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QFT of Many-Particle Systems - Problem Set 13 (Bonus)

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 05.07.2024, 13:30**. Points on this problem set are bonus points not counted towards the 50% of points required for exam admission.

Internet: The problem sets can be downloaded from https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Josephson effect

4+4+4+4+4 (Bonus) Punkte

Electron tunneling between two superconductors has two kinds of contributions. One is due to tunneling of single particles, as in the case between two metals or between a metal and a superconductor. The other is tunneling in pairs, which gives rise to the Josephson effect. The notation is the same as in problem sets 11 and 12. There, we showed that the single particle tunnel current can be expressed as

$$I_{\text{single}} = e\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\left\{e^{ieV(t'-t)}\langle[A(t),A^\dagger(t')]\rangle - e^{ieV(t-t')}\langle[A^\dagger(t),A(t')]\rangle\right\},$$

where the operator A is defined as

$$A = \sum_{\boldsymbol{k},\boldsymbol{p}} T_{\boldsymbol{k},\boldsymbol{p}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{p}}.$$

In the case of tunneling between two superconductors, there is another contribution

$$I_{\text{pair}}(t) = e\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\left\{e^{-ieV(t'+t)}\langle[A(t),A(t')]\rangle - e^{ieV(t+t')}\langle[A^\dagger(t),A^\dagger(t')]\rangle\right\}.$$

In the following, we evaluate the voltage and time dependence of this pair contribution to the tunnel current.

(a) The most unusual feature of this expression is the fact that the time dependence in the exponentials is $t+t'$. Rewrite this as $2t+t'-t$ and change the variable of integration to $t''=t-t'$. Show that the pair current can be expressed as

$$I_{\text{pair}} = -2e\,\mathrm{Im}\left[e^{-2eiVt}C^+_{A,A}(eV)\right].$$

Here, $C^+_{A,A^\dagger}(eV)$ is the Fourier transform of the retarded correlation function

$$C^+_{A,A}(t) = -i\Theta(t)\langle[A(t),A(0)]\rangle.$$

(b) In order to calculate the retarded correlation function $C^+_{A,A}(eV)$, we start from the Matsubara function

$$C^\tau_{A,A}(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n\tau}\langle T_\tau A(\tau)A(0)\rangle.$$

Show that

$$C^{\tau}_{A,A}(i\omega_n) = 2\sum_{\boldsymbol{k},\boldsymbol{p}} T_{\boldsymbol{k},\boldsymbol{p}} T_{-\boldsymbol{k},-\boldsymbol{p}} T \sum_{\epsilon_l} F^{\dagger}_L(i\epsilon_l, \xi_{\boldsymbol{k}}) F_R(i\epsilon_l - i\omega_n, \xi_{\boldsymbol{p}}).$$

Here

$$F(i\epsilon_l, \xi_{\boldsymbol{p}}) = \langle c_{-\boldsymbol{p},\downarrow}(-i\epsilon_l) c_{\boldsymbol{p},\uparrow}(i\epsilon_l)\rangle, \qquad F^{\dagger}(i\epsilon_l, \xi_{\boldsymbol{p}}) = \langle c^{\dagger}_{\boldsymbol{p},\uparrow}(-i\epsilon_l) c^{\dagger}_{-\boldsymbol{p},\downarrow}(i\epsilon_l)\rangle,$$

are the $(1,2)$ and the $(2,1)$ elements of the Gorkov Green function matrix

$$\mathcal{G}(i\epsilon_l, \xi_{\boldsymbol{p}}) = \frac{1}{(i\epsilon_l)^2 - \xi^2_{\boldsymbol{p}} - |\Delta_0|^2} \begin{pmatrix} i\epsilon_l + \xi_{\boldsymbol{p}} & -\Delta_0 \\ -\bar{\Delta}_0 & i\epsilon_l - \xi_{\boldsymbol{p}} \end{pmatrix}.$$

Note that for the evaluation of $F^{\dagger}_L(i\epsilon_l, \xi_{\boldsymbol{p}})$ and $F_R(i\epsilon_l, \xi_{\boldsymbol{p}})$, the order parameter in the Gorkov Green function is respectively $\bar{\Delta}_L$ instead of $\bar{\Delta}_0$ and Δ_R instead of Δ_0.

(c) Use spectral representations

$$F^{\dagger}(i\epsilon_l, \xi_{\boldsymbol{k}}) = \bar{\Delta}_L \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\boldsymbol{k}})}{i\epsilon_l - \omega},$$

and

$$F(i\epsilon_l, \xi_{\boldsymbol{p}}) = \Delta_R \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\boldsymbol{p}})}{i\epsilon_l - \omega},$$

for both Green functions to evaluate the Matsubara sum in the expression for $C^{\tau}_{A,A}(i\omega_n)$. It will be useful to perform a partial fraction decomposition and to make use of the identity

$$T \sum_{\epsilon_l} \frac{e^{i\eta\epsilon_l}}{i\epsilon_l - \xi_{\boldsymbol{k}}} = n_F(\xi_{\boldsymbol{k}}).$$

Show that

$$C^{\tau}_{A,A}(i\omega_n) = 2\bar{\Delta}_L \Delta_R \sum_{\boldsymbol{k},\boldsymbol{p}} T_{\boldsymbol{k},\boldsymbol{p}} T_{-\boldsymbol{k},-\boldsymbol{p}} \int \frac{d\epsilon}{2\pi}\frac{d\epsilon'}{2\pi} A^*(\epsilon, \xi_{\boldsymbol{k}}) A(\epsilon', \xi_{\boldsymbol{p}}) \frac{n_F(\epsilon) - n_F(\epsilon')}{i\omega_n + \epsilon - \epsilon'}.$$

(d) Show that the spectral function depends on $\xi_{\boldsymbol{k}}$ only via $\lambda_{\boldsymbol{k}} = \sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_0|^2}$, and that it is given by

$$A(\epsilon, \lambda_{\boldsymbol{k}}) = 2\pi \frac{1}{2\lambda_{\boldsymbol{k}}} [\delta(\epsilon - \lambda_{\boldsymbol{k}}) - \delta(\epsilon + \lambda_{\boldsymbol{k}})].$$

Use this expression for the spectral function and make use of the momentum independence of tunnel matrix elements to show that, in the limit of zero temperature, and after analytical continuation $i\omega_n \to \omega + i\eta$, the retarded correlation function is given by

$$C^{+}_{A,A}(eV) = \frac{1}{2} |\bar{\Delta}_L \Delta_R T_0^2| e^{i\varphi} \sum_{\boldsymbol{k},\boldsymbol{p}} \frac{1}{\lambda_{\boldsymbol{k}}\lambda_{\boldsymbol{p}}} \left[\frac{1}{eV - \lambda_{\boldsymbol{k}} - \lambda_{\boldsymbol{p}}} - \frac{1}{eV + \lambda_{\boldsymbol{k}} + \lambda_{\boldsymbol{p}}} \right].$$

(e) Assume now that $|\Delta_L| = |\Delta_R| = \Delta_0$. Convince yourself that for $eV < 2\Delta_0$, the δ-function part in $C^{+}_{A,A}$ vanishes and that, for the same reason, there are no singular contributions to the momentum sums. As a consequence, one can write

$$C^{+}_{A,A}(eV) = \frac{1}{2e} J_S(eV) e^{i\varphi},$$

where $J_S(eV)$ depends smoothly on voltage for $eV \to 0$. Show that in this notation the pair contribution to the tunneling current is given by

$$I_{\text{pair}} = J_S(eV) \sin(\omega t + \varphi) \qquad \text{with} \qquad \omega = \frac{2eV}{\hbar}.$$

2. Flux Quantization

4+4 (Bonus) Punkte

In the high temperature (static) limit, the action of a long wave length excitation of the order parameter phase θ (we parametrize $\Delta(\boldsymbol{r}) = \Delta_0 \mathrm{e}^{2i\theta(\boldsymbol{r})}$) in the presence of a vector potential $\boldsymbol{A}$ can be written as

$$\frac{\beta}{2}\int \mathrm{d}^d r \left[\frac{n_s}{m}(\hbar\nabla\theta + e_0\boldsymbol{A})^2 + \frac{1}{\mu_0}(\nabla\times\boldsymbol{A})^2\right] . \tag{1}$$

Here, $-e_0$ and m are the electron charge and mass, n_s is the superfluid density, and μ_0 the vacuum permeability.

(a) By minimizing the above action, derive the equations satisfied by θ and $\boldsymbol{A}$. Show that these equations are consistent with the identification of the gauge invariant (i.e. physical) current as

$$\boldsymbol{j} = \frac{e_0 n_s}{m}\left(\hbar\nabla\theta + e_0\boldsymbol{A}\right) . \tag{2}$$

In terms of this current, your equations should be

$$\nabla\cdot\boldsymbol{j} = 0 \tag{3}$$

$$\nabla\times(\nabla\times\boldsymbol{A}) = \mu_0\boldsymbol{j} . \tag{4}$$

The first equation is the continuity equation expressing charge conservation, the second is Ampere's law.

(b) Now we consider the properties of a vortex configuration in θ. We consider a cylindrical sample with a hole running through the center. Assume now that the phase winds around by $-\pi$ (such that the order parameter $\propto \mathrm{e}^{2i\theta}$ stays single valued) when going once around a loop that encircles the hole, i.e.

$$\int_{\mathcal{C}} \mathrm{d}\boldsymbol{l}\cdot\nabla\theta = -\pi , \tag{5}$$

where the integral is taken along the loop $\mathcal{C}$. Due to the Meissner effect, the magnetic field will extend only a distance λ from the edge of the hole into the superconductor. Deep inside the superconductor, the current will be zero. Show that this implies that there is a magnetic $h/(2e_0)$ associated with this vortex.