Universität Leipzig
Institut für Theoretische Physik
Prof. Dr. B. Rosenow

Statistical Physics, Spring 2011
Problem Set 2

Course Information:

- ☞ Class times: lectures Monday and Thursday, 11:00-12:30 in SR 218, tutorials Friday, 9:15-10:45 in SR 221
- ☞ Final exam: July 11, 13:30 in ThHS
- ☞ The course website is www.uni-leipzig.de/~stp/Statistical_Physics.html
- ☞ In the tutorials you will be expected to present solutions to the class on a volunteer basis. Before each class please decide whether you would like to present any particular problem. If nobody volunteers you may be asked to present. The purpose of this is to gain experience working through problems as a group. Therefore it is informal and need not cause concern. In particular, *please* do not skip a class because you could not complete the problem set. These are the classes you most need to attend!
- ☞ For questions regarding the problem sets, please email Tony at anthony.wright in the uni-leipzig.de

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4. The Energy Shell (4 Marks)

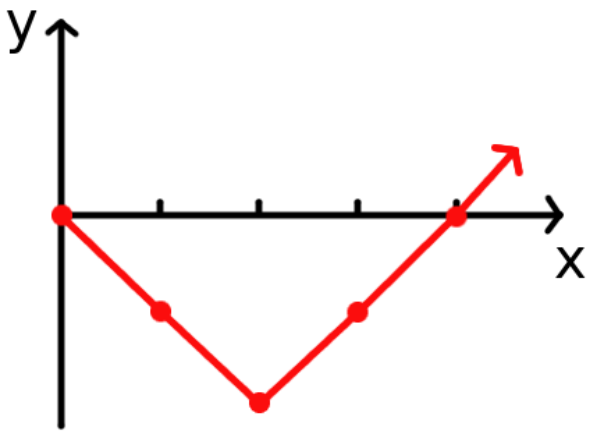
The purpose of this question is to show that for a system with a large number of degrees of freedom, the surface of the energy shell accounts for almost the entire volume of the phase space. In a container of volume V are N non-interacting particles of mass m. The energy of this ideal gas consists solely of kinetic energy. Each point $\vec{X} = (q, p)$ of the phase space represents a microstate of the system. A macrostate $M(E_1, E_2)$ corresponds to the set of all points in the phase space where $E_1 \leq \mathcal{H}(\vec{X}) \leq E_2$.

(a) Calculate the Phase space volume $|\Gamma(M(E_1, E_2))|$.

(b) Calculate the phase space volume of the energy shell $M((1-\epsilon)E, E)$ where $0 < \epsilon < 1$. Express this result as a fraction of the total phase space volume of the sphere $M(0, E)$.

5. Ising-Spin system (2+2 Marks)

The purpose of the next few questions is to demonstrate that for a system with a large number of degrees of freedom, the equilibrium state and small fluctuations around it contains all possible microstates except for a vanishingly small number of exceptions. This in turn tells us that the ergodic principle is not a necessary assumption when we do statistical mechanics, but in fact



that by considering all possible microstates, we are effectively considering the equilibrium case plus small fluctuations about it, and all other states provide a negligible contribution to our calculations.

A large class of physical systems have degrees of freedom which can take only two different values. As a canonical example for this class, we consider the Ising model with degree of freedom S which can take the values $S = \pm 1$. These degrees of freedom can correspond to the z component of a spin $\frac{1}{2}$ particle, by $S_z = \frac{\hbar S}{2}$. Another example would be the occupation in any fermionic system, as the single-particle states in this case, n, can be shown to be given by $n = \frac{1}{2}(1 + S)$. We label a microstate of the Ising system by $S_i = \pm 1$ where i enumerate the individual spins $i = 1, \ldots N$. There are 2^N other such states. The macrostate is defined by the set of all possible states for a given total spin $M = \sum_i S_i$. A macrostate with total spin M has $(N - M)/2$ spins with $S = -1$ and $(N + M)/2$ spins with $S = 1$ (We shall insist that N is even).

(a) Determine the number $W(M)$ of microstates for the macrostate with total magnetization M. Check that the total number of microstates $W_N = \sum_M W(M)$ is equal to 2^N.

(b) Find $W(M)$ in the limit $N \gg M \gg 1$ with the help of Stirling's formula (Gauss' distribution).

6. Directed polymer (1+1+1+1+2 Marks)

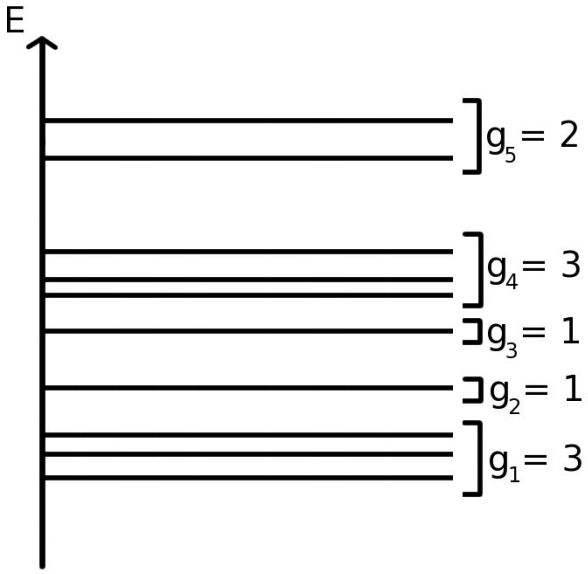
In a simplified model, it is assumed that a molecular chain can only take the following states: the atoms $i = 0, 1, 2, \cdots, N$ are at positions (x_i, y_i) of a square lattice (where x_i and y_i are integers). The atom at the origin is fixed at the position $x_0 = y_0 = 0$, and the other atoms are chained together such that $x_i - x_{i-1} = 1$ and $|y_i - y_{i-1}| = 1$. This is the polymer in the x-oriented direction which, for example, cannot form any loops. A sample polymer is shown in the figure.

(a) Determine the total number of micro-states.

(b) A macrostate (N, y) is the set of all micro-states for which the end of the chain is at $y_N = y$. How many microstates $W(N, y)$ has this macrostate?

(c) A macrostate (i, y) is defined by the position $y_i = y$ of the ith atom. What is the number of microstates $W(i, y)$ contained in this state? Also calculate $\frac{y}{W(i,y)}$.

(d) Now the macro state (i, y, N, y') is defined by both the position $y_i = y$ of the ith atom and $y_N = y'$ of Nth atom. What is the number of respective microstates $W(i, y, N, y')$ contained in this state? Also calculate $\sum_{y,y'} \frac{yy'}{W(i,y,N,y')}$.



(e) Finally, calculate the typical deflection of the chain end $\overline{y_n} = \left(\frac{\sum_y y^2 W(N,y)}{\sum_y W(N,y)} \right)^{1/2}$.

Hint: to perform the sum, try using the "trick" $y = \left| \frac{d}{dz} z^y \right|_{z=1}$.

7. Boson Distribution (1+1+1+2 Marks)

Consider a system of non-interacting bosons, where the single-particle states are labelled by $\nu \geq 1$, and have energy E_ν. A micro-state of the system is then a function of the occupation numbers of all single-particles $n_\nu \geq 0$.

In a coarse grained description, quasi-degenerate single-particles combine to form a group of $g_i \geq 1$ particles, as shown in the figure. The energies ϵ_i and occupation numbers n_i of the groups are $\epsilon_1 \approx E_1 \approx E_2 \approx \cdots \approx E_{g_1}$, $\epsilon_2 \approx E_{g_1+1} \approx E_{g_1+2} \approx \cdots \approx E_{g_1+g_2}$, etc., and $N_1 = n_1 + n_2 + \ldots n_{g_1}$, $N_2 = n_{g_1+1} + n_{g_1+2} + \ldots n_{g_1+g_2}$, and so on. A macrostate M of the system is determined through the occupation numbers n_i.

(a) Calculate the number of microstates $W(N_1, ..., n_i, ...)$ of the macrostate M.

Note that in Task 3 part of this combinatorial problem has already been treated. Approach this as $N_i, g_i \gg 1$.

(b) The number of particles $N = \sum_i n_i$ and the energy $E = \sum_i n_i \epsilon_i$ are both given. Find the occupation numbers $\tilde{n}_i$ for the macro-state at which the largest number of microstates exist.

Note that the method of Lagrange multipliers is useful here as you saw in the lectures.

(c) Find $W(\tilde{N}_1 + \Delta N_1, ..., \tilde{n}_i + \Delta n_i, ...)$ as a function of $\delta n_i = n_i - \tilde{n}_i$ to second order in δn_i.

Use an expansion of $W(\tilde{N}_1 + \Delta N_1, ..., \tilde{n}_i + \delta n_i, ...)$.

(d) Redo (a) - (c) for Fermions whose occupation numbers n_ν, can only take the values 0 or 1.