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Quantum Field Theory of Many-Particle Systems - Problem Set 10

Summer Semester 2017

Due: The problem set will be discussed in the tutorial on **Wednesday, 14.06.2017, 13:30**

Internet: The problem sets can be downloaded from
http://home.uni-leipzig.de/stp/QFT_of_MPS_SS17.html

21. The Cooper Problem   *3+3+2 Punkte*

Consider a pair of electrons in a singlet state, described by the symmetric spatial wave function

$$(1) \qquad \phi(\mathbf{r} - \mathbf{r}') = \int \frac{d^3k}{(2\pi)^3} \chi(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}.$$

In the momentum representation the Schrödinger equation has the form

$$(2) \qquad \left(E - 2\frac{\hbar^2 k^2}{2m} \right) \chi(\mathbf{k}) = \int \frac{d^3k}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}') \chi(\mathbf{k}').$$

We assume that the two electrons interact in the presence of a degenerate free electron gas, whose existence is felt only via the exclusion principle: electron levels with $k < k_F$ are forbidden to each of the two electrons, which gives the constraint:

$$(3) \qquad \chi(\mathbf{k}) = 0, \qquad k < k_F.$$

We take the interaction of the pair to have the simple attractive form

$$(4) \qquad V(\mathbf{k_1}, \mathbf{k_2}) = \begin{cases} -g, & \epsilon_F \leq \frac{\hbar^2 k_i^2}{2m} \leq \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases},$$

with $i = 1, 2$, and look for a bound-state solution to the Schrödinger equation (2) consistent with the constraint (3). Since we are considering only one-electron levels which in the absence of the attraction have energies in the excess of $2\epsilon_F$, a bound state will be one with energy less than $2\epsilon_F$, and the binding energy will be

$$(5) \qquad \Delta = 2\epsilon_F - E.$$

a) Show that a bound state of energy E exists provided that

$$(6) \qquad 1 = g \int_{\epsilon_F}^{\epsilon_F + \hbar\omega_D} d\epsilon \frac{\rho(\epsilon)}{2\epsilon - E},$$

where $\rho(\epsilon)$ is the density of one-electron levels per unit volume for a given spin.

b) Show that Eq. (6) has a solution with $E < 2\epsilon_F$ for arbitrarily weak g, provided that $\rho(\epsilon_F) \neq 0$ and that $\rho(\epsilon)$ is continuous. (Note the crucial role played by the exclusion principle: If the lower cutoff was not ϵ_F, but 0, then since $\rho(0) = 0$, there would *not* be a solution for arbitrarily weak couplings.

c) Assuming that $\rho(\epsilon)$ differs negliguously from $\rho(\epsilon_F) = \rho_F$ in the range $\epsilon_F < \epsilon < \epsilon_F + \hbar\omega_D$, show that the binding energy is given by

(7)
$$\Delta = 2\hbar\omega_D \frac{e^{-\frac{2}{g\rho_F}}}{1 - e^{-\frac{2}{g\rho_F}}},$$

or, in the weak coupling limit:

(8)
$$\Delta = 2\hbar\omega_D e^{-\frac{2}{g\rho_F}}.$$

22. BCS as a mean-field theory   *4+2+4 Punkte*

In this problem we will consider electrons interacting through the same type of interaction as in the previous problem. That is we consider the following Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + \sum_{\mathbf{k_1},\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) c_{\mathbf{k_1}\uparrow}^\dagger c_{-\mathbf{k_1}\downarrow}^\dagger c_{-\mathbf{k_2}\downarrow} c_{\mathbf{k_2}\uparrow},$$

where

$$V(\mathbf{k_1},\mathbf{k_2}) = \begin{cases} -g, & \epsilon_F \leq \frac{\hbar^2 k_i^2}{2m} \leq \epsilon_F + \hbar\omega_D, \\ 0, & \text{otherwise} \end{cases}.$$

a) Proceeding analoguously to the Hartree-Fock mean-field theory in problem 15 convince yourself that a mean-field decoupling of the Hamiltonian yields

$$H = \sum_{\mathbf{k}\sigma} \xi_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} - \sum_{\mathbf{k}} \Delta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^\dagger c_{-\mathbf{k}\downarrow}^\dagger - \sum_{\mathbf{k}} \Delta_{\mathbf{k}}^* c_{\mathbf{k}\downarrow} c_{-\mathbf{k}\uparrow} + \sum_{\mathbf{k_1},\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) \langle c_{\mathbf{k_1}\uparrow}^\dagger c_{-\mathbf{k_1}\downarrow}^\dagger \rangle \langle c_{\mathbf{k_2}\downarrow} c_{-\mathbf{k_2}\uparrow} \rangle,$$

with

(9)
$$\Delta_{\mathbf{k_1}} = -\sum_{\mathbf{k_2}} V(\mathbf{k_1},\mathbf{k_2}) \langle c_{-\mathbf{k_2}\downarrow} c_{\mathbf{k_2}\uparrow} \rangle.$$

b) Show that the Hamiltonian can be diagonalised by the following unitary transformation

$$\begin{pmatrix} \alpha_{\mathbf{k}\uparrow}^\dagger \\ \alpha_{-\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{k}} & \sin\theta_{\mathbf{k}} \\ \sin\theta_{\mathbf{k}} & -\cos\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow}^\dagger \\ c_{-\mathbf{k}\downarrow} \end{pmatrix},$$

such that

$$H = \sum_{\mathbf{k}\sigma} E_{\mathbf{k}} \alpha_{\mathbf{k}\sigma}^\dagger \alpha_{\mathbf{k}\sigma} + \text{constant}.$$

Derive an expression for $E_{\mathbf{k}}$.

c) By writing the original fermionic operators in terms of their transformed counterparts, use Equation (9) to derive a self-consistency equation for $\Delta_{\mathbf{k}}$. The expectation value is with respect to the ground state

$$|BCS\rangle = \prod_{\mathbf{k}} \alpha_{-\mathbf{k}\downarrow}\alpha_{\mathbf{k}\uparrow}|0\rangle \sim \prod_{\mathbf{k}} \left(\cos\theta_{\mathbf{k}} - \sin\theta_{\mathbf{k}} c_{\mathbf{k}\uparrow}^{\dagger}c_{-\mathbf{k}\downarrow}^{\dagger}\right)|0\rangle,$$

where $|0\rangle$ denotes the vacuum.