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Quantum Field Theory of Many-Particle Systems - Problem Set 3

Summer Semester 2017

Due: The problem set will be discussed in the tutorial on **Wednesday, 26.04.2017, 13:30**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_SS17.html

5. Quantisation of the Radiation Field

2+3+3 Punkte

In the absence of charges, and in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the electromagnetic field is described by the Lagrangian

$$L = \frac{1}{2}\int_\Omega d^3x \left[\epsilon_0 \left(\partial_t \mathbf{A}\right)^2 + \frac{1}{\mu_0}\mathbf{A}\nabla^2\mathbf{A}\right].$$

Here ϵ_0 denotes the vacuum dielectric constant, μ_0 is the vacuum permeability, and Ω is a cuboid with extensions L_x, L_y, and L_z. Note that the speed of light is $c = 1/\sqrt{\epsilon_0\mu_0}$.

(a) Derive the equation of motion for $\mathbf{A}$ by extremising the action $S = \int dtL$.

(b) Find eigenfunctions $\mathbf{A}_\mathbf{k}$ and eigenvalues $\omega_\mathbf{k}^2$ of the equation

$$-\nabla^2\mathbf{A} = \frac{\omega_\mathbf{k}^2}{c^2}\mathbf{A},$$

by using periodic boundary conditions. It may be useful to introduce, for each $\mathbf{k}$, a set of orthonormal vectors $\{\xi_{\mathbf{k},1}, \xi_{\mathbf{k},2}\}$ which are both perpendicular to $\mathbf{k}$. The time-dependent solution then has a series expansion

$$\mathbf{A}(\mathbf{x},t) = \frac{1}{\Omega}\sum_{\mathbf{k},i}\alpha_{\mathbf{k},i}(t)\mathbf{A}_{\mathbf{k},i}(\mathbf{x}).$$

Insert this series expansion in the Lagrangian, and find the momenta

$$\pi_{\mathbf{k},i} = \frac{\partial L}{\partial\dot{\alpha}_{\mathbf{k},i}},$$

canonically conjugate to the coordinates $\alpha_{\mathbf{k},i}$. Use the Legendre transform $H = \sum_{\mathbf{k},i}\pi_{\mathbf{k},i}\dot{\alpha}_{\mathbf{k},i} - L(\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i})$ to obtain the Hamiltonian.

(c) The classical Hamiltonian $H(\{\pi_{\mathbf{k},i}, \alpha_{\mathbf{k},i}\})$ can be quantised by imposing canonical commutation relations

$$[\alpha_{\mathbf{k},i}, \alpha_{\mathbf{q},j}] = 0, \quad [\pi_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = 0, \quad [\alpha_{\mathbf{k},i}, \pi_{\mathbf{q},j}] = i\hbar\delta_{\mathbf{k},\mathbf{q}}\delta_{i,j},$$

on the coordinates $\alpha_{\mathbf{k},i}$ and their canonically conjugate momenta $\pi_{\mathbf{k},j}$. In analogy to the one-dimensional harmonic oscillator, we define photon creation and annihilation operators

$$a^\dagger_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0\omega_\mathbf{k}}{2\hbar}}\left(\alpha_{-\mathbf{k},j} - \frac{i}{\epsilon_0\omega_\mathbf{k}}\pi_{\mathbf{k},j}\right), \quad a_{\mathbf{k},j} = \sqrt{\frac{\epsilon_0\omega_\mathbf{k}}{2\hbar}}\left(\alpha_{\mathbf{k},j} + \frac{i}{\epsilon_0\omega_\mathbf{k}}\pi_{-\mathbf{k},j}\right).$$

Show that $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$ obey the commutation relations of harmonic oscillator ladder operators, and express the Hamiltonian in terms of $a_{\mathbf{k},j}$ and $a^\dagger_{\mathbf{k},j}$.

6. Jordan-Wigner transformation *1+2+2+2 Punkte*

In this problem we will show that a spin 1/2 system can be represented in terms of fermionic operators. In particular we represent an up spin as a particle and a down spin as the vacuum $|0\rangle$ such that $|\uparrow\rangle = |1\rangle = c^\dagger|0\rangle$ and $|\downarrow\rangle = |0\rangle = c|1\rangle$, with c denoting a fermionic operator. Naively we might try to construct a representation by writing $\sigma^+ = c^\dagger$ and $\sigma^- = c$, with $\sigma^z = 2c^\dagger c - 1$.

(a) Show that, in this representation, the spins indeed satisfy the spin algebra $[\sigma^x, \sigma^y] = 2i\sigma^z$.

(b) There is however a problem with this representation since spins on different sites commute whilst fermionic operators anticommute. This sign somehow has to be fixed. In one dimension this can be done by the following transformation, known as a Jordan-Wigner transformation:

$$\sigma^+_n = c^\dagger_n e^{-i\pi\sum_{j<n} n_j}, \qquad \sigma^-_n = c_n e^{i\pi\sum_{j<n} n_j}, \qquad \sigma^z_n = 2c^\dagger_n c_n - 1.$$

Here $n_j = c^\dagger_j c_j$ is the number operator. In one dimension the interpretation of this representation is as follows: We order the particles on a line. By then counting the number of particles to the left of a given site we pick up a phase of $+1$ or -1. This allows us to consider the particles as fermions. Using the Jordan-Wigner transformation, show that $\sigma^+_n\sigma^-_{n+1} = c^\dagger_n c_{n+1}$.

(c) Consider now the so-called spin-1/2 XY model in a magnetic field h pointing in the z-direction

$$H = -\sum_{n=1}^{N-1}\left[J_x\sigma^x_n\sigma^x_{n+1} + J_y\sigma^y_n\sigma^y_{n+1}\right] - \sum_{n=1}^{N} h\sigma^z_n.$$

Here $J_x + J_y > 0$ and we will assume periodic boundary conditions. Using the Jordan-Wigner transformation, show that the Hamiltonian can be written, up to unimportant constants, as

$$H = \sum_{n=1}^{N-1}\left[-(J_x+J_y)(c^\dagger_n c_{n+1} + c^\dagger_{n+1}c_n) + (J_x - J_y)(c^\dagger_{n+1}c^\dagger_n + c_n c_{n+1})\right] - \sum_{n=1}^{N} 2hc^\dagger_n c_n.$$

(d) For $J_y = 0$ the Hamiltonian in the previous part reduces to the transverse Ising model. In this case diagonalise the Hamiltonian and thus calculate the spectrum. For which values of J_x does the system become gapless?