Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
Dr. C. Drukier

Quantum Field Theory of Many-Particle Systems - Problem Set 8

Summer Semester 2017

Due: The problem set will be discussed in the tutorial on **Wednesday, 31.05.2017, 13:30**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_SS17.html

16. Charge neutrality and zero momentum component *4 Punkte*

In the Jellium model, the negative charge of the electrons is compensated by a positively charged background of ions, which is assumed to be spatially homogeneous. Show that this positive background is responsible for the absence of the zero momentum component in the interaction term, i.e. show that

$$\frac{1}{2}\int d^d r\, d'^r\,[\hat{\rho}(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\hat{\rho}(\mathbf{r}') - \hat{\rho}(\mathbf{r})V(\mathbf{r}-\mathbf{r}')\bar{\rho}] = \frac{1}{2L^d}\sum_{\mathbf{q}\neq 0}\hat{\rho}_{-\mathbf{q}}V(\mathbf{q})\hat{\rho}_{\mathbf{q}}.$$

Here, $\bar{\rho} = \langle\hat{\rho}(\mathbf{r})\rangle$ is the average charge density of electrons, and the Fourier transform of the density operator and interaction potential are defined as $\hat{\rho}_{\mathbf{q}} = \int d^d r\,\hat{\rho}(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}$ and $V(\mathbf{q}) = \int d^d r\, V(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}$, respectively. All spatial integrals are over a hypercube of volume L^d, and you may assume periodic boundary conditions. *Hint: You may use that* $\frac{1}{L^d}\int d^d r\,\hat{\rho}(\mathbf{r}) \approx \bar{\rho}$.

17. Interaction in frequency space *4 Punkte*

Starting from the interaction

$$\hat{V}_{\text{el}} = \frac{1}{2}\sum_{\sigma,\sigma'}\int d^3 r \int d^3 r' \Psi^\dagger_\sigma(\mathbf{r})\Psi^\dagger_{\sigma'}(\mathbf{r}')\frac{e^2}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|}\Psi_{\sigma'}(\mathbf{r}')\Psi_\sigma(\mathbf{r}),$$

show that the interaction part of the action, in frequency space, is given by

$$S_{\text{el}}[\bar{\Psi},\Psi] = \frac{1}{2}T\sum_{\substack{\epsilon_l,\epsilon'_l\\ \omega_n}}\sum_{\sigma,\sigma'}\frac{1}{L^3}\sum_{\substack{\mathbf{k},\mathbf{k}'\\ \mathbf{q}\neq 0}}\bar{\Psi}_{\mathbf{k}+\mathbf{q},\sigma}(\epsilon_l+\omega_n)\bar{\Psi}_{\mathbf{k}'-\mathbf{q},\sigma'}(\epsilon'_l-\omega_n)\frac{e^2}{\epsilon q^2}\Psi_{\mathbf{k}',\sigma'}(\epsilon'_l)\Psi_{\mathbf{k},\sigma}(\epsilon_l),$$

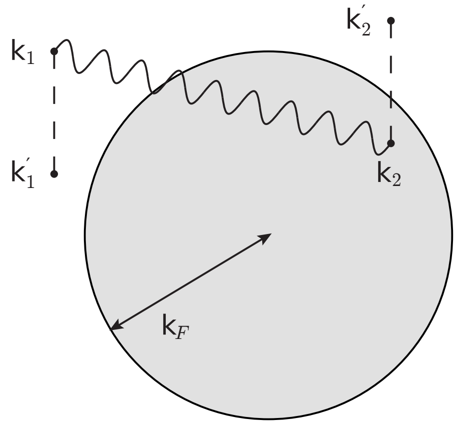
where the field operators in Matsubara space are given by

$$\Psi(\omega_n) = \frac{1}{\sqrt{\beta}}\int_0^\beta d\tau\,\Psi(\tau)e^{i\omega_n\tau},$$
$$\bar{\Psi}(\omega_n) = \frac{1}{\sqrt{\beta}}\int_0^\beta d\tau\,\bar{\Psi}(\tau)e^{-i\omega_n\tau}.$$

18. Fermi liquid quasiparticle lifetime

2+2+2+3+3 Punkte

Fermi liquid theory provides the standard model of simple metals. In it, the low-energy properties of a metallic state can be described in terms of a fermionic liquid of quasiparticles. These quasiparticles are assumed to be adiabatically connected to the states of a non-interacting Fermi gas, and can be labeled by the corresponding quantum numbers. Note, however, that they are not exact eigenstates of the interacting system. Instead, they obtain a finite lifetime. Under certain conditions, the lifetime τ_{qp} of a quasiparticle state is given by the inverse scattering rate which can be determined from Fermi's Golden Rule. To leading order in the interaction V we have



$$\frac{1}{\tau_{qp}} = W = \frac{2\pi}{\hbar} \sum_{\text{out}} |\langle \text{in}|V|\text{out}\rangle|^2 \delta(E_{\text{in}} - E_{\text{out}}).$$

The decay of the quasiparticle state is thus described by the probability per unit time W to have a transition from an “in” - to one of the “out”-states, induced by the interaction V with the constraint of energy conservation.

a) How long does the lifetime of a quasiparticle with energy ϵ above the Fermi surface have to be in order to have a well-defined quantum state?

b) We will consider the interaction of a quasiparticle with wave vector $\mathbf{k_1}$ above the Fermi sea, $|\mathbf{k_1}| > k_F$, with a quasiparticle at $\mathbf{k_2}$, below the Fermi energy, $|\mathbf{k_2}| < k_F$ (there are many of these, see the figure.). The two states after scattering will have wave vector $\mathbf{k_1}'$ and $\mathbf{k_2}'$. Formulate energy and momentum conservation and express $\sum_{\text{out}}$ in terms of independent momentum integrations. Where are the “out”-states located relative to the Fermi surface? Why?

c) For sufficiently short-ranged interactions one finds $W \propto \sum_{\text{out}} \delta(E_{\text{in}} - E_{\text{out}})$. Show that energy and momentum conservation constrain the angular momentum-integrations, such that only the radial momentum-integrations remain.

d) Provide an argument that a quasiparticle “in”-state with momentum close to the Fermi momentum k_F gives $|\mathbf{k_1}| \simeq |\mathbf{k_1}'| + |\mathbf{k_2}'| - |\mathbf{k_2}|$.

e) From the properties of the “in”- and “out”-states specified above and the approximate relation derived in (b), find appropriate bounds for the radial momentum-integrations. Further assume $\int dk\, k^{d-1} \simeq k_F^{d-1} \int d\epsilon_k$. Perform the radial integrations and discuss the behaviour of $1/\tau_{qp}$ upon letting the energy of the initially excited quasiparticle approach the Fermi surface. Using $\epsilon \propto k_B T$, estimate the temperature dependence of the quasiparticle lifetime in a Fermi liquid at temperature T.