbol{J}^T \cdot \boldsymbol{x}} = (2\pi)^{N/2} \mathrm{det} A^{-1/2} e^{\frac{1}{2}\boldsymbol{J}^T A^{-1} \boldsymbol{J}}.$$

(d) What are the analoguous expressions if $\boldsymbol{x}$ and $\boldsymbol{J}$ are complex vectors and A is a complex matrix?

2. Wick's theorem

2+2+2 Punkte

In this problem we will derive a version of Wick's theorem which plays an important role in quantum field theories. To do this let $\boldsymbol{x}$ be an N-dimensional real vector and let A be an $N \times N$ matrix. Furthermore define the "expectation value"

$$\langle \cdots \rangle = \frac{\int d\boldsymbol{x}\, (\cdots) e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}}}{\int d\boldsymbol{x}\, e^{-\frac{1}{2}\boldsymbol{x}^T A \boldsymbol{x}}}.$$

(a) Using the results from the previous problem, or otherwise, show that

$$\langle x_i x_j \rangle = A^{-1}_{ij},$$

where x_i is the i-th element of $\boldsymbol{x}$ and A^{-1}_{ij} is the ij-th element of the matrix A^{-1}.

(b) Likewise show that

$$\langle x_i x_j x_n x_m \rangle = A^{-1}_{ij} A^{-1}_{nm} + A^{-1}_{in} A^{-1}_{jm} + A^{-1}_{im} A^{-1}_{jn}.$$

(c) Generalise the above to the product of $2n$ elements of $\boldsymbol{x}$.

3. Equation of motion for the Green function of an interacting system

3+5 Punkte

(a) Define the time-ordering operator T for a fermionic and a bosonic system.

(b) Consider the following Hamiltonian in second quantization,

$$\begin{aligned} H - \mu N = &\int d^3r\, \psi^\dagger(\boldsymbol{r})\epsilon(\boldsymbol{r})\psi(\boldsymbol{r}) \\ &+ \frac{1}{2}\int d^3r \int d^3r'\, \psi^\dagger(\boldsymbol{r})\psi^\dagger(\boldsymbol{r}')V(\boldsymbol{r}-\boldsymbol{r}')\psi(\boldsymbol{r}')\psi(\boldsymbol{r}), \end{aligned}$$

where $\epsilon(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2_{\boldsymbol{r}} + U(\boldsymbol{r}) - \mu$ represents the one-particle contribution to the Hamiltonian, and $V(\boldsymbol{r}-\boldsymbol{r}')$ is a two-body interaction. Using the Heisenberg equation of motion $\hbar\dot{\psi}(\boldsymbol{r},t) = i[H, \psi(\boldsymbol{r},t)]$ for the (either fermionic or bosonic) field operator $\psi(\boldsymbol{r},t)$, show that the time-ordered Green function $G(\boldsymbol{r},t;\boldsymbol{r}',t') = -i\langle T\psi(\boldsymbol{r},t)\psi^\dagger(\boldsymbol{r}',t')\rangle$ obeys the following equation of motion

$$\begin{aligned} \left[i\hbar\frac{\partial}{\partial t} - \epsilon(\boldsymbol{r})\right] G(\boldsymbol{r},t;\boldsymbol{r}',t') = &\hbar\delta(t-t')\delta(\boldsymbol{r}-\boldsymbol{r}') \\ &- i\int d^3r'' V(\boldsymbol{r}-\boldsymbol{r}'')\langle T\psi^\dagger(\boldsymbol{r}'',t)\psi(\boldsymbol{r}'',t)\psi(\boldsymbol{r},t)\psi^\dagger(\boldsymbol{r}',t')\rangle. \end{aligned}$$

Hint: The following identity for commutators may be useful:

$$[AB, \psi] = A[B,\psi]_\pm \mp [A,\psi]_\pm B.$$