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Quantum Field Theory of Many-Particle Systems - Problem Set 13

Summer Semester 2017

Due: The problem set will be discussed in the tutorial on **Wednesday, 05.07.2017, 13:30**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_SS17.html

26. Josephson effect

4+4+4+4+4 Punkte

Electron tunneling between two superconductors has two kinds of contributions. One is due to tunneling of single particles, as in the case between two metals or between a metal and a superconductor. The other is tunneling in pairs, which gives rise to the Josephson effect. The notation is the same as in Problem 23. There, we showed that the single particle tunnel current can be expressed as

$$I_{\text{single}} = e\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\left\{e^{ieV(t'-t)}\langle[A(t),A^\dagger(t')]\rangle - e^{ieV(t-t')}\langle[A^\dagger(t),A(t')]\rangle\right\},$$

where the operator A is defined as

$$A = \sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} c^\dagger_{\mathbf{k}} c_{\mathbf{p}}.$$

In the case of tunneling between two superconductors, there is another contribution

$$I_{\text{pair}}(t) = e\int_{-\infty}^{\infty} dt'\,\Theta(t-t')\left\{e^{-ieV(t'+t)}\langle[A(t),A(t')]\rangle - e^{ieV(t+t')}\langle[A^\dagger(t),A^\dagger(t')]\rangle\right\}.$$

In the following, we evaluate the voltage and time dependence of this pair contribution to the tunnel current.

a) The most unusual feature of this expression is the fact that the time dependence in the exponentials is $t+t'$. Rewrite this as $2t+t'-t$ and change the variable of integration to $t''=t-t'$. Show that the pair current can be expressed as

$$I_{\text{pair}} = -2e\,\mathrm{Im}\left[e^{-2eiVt} C^+_{A,A}(eV)\right].$$

Here, $C^+_{A,A^\dagger}(eV)$ is the Fourier transform of the retarded correlation function

$$C^+_{A,A}(t) = -i\Theta(t)\langle[A(t),A(0)]\rangle.$$

b) In order to calculate the retarded correlation function $C^{+}_{A,A}(eV)$, we start from the Matsubara function

$$C^{\tau}_{A,A}(i\omega_n) = \int_0^{\beta} d\tau\, e^{i\omega_n\tau} \langle T_\tau A(\tau) A(0)\rangle.$$

Show that

$$C^{\tau}_{A,A}(i\omega_n) = 2\sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} T_{-\mathbf{k},-\mathbf{p}} T \sum_{\epsilon_l} F^{\dagger}_{L}(i\epsilon_l, \xi_{\mathbf{k}}) F_R(i\epsilon_l - i\omega_n, \xi_{\mathbf{p}}).$$

Here

$$F(i\epsilon_l, \xi_{\mathbf{p}}) = \langle c_{-\mathbf{p},\downarrow}(-i\epsilon_l) c_{\mathbf{p},\uparrow}(i\epsilon_l)\rangle, \qquad F^{\dagger}(i\epsilon_l, \xi_{\mathbf{p}}) = \langle c^{\dagger}_{\mathbf{p},\uparrow}(-i\epsilon_l) c^{\dagger}_{-\mathbf{p},\downarrow}(i\epsilon_l)\rangle,$$

are the $(1,2)$ and the $(2,1)$ elements of the Gorkov Green function matrix

$$\mathcal{G}(i\epsilon_l, \xi_{\mathbf{p}}) = \frac{1}{(i\epsilon_l)^2 - \xi_{\mathbf{p}}^2 - |\Delta_0|^2} \begin{pmatrix} i\epsilon_l + \xi_{\mathbf{p}} & -\Delta_0 \\ -\bar{\Delta}_0 & i\epsilon_l - \xi_{\mathbf{p}} \end{pmatrix}.$$

Note that for the evaluation of $F^{\dagger}_{L}(i\epsilon_l, \xi_{\mathbf{p}})$ and $F_R(i\epsilon_l, \xi_{\mathbf{p}})$, the order parameter in the Gorkov Green function is respectively $\bar{\Delta}_L$ instead of $\bar{\Delta}_0$ and Δ_R instead of Δ_0.

c) Use spectral representations

$$F^{\dagger}(i\epsilon_l, \xi_{\mathbf{k}}) = \bar{\Delta}_L \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\mathbf{k}})}{i\epsilon_l - \omega},$$

and

$$F(i\epsilon_l, \xi_{\mathbf{p}}) = \Delta_R \int \frac{d\omega}{2\pi} \frac{A(\omega, \xi_{\mathbf{p}})}{i\epsilon_l - \omega},$$

for both Green functions to evaluate the Matsubara sum in the expression for $C^{\tau}_{A,A}(i\omega_n)$. It will be useful to perform a partial fraction decomposition and to make use of the identity

$$T\sum_{\epsilon_l} \frac{e^{i\eta\epsilon_l}}{i\epsilon_l - \xi_{\mathbf{k}}} = n_F(\xi_{\mathbf{k}}).$$

Show that

$$C^{\tau}_{A,A}(i\omega_n) = 2\bar{\Delta}_L \Delta_R \sum_{\mathbf{k},\mathbf{p}} T_{\mathbf{k},\mathbf{p}} T_{-\mathbf{k},-\mathbf{p}} \int \frac{d\epsilon}{2\pi}\frac{d\epsilon'}{2\pi} A^{*}(\epsilon, \xi_{\mathbf{k}}) A(\epsilon', \xi_{\mathbf{p}}) \frac{n_F(\epsilon) - n_F(\epsilon')}{i\omega_n + \epsilon - \epsilon'}.$$

d) Show that the spectral function depends on $\xi_{\mathbf{k}}$ only via $\lambda_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_0|^2}$, and that it is given by

$$A(\epsilon, \lambda_{\mathbf{k}}) = 2\pi \frac{1}{2\lambda_{\mathbf{k}}} [\delta(\epsilon - \lambda_{\mathbf{k}}) - \delta(\epsilon + \lambda_{\mathbf{k}})].$$

Use this expression for the spectral function and make use of the momentum independence of tunnel matrix elements to show that, in the limit of zero temperature, and after analytical continuation $i\omega_n \to \omega + i\eta$, the retarded correlation function is given by

$$C^{+}_{A,A}(eV) = \frac{1}{2} |\bar{\Delta}_L \Delta_R T_0^2| e^{i\varphi} \sum_{\mathbf{k},\mathbf{p}} \frac{1}{\lambda_{\mathbf{k}}\lambda_{\mathbf{p}}} \left[\frac{1}{eV - \lambda_{\mathbf{k}} - \lambda_{\mathbf{p}}} - \frac{1}{eV + \lambda_{\mathbf{k}} + \lambda_{\mathbf{p}}} \right].$$

e) Assume now that $|\Delta_L| = |\Delta_R| = \Delta_0$. Convince yourself that for $eV < 2\Delta_0$, the δ-function part in $C^{+}_{A,A}$ vanishes and that, for the same reason, there are no singular contributions to the momentum sums. As a consequence, one can write

$$C^{+}_{A,A}(eV) = \frac{1}{2e} J_S(eV) e^{i\varphi},$$

where $J_S(eV)$ depends smoothly on voltage for $eV \to 0$. Show that in this notation the pair contribution to the tunneling current is given by

$$I_{\text{pair}} = J_S(eV) \sin(\omega t + \varphi) \qquad \text{with} \qquad \omega = \frac{2eV}{\hbar}.$$