Universität Leipzig
Institut für Theoretische Physik
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Statistical Physics, Spring 2011
Problem Set 11

Course Information:

- ☞ Class times: lectures Monday and Thursday, 11:00-12:30 in SR 218, tutorials Friday, 9:15-10:45 in SR 221
- ☞ Final exam: July 11, 13:30 in ThHS
- ☞ The course website is www.uni-leipzig.de/~stp/Statistical_Physics.html

Problem 32: Ideal Fermions at low temperatures (4 Marks)

With the aid of the Sommerfeld-expansion, one can calculate various properties of fermions at low temperatures.

Calculate $E(T, N, V)$ up to $O(T^2)$. Derive $p(T, N, V)$ to $O(T^2)$ as well as $C_V(T, N, V)$ to $O(T)$.

Problem 33: Specific heat of a solid (4 Marks)

We consider the Debye model which describes the contribution of lattice vibrations (phonons) to the specific heat of a solid. In this model, phonons (which have Bose statistics) with momentum $\vec{p}$ and energy $\varepsilon_i(\vec{p}) = c_i|\vec{p}|$ can be considered, where c_i is the speed of sound. In an isotropic solid, there is one longitudinal phonon mode for each possible momentum (with sound velocity c_l), and two transverse modes (with velocity c_t). The phonon momentum is limited by the lattice constant a, such that the upper limit is $|\vec{p}| \leq h/a$.

Find the temperature dependence of the phonon contribution to the specific heat C_V in the limit of low temperature ($T \ll T_D$), and high temperature ($T \gg T_D$), where $T_D h\bar{c}/ak_B$ is the Debye temperature, and the $\bar{c}$ is the mean speed of sound, $3\bar{c}^{-3} = c_l^{-3} + 2c_t^{-3}$.

Problem 34: Pauli-Paramagnetism (4 Marks)

In an external magnetic field B, the single particle energies $\epsilon_{\vec{p},\sigma}$ of conduction electrons differ according to the spin orientation $\sigma = \pm$ (ie. parallel or antiparallel to the field):

$$\epsilon_{\vec{p},\sigma} = \frac{1}{2m}\vec{p}^2 - \sigma\mu_B B.$$

where $\mu_B = \frac{|e|\hbar}{2mc}$ is the Bohr magneton. For the two spin orientations, this leads to different occupation numbers $\langle n_{\vec{p},\sigma}\rangle$ with the same chemical potential μ.
Magnetization and susceptibility are defined by

$$M = \mu_B \cdot (\langle N_+\rangle - \langle N_-\rangle) \quad \text{with} \quad N_\sigma = \sum_{\vec{p}} n_{\vec{p},\sigma}\,, \qquad \chi = \chi(T, B) = \left.\frac{\partial M}{\partial B}\right|_{T,N}.$$

(a) Show that $\chi(T,0) = N\mu_B^2 \int_{-\infty}^{\infty} d\epsilon \rho'(\epsilon) n(\epsilon)$. The values on the right hand side for $B = 0$ are known from lectures. $\rho(\epsilon)$ is the single particle density of states, $n(\epsilon)$ is the Fermi distribution function at the chemical potential μ.

(b) Develop, using the method given in the lectures, $\chi(T,0)$ up to and including $O(T^2)$. The total number of particles N is given, and note that μ is dependent on temperature. Obtain an explicit result for the case $\epsilon_{\vec{p}} = \frac{p^2}{2m}$.

Problem 35: Thermodynamic relations (8 Marks)

We consider a system of N identical particles. E, T, V and μ are the energy, temperature, volume, and chemical potential, respectively. Prove the following relations:

$$\begin{aligned}
\left(\frac{\partial E}{\partial N}\right)_{T,V} &= \mu - T\left(\frac{\partial \mu}{\partial T}\right)_{V,N} \\
\left(\frac{\partial N}{\partial T}\right)_{V,\mu/T} &= \left(\frac{\partial N}{\partial \mu}\right)_{T,V}\left(\frac{\partial E}{\partial N}\right)_{T,V}/T \\
\left(\frac{\partial E}{\partial T}\right)_{V,(\mu/T)} &= \left(\frac{\partial E}{\partial T}\right)_{V,N} + \left(\frac{\partial N}{\partial \mu}\right)_{T,V}\left(\frac{\partial E}{\partial N}\right)_{T,V}^2/T
\end{aligned} \tag{1}$$

In addition, prove for systems with fixed particle number:

$$\begin{aligned}
\left(\frac{\partial E}{\partial V}\right)_{T} &= T\left(\frac{\partial S}{\partial V}\right)_{T} - p \\
\left(\frac{\partial S}{\partial p}\right)_{T} &= -\left(\frac{\partial V}{\partial T}\right)_{p} \\
K_T - K_S &= V\frac{T\alpha_p^2}{C_p} \\
\frac{C_p}{C_V} &= \frac{K_T}{K_S}
\end{aligned} \tag{2}$$

Where

$$\begin{aligned}
C_V &= \left(\frac{\partial E}{\partial T}\right)_{V} = T\left(\frac{\partial S}{\partial T}\right)_{V} \\
C_p &= \left(\frac{\partial H}{\partial T}\right)_{p} = T\left(\frac{\partial S}{\partial T}\right)_{p} \\
\alpha_p &= \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p} \\
K_T &= -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T} \\
K_S &= -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{S}
\end{aligned} \tag{3}$$