Universität Leipzig
Institut für Theoretische Physik
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Statistical Physics, Spring 2011
Problem Set 10

Course Information:

- ☞ Class times: lectures Monday and Thursday, 11:00-12:30 in SR 218, tutorials Friday, 9:15-10:45 in SR 221
- ☞ Final exam: July 11, 13:30 in ThHS
- ☞ The course website is www.uni-leipzig.de/˜stp/Statistical_Physics.html

Problem 30: Bose-Condensation I (8 Marks)

The grand canonical potential of the ideal gas is

$$J(T,\mu,V) = T\ln(1-z) - T\frac{V}{\lambda^3}g_{5/2}(z)$$

where $z = e^{\mu/T}$ and $\lambda = h/\sqrt{2\pi m T}$.

Calculate the entropy S and the specific heat C_V above and just below the point of Bose-condensation. Calculate the isothermal compressibility as a function of the specific volume near v_c, $v - v_c \ll v_c$. $v_c(T) = \lambda_T^3/\zeta(3/2)$ is the critical value of v where Bose-condensation begins.

If you got this right, then S is continuous on the condensation point and vanishes at $v = 0$ ($T = 0$). The specific heat shows a jump in the first derivative at this point, and K_T diverges.

If one considers the region $v < v_c$ as a new phase (and not as a mixed phase of condensate and gas in excited states), then the Bose condensation is a second order phase transition. But if we consider the state in which all particles are assumed to be in the condensate as a new phase (due to the lack of repulsion, this state has specific volume $v = 0$), then the Bose condensation is a first order isothermal phase transition.

Show that the formula for the pressure $p(T) = [\zeta(5/2)/\hbar^3](2\pi m)^{3/2}(k_B T)^{5/2}$ of the Bose gas with partial condensation at $T_c(v)$ satisfies the Clausius-Clapeyron equation

$$\frac{\mathrm{d}P}{\mathrm{d}T} = \frac{\Delta S}{\Delta V}.$$

Where V is the volume of the condensate, $V = N v_c$.

Note: $g_{3/2}(z)/g_{3/2}(1) \approx 1 - 1.36\sqrt{1-z} + O(1-z)$, where $g_\alpha(z) = \sum_{n=1}^{\infty} n^{-a} z^n$. *Hint:* Remember that below the condensation point (ie. $T < T_c$ and $v < v_c$), $z = 1$. Use the expansion of $g_{3/2}(z)$ to get the entropy above the condensation point.

Problem 31: Bose Condensation II (4 Marks)

For an ideal gas of N bosons at $T = T_c$, the following properties hold:

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\begin{aligned}
\frac{\langle n_0 \rangle}{N} &= \Psi_{3/2}(z) \\
z &= \frac{\langle n_0 \rangle}{1 + \langle n_0 \rangle} \\
\Psi_{3/2}(1-\delta) &= 1.36(1-\delta)[-\ln(1-\delta)]^{1/2} + \delta + 0.38(1-\delta)\ln(1-\delta) + \mathcal{O}(\delta^2)
\end{aligned}
$$

for $0 \leq \delta \ll 1$.

Show that for large N, $\langle n_0 \rangle \approx 1.22 N^{2/3}$ holds.

Calculate $\langle \alpha \rangle$ for large N and excited states α.

Problem 32: Ideal Fermions at low temperatures (4 Marks)

With the aid of the Sommerfeld-expansion, one can calculate various properties of fermions at low temperatures.

Calculate $E(T, N, V)$ up to $O(T^2)$. Derive $p(T, N, V)$ to $O(T^2)$ as well as $C_V(T, N, V)$ to $O(T)$.