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QFT of Many-Particle Systems - Problem Set 10

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 14.06.2024, 13:30**.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. The Cooper Problem

3+3+2 Punkte

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

$$\phi(\boldsymbol{r}-\boldsymbol{r}') = \int \frac{d^3k}{(2\pi)^3} \chi(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}. \tag{1}$$

In the momentum representation the Schrödinger equation has the form

$$\left(E - 2\frac{\hbar^2 k^2}{2m}\right)\chi(\boldsymbol{k}) = \int \frac{d^3k}{(2\pi)^3} V(\boldsymbol{k},\boldsymbol{k}')\chi(\boldsymbol{k}'). \tag{2}$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

$$\chi(\boldsymbol{k}) = 0, \qquad k < k_F. \tag{3}$$

We take the interaction of the pair to have the simple attractive form

$$V(\boldsymbol{k}_1,\boldsymbol{k}_2) = \begin{cases} -g, & \epsilon_F \leq \frac{\hbar^2 k_i^2}{2m} \leq \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases}, \tag{4}$$

with $i = 1, 2$, and look for a bound-state solution to the Schrödinger equation (2) consistent with the constraint (3). Since we are considering only one-electron levels which in the absence of the attraction have energies in the excess of $2\epsilon_F$, a bound state will be one with energy less than $2\epsilon_F$, and the binding energy will be

$$\Delta = 2\epsilon_F - E. \tag{5}$$

(a) Show that a bound state of energy E exists provided that

$$1 = g\int_{\epsilon_F}^{\epsilon_F+\hbar\omega_D} d\epsilon \frac{\rho(\epsilon)}{2\epsilon - E}, \tag{6}$$

where $\rho(\epsilon)$ is the density of one-electron levels per unit volume for a given spin.

(b) Show that Eq. (6) has a solution with $E < 2\epsilon_F$ for arbitrarily weak g, provided that $\rho(\epsilon_F) \neq 0$ and that $\rho(\epsilon)$ is continuous. (Note the crucial role played by the exclusion principle: If the lower cutoff was not ϵ_F, but 0, then since $\rho(0) = 0$, there would *not* be a solution for arbitrarily weak couplings.)

(c) Assuming that $\rho(\epsilon)$ differs negligibly from $\rho(\epsilon_F) = \rho_F$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega_D$, show that the binding energy is given by

$$\Delta = 2\hbar\omega_D \frac{e^{-\frac{2}{g\rho_F}}}{1 - e^{-\frac{2}{g\rho_F}}}, \tag{7}$$

or, in the weak coupling limit:

$$\Delta = 2\hbar\omega_D e^{-\frac{2}{g\rho_F}}. \tag{8}$$

2. BCS as a mean-field theory

4+2+4 Punkte

In this problem we will consider electrons interacting through the same type of interaction as in the previous problem. That is we consider the following Hamiltonian

$$H = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} V(\boldsymbol{k}_1,\boldsymbol{k}_2) c^\dagger_{\boldsymbol{k}_1\uparrow} c^\dagger_{-\boldsymbol{k}_1\downarrow} c_{-\boldsymbol{k}_2\downarrow} c_{\boldsymbol{k}_2\uparrow},$$

where

$$V(\boldsymbol{k}_1,\boldsymbol{k}_2) = \begin{cases} -g, & \epsilon_F \le \frac{\hbar^2 k_i^2}{2m} \le \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases}.$$

(a) Proceeding analoguously to the Hartree-Fock mean-field theory in problem 15 convince yourself that a mean-field decoupling of the Hamiltonian yields

$$H = \sum_{\boldsymbol{k}\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \sum_{\boldsymbol{k}} \Delta_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} - \sum_{\boldsymbol{k}} \Delta^*_{\boldsymbol{k}} c_{\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}\uparrow} + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} V(\boldsymbol{k}_1,\boldsymbol{k}_2) \langle c^\dagger_{\boldsymbol{k}_1\uparrow} c^\dagger_{-\boldsymbol{k}_1\downarrow}\rangle \langle c_{\boldsymbol{k}_2\downarrow} c_{-\boldsymbol{k}_2\uparrow}\rangle,$$

with

$$\Delta_{\boldsymbol{k}_1} = -\sum_{\boldsymbol{k}_2} V(\boldsymbol{k}_1,\boldsymbol{k}_2) \langle c_{-\boldsymbol{k}_2\downarrow} c_{\boldsymbol{k}_2\uparrow}\rangle. \tag{9}$$

(b) Show that the Hamiltonian can be diagonalised by the following unitary transformation

$$\begin{pmatrix} \alpha^\dagger_{\boldsymbol{k}\uparrow} \\ \alpha_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\boldsymbol{k}} & \sin\theta_{\boldsymbol{k}} \\ \sin\theta_{\boldsymbol{k}} & -\cos\theta_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} \\ c_{-\boldsymbol{k}\downarrow} \end{pmatrix},$$

such that

$$H = \sum_{\boldsymbol{k}\sigma} E_{\boldsymbol{k}} \alpha^\dagger_{\boldsymbol{k}\sigma} \alpha_{\boldsymbol{k}\sigma} + \text{constant}.$$

Derive an expression for $E_{\boldsymbol{k}}$.

(c) By writing the original fermionic operators in terms of their transformed counterparts, use Equation (9) to derive a self-consistency equation for $\Delta_{\boldsymbol{k}}$. The expectation value is with respect to the ground state

$$|BCS\rangle = \prod_{\boldsymbol{k}} \alpha_{-\boldsymbol{k}\downarrow} \alpha_{\boldsymbol{k}\uparrow} |0\rangle \sim \prod_{\boldsymbol{k}} \left(\cos\theta_{\boldsymbol{k}} - \sin\theta_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow}\right) |0\rangle,$$

where $|0\rangle$ denotes the vacuum.