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QFT of Many-Particle Systems - Problem Set 12

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 28.06.2024, 13:30**.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Tunneling between metal and superconductor *3+3+3+3 Punkte*

The most important verification of the BCS theory of superconductivity came from electron tunneling experiments, in which the energy gap as a function of temperature was measured and showed excellent agreement with the BCS theory. In this problem electron tunneling between a normal metal and a superconductor will be explored. The notation is the same as in Problem Set 11. There, we showed that the single particle tunnel current can be expressed as

$$\begin{aligned} I_{\text{single}} &= ie\left[C^{+}_{A,A^\dagger}(-eV) - \left(C^{+}_{A,A^\dagger}(-eV)\right)^*\right] \\ &= -2e\text{Im}\left[C^{+}_{A,A^\dagger}(-eV)\right]. \end{aligned}$$

Here, the Fourier transform

$$C^{+}_{A,A^\dagger}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} C^{+}_{A,A^\dagger}(t),$$

of the retarded correlation function

$$C^{+}_{A,A^\dagger}(t) = -i\Theta(t)\langle[A(t), A(0)]\rangle$$

is used, and the operator A is defined as

$$A = \sum_{\boldsymbol{k},\boldsymbol{p}} T_{\boldsymbol{k},\boldsymbol{p}} c^\dagger_{\boldsymbol{k}} c_{\boldsymbol{p}}.$$

(a) We want to obtain the retarded correlation function $C^{+}_{A,A^\dagger}(\omega)$ via analytic continuation from the imaginary time correlation function $C^{+}_{A,A^\dagger}(i\omega_n)$.
Show that

$$C^{\tau}_{A,A^\dagger}(i\omega_n) = \sum_{\boldsymbol{k},\boldsymbol{p}} |T_{\boldsymbol{k},\boldsymbol{p}}|^2 T \sum_{i\epsilon_l} G_L(i\epsilon_l, \xi_{\boldsymbol{k}}) G_R(i\epsilon_l + i\omega_n, \xi_{\boldsymbol{p}}).$$

(b) Use the spectral representation

$$G_{L,R}(i\epsilon_l, \xi_{\boldsymbol{k},\boldsymbol{p}}) = \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\boldsymbol{k},\boldsymbol{p}})}{i\epsilon_l - \omega},$$

for both Green functions to evaluate the Matsubara sum in the expression for $C^{\tau}_{A,A^\dagger}(i\omega_n)$. It will be useful to perform a partial fraction decomposition and to make use of the identity

$$T\sum_{\epsilon_l}\frac{e^{i\eta\epsilon_l}}{i\epsilon_l-\xi_{\boldsymbol{k}}}=n_F(\xi_{\boldsymbol{k}}).$$

Analytically continue $i\omega_n\to\omega+i\eta$ to derive the retarded correlation function, and show that the current is given by

$$I=2e\sum_{\boldsymbol{k},\boldsymbol{p}}|T_{\boldsymbol{k},\boldsymbol{p}}|^2\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}A_L(\epsilon,\xi_{\boldsymbol{k}})A_R(\epsilon-eV,\xi_{\boldsymbol{p}})[n_F(\epsilon)-n_F(\epsilon-eV)].$$

(c) Now make use of the fact that the tunneling matrix elements can be approximated as being independent of momenta, i.e.

$$T_{\boldsymbol{k},\boldsymbol{p}}=T_0,$$

and introduce the tunneling densities of states

$$\rho_L(\epsilon)=\frac{1}{2\pi\Omega_L}\sum_{\boldsymbol{k}}A_L(\epsilon,\xi_{\boldsymbol{k}}),\qquad \rho_R(\epsilon)=\frac{1}{2\pi\Omega_R}\sum_{\boldsymbol{p}}A_R(\epsilon,\xi_{\boldsymbol{p}}),$$

to show that the tunneling current is given by

$$I_{\text{single}}=4\pi e\Omega_L\Omega_R|T_0|^2\int d\epsilon\,\rho_L(\epsilon)\rho_R(\epsilon-eV)[n_F(\epsilon)-n_F(\epsilon-eV)].$$

Here Ω_L and Ω_R are the volumes of the left and the right system, respectively.

(d) Finally, use that for a normal metal $\rho_L(\epsilon)=\rho_F$ and that for the superconductor

$$\rho_R(\epsilon)=\rho_F\frac{|\epsilon|}{\sqrt{\epsilon^2-|\Delta|^2}}\Theta(\epsilon^2-\Delta^2),$$

to evaluate the current.

2. Meissner effect

3+3+3 Punkte

In the high temperature (static) limit, the effective action of a vector potential in a superconductor is

$$S_{\text{eff}}[\boldsymbol{A}]=\frac{\beta}{2}\int d^d r\,\boldsymbol{A}^{\perp}(\boldsymbol{r})\cdot\left(-\frac{1}{\mu_0}\nabla^2+\frac{n_s}{m}\right)\boldsymbol{A}^{\perp}(\boldsymbol{r}),$$

where in Fourier space the transverse component of $\boldsymbol{A}$ is defined by

$$\boldsymbol{A}^{\perp}(\boldsymbol{q})=\boldsymbol{A}(\boldsymbol{q})-\frac{\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{A}(\boldsymbol{q}))}{q^2}.$$

In the above action, μ_0 is the vacuum permeability, n_s is the superfluid density, and m is the electron mass.

(a) Show that the gradient term in the effective action is equivalent to the standard magnetic field energy

$$\frac{1}{2\mu_0}\int d^d r\,\boldsymbol{B}(\boldsymbol{r})^2,$$

where $\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r})$.

(b) Show that the equation of motion for $\boldsymbol{A}(\boldsymbol{r})$ becomes

$$\nabla^2 \boldsymbol{A} = \frac{1}{\lambda^2}\boldsymbol{A} \ .$$

Derive an expression for the length scale λ in terms of n_s. This equation implies that a weak magnetic field will penetrate into a superconductor only up to the length scale λ.

(c) To get a feeling for the penetration of a magnetic field into a superconductor, consider a semi-infinite slab of superconductor which occupies all of space for $x > 0$. Assume that for $x < 0$, there is a magnetic field $\boldsymbol{B} = (0, 0, B_0)$ pointing along the z-axis. Solve the equation derived in (b) to get the profile of magnetic field inside the superconductor for $x > 0$.