Universität Leipzig
Institut für Theoretische Physik
Prof. Dr. B. Rosenow

Statistical Physics, Spring 2011
Problem Set 3

Course Information:

- ☞ Class times: lectures Monday and Thursday, 11:00-12:30 in SR 218, tutorials Friday, 9:15-10:45 in SR 221
- ☞ Final exam: July 11, 13:30 in ThHS
- ☞ The course website is www.uni-leipzig.de/˜stp/Statistical_Physics.html
- ☞ In the tutorials you will be expected to present solutions to the class on a volunteer basis. Before each class please decide whether you would like to present any particular problem. If nobody volunteers you may be asked to present. The purpose of this is to gain experience working through problems as a group. Therefore it is informal and need not cause concern. In particular, *please* do not skip a class because you could not complete the problem set. These are the classes you most need to attend!
- ☞ For questions regarding the problem sets, please email Tony at anthony.wright in the uni-leipzig.de

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8. Expansion of a gas (2+1+1 Marks)

N indistinguishable classical particles form an ideal gas in the volume V. Each atom has a momentum in the range $p_{\min} < |\mathbf{p}| < p_{\max}$.

The macro-state is defined by separating the volume of the container B into m equally sized sub-volumes, each containing some number of the total particles such that $m << N$. A cell i of the μ-space contains the single particle states in the sub-volume i.

(a) Determine the cell phase-space volume ω_i, and the entropy of the equilibrium state.

The container B is now joined to a second container B'. The $2m$ cells of the μ-space are now defined. The particles return to equilibrium.

(b) What is the entropy of the gas the moment the container size is doubled, yet all particles are still in B?

(c) What is the entropy of the gas after the particles have filled the doubled space and returned to equilibrium?

9. The continuum limit (4 Marks)

In the calculation of the particle fluctuations we can convert a sum to an integral by writing

$$\langle(\Delta N)^2\rangle = \frac{\sum_N (\Delta N)^2 e^{\Delta S(N)/k_B}}{\sum_N e^{\Delta S(N)/k_B}} \approx \frac{\int dN (\Delta N)^2 e^{\Delta S(N)/k_B}}{\int dN e^{\Delta S(N)/k_B}}$$

where $\Delta S(N) \approx -k_B \frac{c}{2}(\Delta N)^2$. Analyse when it is reasonable to take the continuum limit. You may find the Poisson summation formula helpful:

$$\sum_{N-=\infty}^{\infty} f(N) = \int_{-\infty}^{\infty} dN f(N)\big(1 + 2\sum_{k=1}^{\infty}\cos(2\pi k N)\big)$$

10. Maxwell Distribution (1+2+2+1+1+1 Marks)

This is a somewhat difficult but valuable exercise.

A classical nonrelativistic particle has energy $\epsilon(\mathbf{x}) = \frac{1}{2m}p^2$ at the phase-space point $\mathbf{x} = (\mathbf{r}, \mathbf{p})$. A non-interacting gas of such particles resides in a container of volume V.

(a) In the lectures, the particle density $n(\mathbf{x}) = Ne^{-\beta\epsilon(\mathbf{x})}/Z_1 h^3$ was obtained with the method of Lagrange-Multipliers. It is normalised only if the momentum integral spans all of $\mathbb{R}^3$. At a given energy of the gas, E, $|\mathbf{p}| \leq p_{\max} = \sqrt{2mE}$. Under what conditions is the contribution from $|\mathbf{p}| < p_{\max}$ negligible?

Next we will construct a rigorous derivation of $n(\mathbf{x})$.

(b) Determine, for a single particle, the phase-space volume $|\Omega_1(E)|dE$ for a state with energy $E \leq \epsilon(\mathbf{x}) \leq E + dE$. Use $|\Omega_1(E)| = \int d^6x_1 \delta(E - \epsilon(\mathbf{x}_1))$, where $\delta(z)$ is the Dirac-Delta function, together with the relation

$$\int dz f(z)\delta(g(z)) = \sum_\nu \frac{f(z_\nu)}{g'(z_\nu)}$$

Where z_ν are the set of solutions to $g(z) = 0$.

(c) Calculate now, by gradually adding individual particles, the N-particle phase-space volume $|\Omega_N(E)|dE$ for states with energy $E \leq \sum_{i=1}^N \epsilon(\mathbf{x}_i) \leq E + dE$. Using

$$|\Omega_N(E)| = \Pi_i^N\Big(\int d^6x_i\Big)\delta(E - \sum_i^N \epsilon(\mathbf{x}_i)),$$

identify $|\Omega_N(E)|dE = |\Gamma(M(E, E + dE))|$ from exercise 4. Also useful is the recursion relation

$$|\Omega_N(E_N)| = \int dE_{N-1}|\Omega_1(E_N - E_{N-1})||\Omega_{N-1}(E_{N-1})|.$$

For the following it is not important to calculate numerical factors explicitly.

(d) Find the particle distribution in μ-space. For this you can pick out, for example, the N^{th} particle, and use

$$n(\mathbf{x}_N) = N\Big(\Pi_{i=1}^{N-1}\int dE_i\Big)\delta(E - \sum_{i=1}^N \epsilon(\mathbf{x}_i))/|\Omega_N(E)|.$$

(e) Sketch $n(\mathbf{x})/N$ for a small N ($N = 1, 2, 3$), at fixed energy ϵ_f per particle. How does $n(\mathbf{x})$ vary qualitatively with increasing N?

(f) Discuss $n(\mathbf{x})/N$ in the large N limit. Compare your result with that obtained in the lectures. Calculate the Lagrange multiplier β as a function of ϵ_f.

11. Fluctuations in an ideal gas (0 Marks)

This is an optional question worth no marks

Consider an ideal gas in three statistical cases: those with Boltzmann, Bose, and Fermi distributions. As in task 7, cells enumerated by i are given, and their occupation numbers n_i fluctuate around their most probable values $\tilde{N}_i$.

(a) First, determine the entropy change upon change of occupation numbers for $\Delta N_i = N_i - \tilde{N}_i$ to second order in ΔN_i. Why does the linear term in ΔN_i disappear?

(b) Now, determine the mean occupation $\langle N_i \rangle$ in the cell.

(c) Calculate the average quadratic fluctuations, $\langle (\Delta N_i)^2 \rangle$ with help from the Gaussian approximation given in lectures. When are these average quadratic fluctuations large or small?