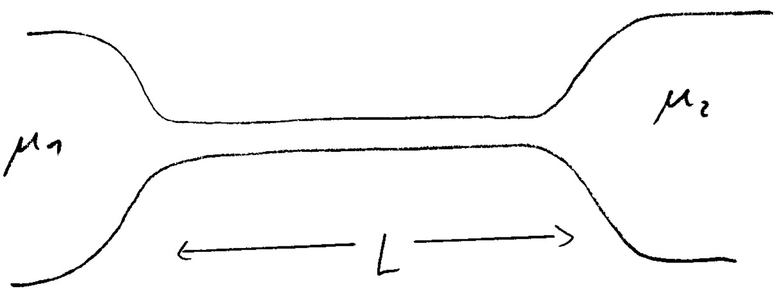
§ 5 The Landauer formulation and applications

There are different ways to measure the conductivity of a onedimensional sample. One possibility is to irradiate the sample with microwaves and measure the absorption of the radiation. This conductivity is theoretically described by the Kubo formula, taking into account the discreteness of levels (which is important in the limit of vanishing frequency).

Another possibility is connecting the sample to electron reservoirs at chemical potentials μ_1 and μ_2. We first consider an ideal onedimensional conductor (quantum point contact)



$\mu_1 > \mu_2$

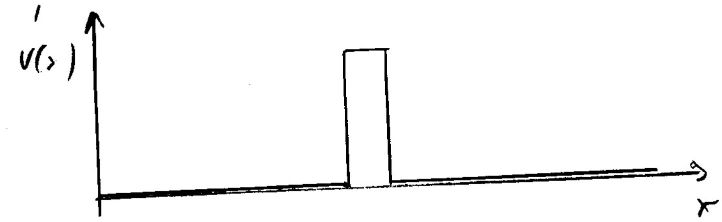
The quantization condition for momentum eigenstates is $\Delta k = \frac{2\pi}{L}$, each eigenstate carries a current $\frac{e}{L} \cdot v_k = \frac{e}{L} \frac{\hbar k}{m}$. In the energy intervall $\mu_1 - \mu_2$ there are $e(\mu_1 - \mu_2) / \left(\frac{\hbar^2 k_F}{m} \cdot \Delta k \right) = e(\mu_1 - \mu_2) / \left(\frac{\hbar^2 k_F \cdot 2\pi}{Lm} \right)$ eigenstates

$$\rightarrow J = e(\mu_1 - \mu_2) \frac{\cancel{Lm}}{\hbar^2 k_F \cdot 2\pi} \cdot \frac{e \hbar k_F}{\cancel{mL}} = \frac{e^2}{h} (\mu_1 - \mu_2)$$

The conductance of an ideal 1d wire connected to two reservoirs is

$$\boxed{G_0 = \frac{e^2}{h}} \qquad \text{quantum conductance}$$

If there is an energy barrier in the wire, the conductance is $G = G_0 \cdot T$, where T is the transmission probability of the barrier

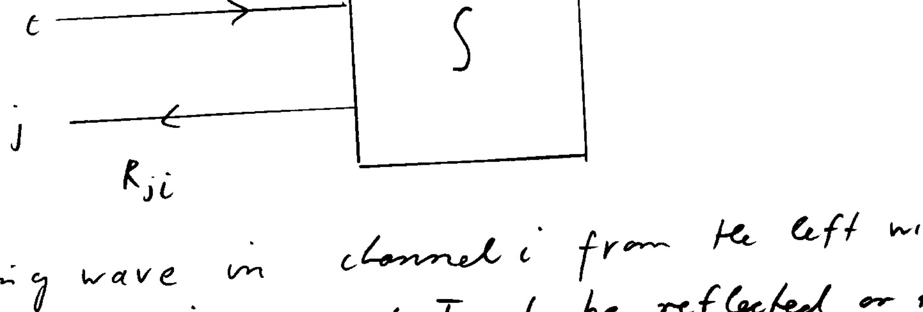


A general multichannel system is described by a scattering matrix S relating in- and outgoing states in the wire. The leads feeding into the elastic scattering system S are ideal wires with a finite cross section A. Quantization in the transverse direction leads to discrete transverse energies E_i for the $N_\perp$ conducting channels. At the Fermi energy one has $E_i + \frac{\hbar^2 k_i^2}{2m} = E_F \quad i = 1, \dots, N_\perp$

$N_\perp = A \frac{k_F^2}{2\pi}$ for a 2D cross section, and $N_\perp = 2 W k_F / \pi$ for a 1d cross section (both including spin). Note that the scattering matrix S describes scattering between channels (not wires).

Channels in the same wire have the same chemical potential.

The incoming channels are fed from electron baths with chemical potentials μ_1, μ_2, and the overall temperature is T.



An incoming wave in channel i from the left with amplitude one has probabilities R_{ji} and T_{ji} to be reflected or transmitted into the jth l.h.s. or r.h.s. channel, respectively.

We assume that the outgoing channels from each reservoir are fed up to a thermal equilibrium population. It is convenient to assume that particles reaching the "sink" reservoir via the ideal lead are totally absorbed there. This is not obviously correct for electrons reaching the reservoir much below its Fermi energy. However, those reflected electrons contribute to the outgoing current from the sink reservoir and reduce the rate of electrons emanating from it. We also assume that different channels behave as incoherent sources, i.e there is no definite phase relationship between electrons in different channels.

Transmission amplitudes and probabilities are related via $T_{ij} = |t_{ij}|^2$ ($R_{ij} = |r_{ij}|^2$). The analogous quantities for incoming waves from the r.h.s. are denoted by primes.
The $2N_\perp \times 2N_\perp$ matrix S is then given by

$$S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}.$$

S is unitary due to current conservation. In the case of time reversal symmetry we have

$$S S^* = \mathbb{1} \quad , \quad S = S^T$$

In the presence of a magnetic field, the second relation becomes $S(H) = S^T(-H)$.

The total transmission and reflection probabilities into the ith channel are defined by

$$T_i = \sum_j T_{ij} \qquad R_i = \sum_j R_{ij}$$

Due to current conservation (unitarity) we have

$$\sum_i T_i = \sum_i (1 - R_i) \quad \text{and} \quad R_i' + T_i = 1 \,, \quad R_i + T_i' = 1$$

As the quantum conductance including spin is $\frac{e^2}{\pi \hbar}$, we can write the total current as

$$J = \frac{e^2}{\pi \hbar} \sum_i \int dE \left[f_1(E)\, T_i(E) - f_2(E)\left(1 - R_i'(E)\right) \right]$$

$$= \frac{e^2}{\pi \hbar} \sum_i \int dE \; T_i(E) \underbrace{\left[f_1(E) - f_2(E) \right]}_{\approx -(\mu_1 - \mu_2) \left.\frac{\partial f}{\partial E}\right|_{E_F}}$$

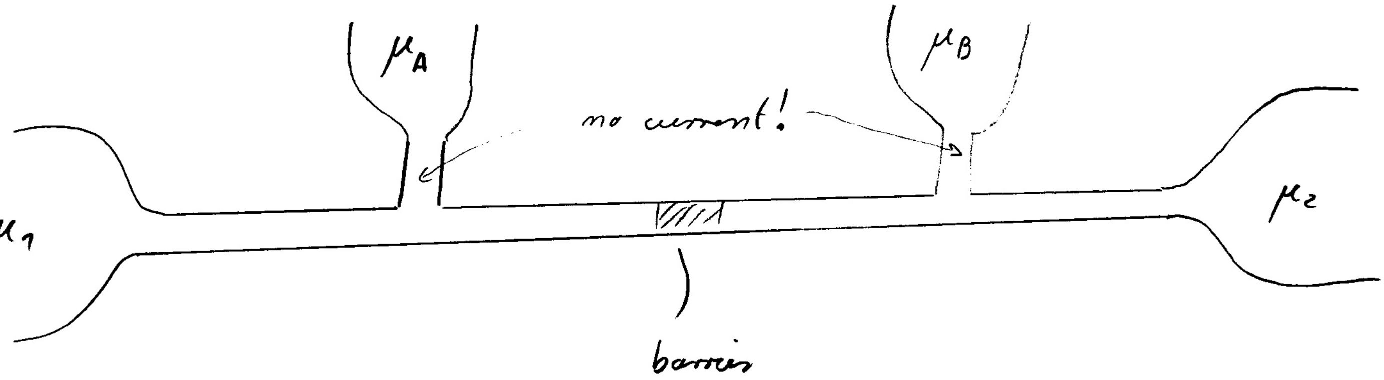
$$\boxed{J = (\mu_1 - \mu_2) \frac{e^2}{\pi \hbar} \int dE \left(- \frac{\partial f}{\partial E} \right) \sum_i T_i(E)}$$

The conductance is $G \equiv \frac{J}{\mu_1 - \mu_2} = \frac{e^2}{\pi \hbar} \int dE \left(- \frac{\partial f}{\partial E} \right) \sum_i T_i(E)$

$$\underset{T \to 0}{=} \frac{e^2}{\pi \hbar} \sum_{ij} T_{ij} = \frac{e^2}{\pi \hbar} \mathrm{Tr}\left[t t^+ \right]$$

This prediction of a quantized conductance has been verified in many experiments.

As there is already a finite voltage necessary to drive current through a perfect wire, it is clear that in the case of a barrier only part of the total voltage will be measured across the barrier. To address the question of the voltage across the barrier, we need to consider a <u>four terminal measurement</u>



The chemical potentials μ_A and μ_B are chosen in such a that no current flows into or out of the additional reservoirs.

The simplest way to arrive at the idealized conductance of the sample is to define it as

$$G_s = \frac{J}{\mu_A - \mu_B}$$

and to define μ_A and μ_B as follows:

The electron density on the l. h. s. of the barrier is ($\frac{1}{2\pi\hbar v_i}$ is the density of states per volume in channel i)

$$n_\ell = \frac{1}{2\pi\hbar} \int dE \sum_i \frac{1}{v_i} \left[(1+R_i)\, f_1(E) + T_i' f_2(E) \right]$$

The electron density in equilibrium with a chemical potential μ_A would be

$$n_A = \frac{1}{\pi\hbar} \int dE \sum_i f_A(E)$$

Define μ_A by the condition $n_e = n_A \quad \rightarrow n_e - n_r = n_A - n_B$

$$n_e - n_r = \frac{1}{2\pi\hbar} \int dE \sum_i \frac{1}{v_i} \left[(1+R_i)(f_1 - f_2) + T_i'(f_2 - f_1) \right]$$

$$= (\mu_1 - \mu_2) \frac{1}{2\pi\hbar} \int dE \left(-\frac{\partial f}{\partial E}\right) \sum_i \frac{1}{v_i} \left[1 + R_i(E) - T_i'(E) \right]$$

$$n_A - n_B = \frac{1}{\pi\hbar} \int dE \, (\mu_A - \mu_B) \left(-\frac{\partial f}{\partial E}\right) \sum_i \frac{1}{v_i} = \frac{1}{\pi\hbar} (\mu_A - \mu_B) \sum_i \frac{1}{v_i}$$

To calculate the ratio $G_s = \dfrac{J}{\mu_A - \mu_B}$ we express J by $(\mu_1 - \mu_2)$

$$J = \frac{2e^2}{h} (\mu_1 - \mu_2) \int dE \left(-\frac{\partial f}{\partial E}\right) \sum_i T_i(E)$$

$$\rightarrow \frac{J}{\mu_A - \mu_B} = 2 \cdot \frac{2e^2}{h} \frac{\left(\sum_i \frac{1}{v_i}\right) \int dE \left(-\frac{\partial f}{\partial E}\right) \sum_i T_i(E)}{\int dE \left(-\frac{\partial f}{\partial E}\right) \sum_i \frac{1}{v_i} \left[1 + R_i(E) - T_i'(E) \right]}$$

Use now the relation $R_i + T_i' = 1 \quad \rightarrow T_i' = 1 - R_i$ and take the limit of zero temperature

$$G_s = \frac{2e^2}{h} \frac{\sum_i \frac{1}{v_i} \sum_i T_i}{\sum_i R_i / v_i}$$

In the limit of one channel one obtains

$$\boxed{G_s = \frac{e^2}{\pi\hbar} \frac{T}{R}}$$

for the conductance of the barrier itself.

Interpretation of this result: the two-terminal conductance

$G = 2\frac{e^2}{h}T$ can be obtained via the series addition of quantum resistors. Each contact has a resistance $\frac{h}{4e^2}$, the impurity region has resistance $\frac{h}{2e^2}\frac{R}{T}$

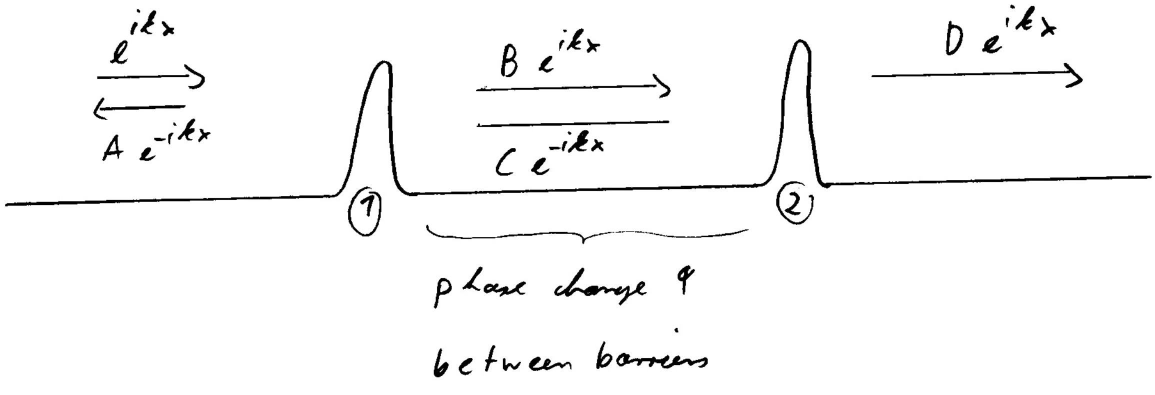
$$\Rightarrow \frac{1}{G} = \frac{h}{4e^2} + \frac{h}{4e^2} + \frac{h}{2e^2}\frac{R}{T} = \frac{h}{2e^2}\left(1 + \frac{R}{T}\right) = \frac{h}{2e^2}\frac{R+T}{T}$$

$$\Rightarrow \frac{1}{G} = \frac{h}{2e^2}\cdot\frac{1}{T} \quad \Rightarrow \quad G = 2\frac{e^2}{h}\cdot T$$

5.2 Applications of the Landauer Formulation

Series addition of quantum resistors

Consider a system of two quantum resistors



barrier equations: $A = r_1 + C t_1$ $\quad B = t_1 + C r_1'$

$C e^{-i\varphi} = B e^{i\varphi} r_2 \quad D = B e^{i\varphi} t_2$

Solving the equations one finds

$$D = \frac{e^{i\varphi} t_1 t_2}{1 - e^{2i\varphi} r_2 r_1'}$$

$\rightarrow$ transmittance $$T_{12} = \frac{T_1 \quad T_2}{1 + R_1 R_2 - 2\sqrt{R_1 R_2}\cos\theta}$$

$$\theta = 2\varphi + \arg(r_2 r_1')$$

Denote $T_{12} = T$, $R_{12} = R$

$$\rightarrow \quad \frac{R}{T} = \frac{R_1 + R_2 - 2\sqrt{R_1 R_2}\cos\theta}{T_1 \quad T_2}$$

Consider an ensemble with similar R_1 and R_2 but phase differences φ uniformly distributed over many multiples of 2π

$\rightarrow$ $\langle \cos\theta \rangle = 0$ (is true for distances between obstacles much larger than the electron wavelength; the ratio $L/\lambda \sim 10^2$ for 300 Å of a ballistic wire)

Define the dimensionless conductance by $g \equiv \frac{G}{e^2/\pi\hbar}$

$$\rightarrow \quad \langle g^{-1} \rangle = \frac{R_1 + R_2}{(1-R_1)(1-R_2)}$$

Ohm's law for the series addition of resistances,

$$g^{-1} = g_1^{-1} + g_2^{-1} = \frac{R_1}{1-R_1} + \frac{R_2}{1-R_2} = \frac{R_1(1-R_2) + R_2(1-R_1)}{(1-R_1)(1-R_2)}$$

$$= \frac{R_1 + R_2 - 2R_1R_2}{(1-R_1)(1-R_2)}$$

is not valid in general!

Combine good transmittances $(R \ll 1,\ T \sim 1)$ in series

$\rightarrow$ in the beginning G^{-1} increases linearly with n as one would expect, but for a total transmittance smaller than unity, one finds

$$\langle g^{-1} \rangle_{n+1} = \frac{R_n + R}{T_n} = \langle g^{-1} \rangle_n + \frac{R}{T_n}$$

$\rightarrow$ adding a good transmittance $(R \ll 1)$ as the $(n+1)$th element to a chain of n such elements increases the resistance by $\frac{R}{T_n} > R$

Derive a recursive (RG) equation for $\langle g^{-1} \rangle_n$

$$\frac{\langle g^{-1} \rangle_{n+1} - \langle g^{-1} \rangle_n}{\Delta n} = \frac{R}{T_n}$$

Use now $\langle g^{-1} \rangle_n = \frac{R_n}{T_n} = \frac{1-T_n}{T_n} = \frac{1}{T_n} - 1$

$$\frac{1}{T_n} = \langle g^{-1} \rangle_n + 1$$

$$\rightarrow \boxed{\frac{d}{dn} \langle g^{-1} \rangle_n = R [\langle g^{-1} \rangle_n + 1]}$$

For $\langle g^{-1} \rangle \ll 1$ one finds a linear increase, whereas for $\langle g^{-1} \rangle \gg 1$ the resistance grows exponentially. This result confirms the scaling theory of localization.

This consideration is problematic when applied to realistic systems: the distribution of resistances in a sample is not narrow $\rightarrow$ it is important which quantity is being averaged.

Anderson, Thouless, Abrahams, Fisher (1980) point out that the result depends on the type of quantity being averaged. One needs an object that behaves like an ordinary extensive quantity with both average and variance increasing linearly with n. Such a quantity is $\ln(1+g^{-1})$ as

$$1 + g^{-1} = 1 + \frac{R}{T} = \frac{1}{T} \rightarrow \ln(1+g^{-1}) = -\ln T$$

$-\ln T$ plays the role of an extinction coefficient, one expects it to be additive if the relative phase is averaged over.

Using $$\int_0^{2\pi} d\theta \, \ln(a + b\cos\theta) = \pi \ln \tfrac{1}{2} [a + \sqrt{a^2 - b^2}]$$

one finds $\langle \ln T_{12} \rangle = \ln T_1 + \ln T_2$

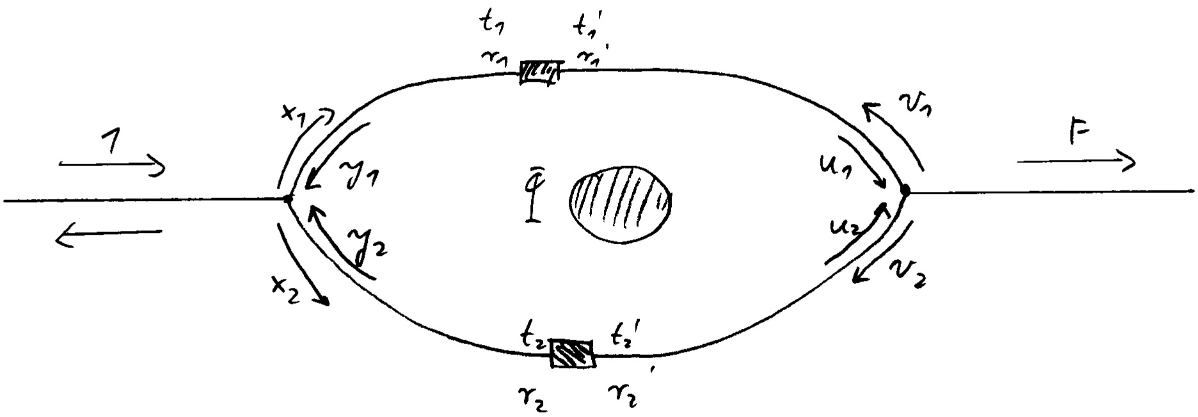
→ the exact scaling of the 1D resistance with n is given by

$$\boxed{\langle \ln(1+g_n^{-1}) \rangle = \rho_1 \cdot n}$$

ρ_1 resistance in units of $\frac{\pi \hbar}{e^2}$ of a single obstacle

grows first linearly and then exponentially.

Parallel addition of quantum resistors, A-B oscillations of the conductance



All phases and scattering effects along the channels are absorbed in the parameters describing each scatterer.

t_i, t_i' transmission amplitudes from the left and from the right

r_i, r_i' reflection amplitudes

time reversal $\qquad t_i = t_i'$

current conservation $\qquad -t_i/t_i'^{*} = r_i/r_i'^{*}$

An AB - flux Φ can be incorporated into the transmission and reflection amplitudes via the usual gauge transformation

$$t_1 \to t_1 e^{-i\theta} \quad , \quad t_1' \to t_1' e^{i\theta} \qquad \theta = \pi \Phi/\Phi_0$$

$$t_2 \to t_2 e^{i\theta} \quad , \quad t_2' \to t_2' e^{-i\theta}$$

$$r_i \to r_i \quad , \quad r_i' \to r_i'$$

Each three terminal junction can be described by a 3×3 scattering matrix

$$S = \begin{pmatrix} 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

S_{ii} denotes the reflection amplitude of the ith channel

Channel 1 of the l.h.s. junction is the ingoing amplitude (unity), whereas channel 1 of the r.h.s. junction is the outgoing amplitude F.

After some algebra one finds

$$F = 2\,\frac{t_1 t_2 (t_1' + t_2') + t_1 (r_2 - 1)(1 - r_2') + t_2 (r_1 - 1)(1 - r_1')}{(t_1 + t_2)(t_1' + t_2') - (2 - r_1 - r_2)(2 - r_1' - r_2')}$$

Before we look at the expression for F in detail, we look for a general parametrization of the t_i, r_i of one individual scatterer. If time reversal invariance holds, s has the form

$$s = \begin{pmatrix} r & t \\ t & r \end{pmatrix}$$

Unitarity condition: $\quad -t/t^* = r/r^* \quad$ and $\quad |t|^2 + |r|^2 = 1$

If we assume r to be real, then t must be imaginary. The equation for probability conservation is satisfied for $r = \cos\theta$, $t = i\sin\theta$. The most general parametrization of s is

$$s = e^{i\varphi} \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}$$

Now we evaluate the formula for F in special situations

i) the lower arm is completely blocked, $t_2 = 0$ 

the upper arm is open, $t_1 = e^{i\varphi}$, $r_1 = 0$
time reversal (no magnetic flux)

$$\rightarrow F = 2\,\frac{-t_1(1-r_2)^2}{|t_1|^2 - (2-r_1-r_2} = 2\,\frac{-e^{i\varphi}(1-r_2)^2}{1-(2-r_2)^2} \qquad |r_2| = 1$$

$\underline{r_2 = 1}$ $\quad \rightarrow F = -2e^{i\varphi}\,\frac{0}{1-1^2}$ is indeterminate

more careful calculation: $\quad r_2 = e^{i\delta}$ with $\delta \rightarrow 0$

$$r_2 \approx 1 + i\delta$$

$$F = -2e^{i\varphi} \frac{(1-1-i\delta)^2}{1-(2-1-i\delta)^2}$$

$$= -2e^{i\varphi} \frac{-\delta^2}{2i\delta + \delta^2} \rightarrow 0$$

$$r_2 = -1 \qquad F = -2e^{i\varphi} \frac{2^2}{1-3^2} = e^{i\varphi}$$

ii) both arms are completely open, no magnetic field

$t_1 = e^{i\varphi} \qquad t_2 = 1$

$t_1' = e^{i\varphi} \qquad t_2' = 1$

$r_1 = 0 \qquad r_2 = 0$

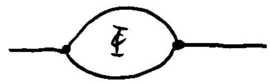
$$F = 2 \frac{e^{i\varphi}(1+e^{i\varphi}) - e^{i\varphi} - 1}{(1+e^{i\varphi})(1+e^{i\varphi}) - 4} = 2 \frac{e^{i\varphi} + e^{2i\varphi} - e^{i\varphi} - 1}{(1+e^{i\varphi})^2 - 4}$$

$$= 2 \frac{e^{2i\varphi} - 1}{1 + 2e^{i\varphi} + e^{2i\varphi} - 4}$$

$\underline{t_1 = 1}$ can be studied in the limit $\varphi \rightarrow 0$

$$F = 2 \frac{1 + 2i\varphi - 1}{1 + 2 + 2i\varphi + 1 + 2i\varphi - 4} = 2 \frac{2i\varphi}{4i\varphi} = 1$$

$$\underline{t_1 = -1} \qquad F = 2\,\frac{0}{1-2+1-4} = 0$$

iii) "perfect interferometer" with flux 

$$t_1 = e^{i\varphi}e^{-i\theta} \qquad t_2 = e^{i\theta}$$

$$t_1' = e^{i\varphi}e^{i\theta} \qquad t_2' = e^{-i\theta} \qquad \text{in the limit } \varphi \to 0$$

$$r_1 = 0 \qquad r_2 = 0$$

$$F = 2\,\frac{e^{i\varphi}\left(e^{i\varphi}e^{i\theta} + \cancel{e^{-i\theta}}\right) - e^{i\varphi}\cancel{e^{-i\theta}} - e^{i\theta}}{\left(e^{i\varphi}e^{-i\theta} + e^{i\theta}\right)\left(e^{i\varphi}e^{i\theta} + e^{-i\theta}\right) - 4}$$

$$= 2\,\frac{e^{2i\varphi}e^{i\theta} - e^{i\theta}}{e^{2i\varphi} + e^{i\varphi}e^{-2i\theta} + e^{i\varphi}e^{2i\theta} - 4}$$

$$= 2\,\frac{e^{i\theta}\left(e^{2i\varphi} - 1\right)}{e^{i\varphi}\left(e^{i\varphi} + 2\cos 2\theta\right) - 3} \qquad \text{expand in powers of } \varphi$$

$$\to F = 2\,\frac{e^{i\theta}\left(1 + 2i\varphi - 1\right)}{(1+i\varphi)(1+i\varphi+2\cos 2\theta) - 3}$$

$$= 2e^{i\theta}\,\frac{2i\varphi}{(2\cos 2\theta - 2) + 2i\varphi(1+\cos 2\theta)} \to \begin{cases} 0 & \text{for } \theta \neq 0 \\ 1 & \text{for } \theta = 0 \end{cases}$$

This corresponds to a very sharp resonance $|F|^2 \sim \delta(2\theta)$ containing all higher harmonics in Φ/Φ_0.

iv) upper arm open, lower arm almost closed, with flux

$$t_1 = e^{-i\theta} \qquad t_2 = i\sin\delta\, e^{i\theta}$$

$$t_1' = e^{i\theta} \qquad t_2' = i\sin\delta\, e^{-i\theta}$$

$$r_1 = 0 \qquad r_2 = -\cos\delta = r_2'$$

$$F = 2\,\frac{i\sin\delta\left(\cancel{e^{i\theta}} + i\sin\delta\, e^{-i\theta}\right) - e^{-i\theta}(1+\cos\delta)^2 - i\sin\delta\, \cancel{e^{i\theta}}}{\left(e^{-i\theta} + i\sin\delta\, e^{i\theta}\right)\left(e^{i\theta} + i\sin\delta\, e^{-i\theta}\right) - (2+\cos\delta)^2}$$

$$= 2\,\frac{-e^{-i\theta}(1+\cos\delta)^2 - \sin^2\delta\, e^{-i\theta}}{1 + 2i\sin\delta\cos 2\theta - \sin^2\delta - (2+\cos\delta)^2}$$

$$= 2\,e^{-i\theta}\,\frac{(1+\cos\delta)^2 + \sin^2\delta}{(2+\cos\delta) - 1 + \sin^2\delta - 2i\sin\delta\cos 2\theta}$$

To order δ we find

$$F \approx e^{-i\theta}\,\frac{8}{8 - 2i\sin\delta\cos 2\theta}$$

$$= e^{-i\theta}\left(1 + \frac{i}{4}\sin\delta\cos 2\theta\right)$$

The flux-dependent part of $|F|^2$ to $O(\delta^2)$ is given by

$$\frac{\sin^2\delta}{16}\cos^2 2\theta = 1 + \frac{\sin^2\delta}{32}\left(1 + \cos 4\theta\right)$$

$\Rightarrow$ the dominant oscillations have period $2\Phi/\Phi_0$!

However, the choice r_2 close to -1 is not generic.

Claim: the AB-oscillations of period Φ/Φ_0 are missing due to the specific choice of the backscattering phase $r_2 \approx -1$. Hence we consider the situation

v) $t_1 = e^{-i\theta}$ $\quad t_2 = -\sin\delta\, e^{i\theta}$

$t_1' = e^{i\theta}$ $\quad t_2' = -\sin\delta\, e^{-i\theta}$ $\quad S_2 = i\begin{pmatrix} \cos\delta & i\sin\delta \\ i\sin\delta & \cos\delta \end{pmatrix}$

$r_1 = 0$ $\quad r_2 = i\cos\delta = r_2'$

$$\rightarrow \quad F = 2\,\frac{-\sin\delta\left(\cancel{e^{i\theta}} - \sin\delta\, e^{-i\theta}\right) - e^{-i\theta}(1-i\cos\delta)^2 + \sin\delta\,\cancel{e^{i\theta}}}{\left(e^{-i\theta}-\sin\delta\, e^{i\theta}\right)\left(e^{i\theta}-\sin\delta\, e^{-i\theta}\right) - (2-i\cos\delta)^2}$$

$$= 2\,\frac{\sin^2\delta\, e^{-i\theta} - e^{-i\theta}(1-i\cos\delta)^2}{1-2\sin\delta\cos 2\theta + \sin^2\delta - (2-i\cos\delta)^2}$$

$$= 2\,e^{-i\theta}\,\frac{\sin^2\delta - (1-i\cos\delta)^2}{1-2\sin\delta\cos 2\theta + \sin^2\delta - (2-i\cos\delta)^2}$$

expand to lowest order in δ

$$F \simeq 2\,e^{-i\theta}\,\frac{-(1-i)^2}{1-(2-i)^2 - 2\delta\cos 2\theta}$$

$$= -2\,e^{-i\theta}\,\frac{1-2i-1}{-2+4i-2\delta\cos 2\theta}$$

$$= -2\,e^{-i\theta}\,\frac{i}{-(1+\delta\cos 2\theta)+2i}$$

$$\to |F|^2 = 4 \, \frac{1}{(1+\delta\cos 2\theta)^2 + 4} = \frac{4}{5 + 2\delta\cos 2\theta}$$

$$= \frac{4}{5} \, \frac{1}{1 + \frac{2}{5}\delta\cos 2\theta} \approx \frac{4}{5}\left(1 - \frac{2}{5}\delta\cos 2\theta\right)$$

$|F|^2$ is periodic in Φ/Φ_0.

This result is generic for the situation $|t_2| \ll |t_1|$.

For $|t_1| \sim |t_2| \sim t \ll 1$ one finds Φ/Φ_0 - periodic oscillations with amplitude $|t|^2$ appearing on a constant background $\sim |t|^2$; in addition, there are oscillations with period $\Phi/2$ and amplitude $\sim |t|^4$.

Remark: in experiments involving ensemble averaging, the lowest harmonic Φ/Φ_0 vanishes. Ensemble average as well as thermal average on scales $k_B T \gg \Delta$ (in 1d an energy difference Δ corresponds to a phase change 2π around the ring) introduces a phase uncertainty $> 2\pi$ and destroys the interference pattern.

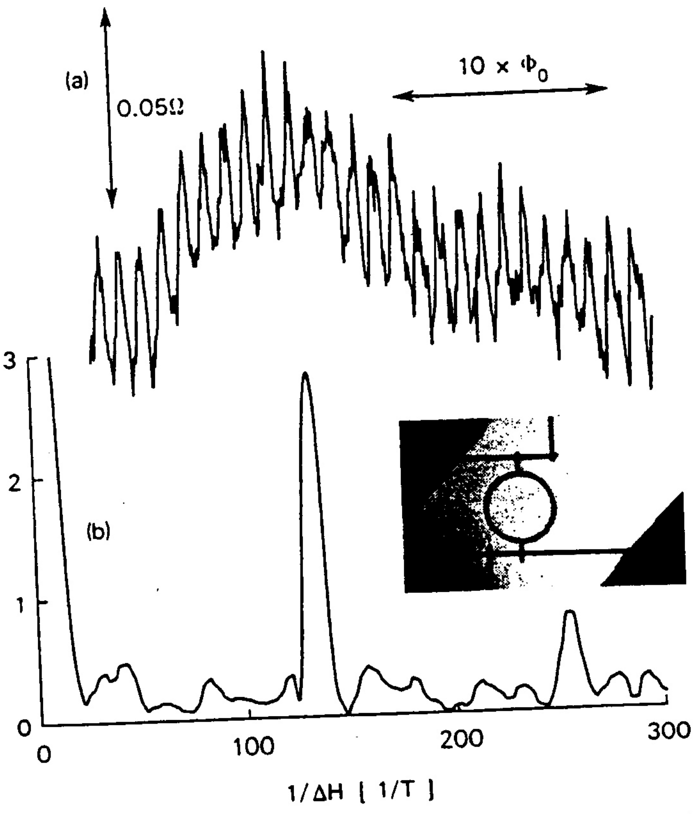
There are weak localization corrections with a periodicity $\Phi/2\Phi_0$. These corrections are due to pairs of time reversed paths, each of them encircling the flux $\Phi \to$ total flux 2Φ is enclosed $\to$ periodicity of oscillations is $2\Phi/\Phi_0$.

What is the effect of a <u>magnetic field in the material</u>?

Sample specific fluctuations on a scale Φ_c/Φ_0 are observed, where Φ_c is the flux through the arms of the sample.

$\Phi/\Phi_c = A$ — A is the aspect ratio of the sample, the ratio of the area of the hole to the area of the arms.

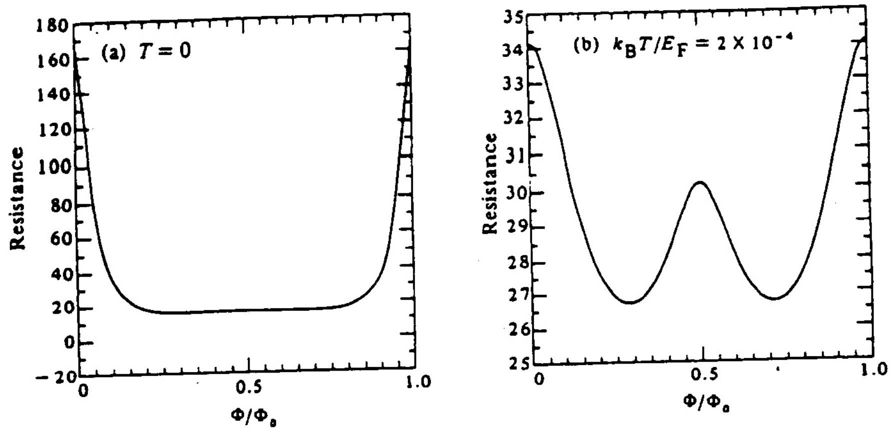
The Φ/Φ_0 - periodic oscillations coexist with the sample specific Φ_c/Φ_0 - structure. The Φ/Φ_0 oscillations are damped by the flux Φ_c due to a crossover to a different universality class (orthogonal with time reversal symmetry to unitary without time reversal invariance).



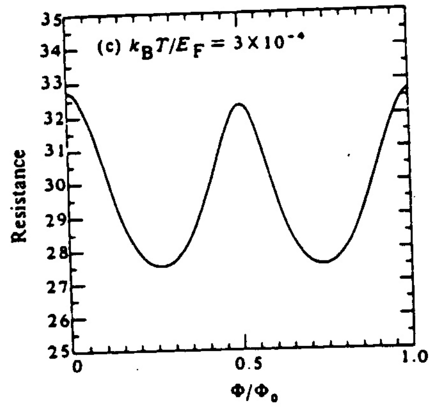
a) Φ/Φ_0 oscillation in the magnetoresistance of a gold ring. The arrow corresponds to 10 flux quanta in the hole of the ring.

b) Fourier spectrum; the low frequency peak corresponds to the slow modulation due to flux in the "arms" of the ring.

From R. A. Webb et al., PRL <u>54</u>, 2696 (1985)



From M. Murat et al., Phys. Rev. B 34, 659 (1986)



Resistance as a function of flux for a 1D-ring with long arms with scatterers. At higher temperatures the Φ/Φ_0-periodic oscillations are averaged out and the $\Phi_0/2$-periodic weak-localization oscillations remain.

5.3 Universal Conductance Fluctuations

In many experiments one finds reproducible fluctuations of the conductance upon changing the magnetic field or the Fermi energy. We will concentrate on a changing Fermi energy, as this situation can be more easily analyzed.

In the Thouless picture, the conductance is given by

$$G = \frac{e^2}{\hbar} \frac{E_c}{\Delta}$$

$$\Delta = \frac{1}{N(E_F)} \quad \text{mean level spacing}$$

$$E_c = \pi^2 \left| \overline{\frac{\partial^2 E_i}{\partial q^2}} \right| \quad \text{Thouless energy}$$

Here, E_c/Δ can be interpreted as the number of levels in an energy interval of width E_c, short $N(E_c)$

$$\rightarrow \quad \langle G \rangle = \frac{e^2}{\hbar} \langle N(E_c) \rangle$$

The fluctuations in $\langle G \rangle$ are related to fluctuations in $N(E_c)$ as

$$\frac{\delta G}{\langle G \rangle} \approx \frac{\delta N(E_c)}{\langle N(E_c) \rangle}$$

Naively, one would expect $\delta N(E_c) \simeq \sqrt{N(E_c)}$, experimentally one finds $\delta G \simeq \frac{e^2}{\hbar}$ or $\delta N(E_c) \simeq 1$.

This result can be explained by correlation (repulsion) between energy levels.

Under the assumption of a random Hamilton operator from the Gaussian

$\beta = 1$	orthogonal ensemble	time reversal invariance
$\beta = 2$	unitary ensemble	no time reversal invariance
$\beta = 4$	symplectic ensemble	spin-orbit scattering

the fluctuation strength for an isolated sample is given by

$$\langle [\delta N(E)]^2 \rangle = \frac{2 k s^2}{\pi^2 \beta} \ln \langle N(E) \rangle$$

k number of "noninteracting" series of levels (with different quantum numbers

s s-fold degeneracy due to spin degrees of freedom

In the following, we make this result plausible and understand how a coupling of the sample to reservoirs changes the result.

As a first step, we calculate the joint probability density of eigenvalues for an orthogonal 2×2-matrix.

$$H = \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix}$$ has eigenvalues

$$E_{1,2} = \frac{1}{2}\left[H_{11} + H_{22} \pm \sqrt{(H_{11}-H_{22})^2 + 4 H_{12}^2} \right]$$

Assume a Gaussian distribution for the matrix elements

$$e^{-\frac{1}{2}\mathrm{Tr}[HH^T]/H_0^2} \frac{dH_{11}\, dH_{22}\, dH_{12}}{(\sqrt{\pi} H_0)^3 \cdot 2}$$

$$\mathrm{Tr}[H^2] = H_{11}^2 + H_{22}^2 + 2H_{12}^2 = \underbrace{\varepsilon_1^2 + \varepsilon_2^2}_{\text{eigenvalues}}$$

Evaluate $P(\varepsilon_1, \varepsilon_2) = \int \frac{dH_{11}\, dH_{22}\, dH_{12}}{(\sqrt{\pi} H_0)^3\, 2} \delta(\varepsilon_1 - E_1)\delta(\varepsilon_2 - E_2)\, e^{-\frac{1}{2}\mathrm{Tr}[HH^T]/H_0^2}$

The integral is most easily evaluated in "polar coordinates" $\varepsilon_1, \varepsilon_2$, and θ. We use the fact that a symmetric and real matrix is diagonalized by an orthogonal transformation.
Orthogonal 2×2-matrices can be parametrized as

$$O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_1 \cos\theta & \varepsilon_2 \sin\theta \\ -\varepsilon_1 \sin\theta & \varepsilon_2 \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta & -\varepsilon_1 \cos\theta \sin\theta + \varepsilon_2 \sin\theta\cos\theta \\ -\varepsilon_1 \sin\theta\cos\theta + \varepsilon_2 \sin\theta\cos\theta & \sin^2\theta\, \varepsilon_1 + \cos^2\theta\, \varepsilon_2 \end{pmatrix}$$

$$\rightarrow \left.\begin{aligned} H_{11} &= \varepsilon_1 \cos^2\theta + \varepsilon_2 \sin^2\theta \\ H_{22} &= \varepsilon_1 \sin^2\theta + \varepsilon_2 \cos^2\theta \\ H_{12} &= (\varepsilon_2 - \varepsilon_1)\sin\theta\cos\theta \end{aligned}\right\}$$

Use these transformation equations to calculate the Jacobian

$$\rightarrow \left| \frac{\partial (H_{11}, H_{22}, H_{12})}{\partial (\varepsilon_1, \varepsilon_2, \theta)} \right| = \begin{vmatrix} \cos^2\theta & \sin^2\theta & -\varepsilon_1 2\cos\theta\sin\theta + \varepsilon_2 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & \varepsilon_1 2\cos\theta\sin\theta - 2\varepsilon_2\cos\theta\sin\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & (\varepsilon_2 - \varepsilon_1)(\cos^2\theta - \sin^2\theta) \end{vmatrix}$$

$$= \begin{vmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{vmatrix} |\varepsilon_2 - \varepsilon_1|$$

$$= |\varepsilon_2 - \varepsilon_1| \, \Big| \cos^4\theta \, (\cos^2\theta - \sin^2\theta) + 2\sin^4\theta\cos^2\theta + 2\sin^4\theta\cos^2\theta + 2\cos^4\theta\sin^2\theta$$

$$+ 2\cos^4\theta \sin^2\theta - \sin^4 (\cos^2\theta - \sin^2\theta) \Big|$$

$$= |\varepsilon_2 - \varepsilon_1| \, \Big| \cos^6\theta + \cos^4\theta\sin^2\theta \, (-1+2+2) + \sin^4\theta\cos^2\theta \, (2+2-1) + \sin^6\theta \Big|$$

$$= |\varepsilon_2 - \varepsilon_1| \, (\cos^2\theta + \sin^2\theta)^3 = |\varepsilon_2 - \varepsilon_1|$$

$$\rightarrow \quad P(\varepsilon_1, \varepsilon_2) = \int \frac{dH_{11}\, dH_{22}\, dH_{12}}{(\sqrt{\pi} H_0)^3 \, 2} \, \delta(\varepsilon_1 - E_1)\, \delta(\varepsilon_2 - E_2)\, e^{-\frac{1}{2} Tr [H^2] / H_0^2}$$

$$= \int_{-\infty}^{\infty} dE_1\, dE_2 \int_0^{2\pi} d\theta \, |E_1 - E_2| \, \delta(\varepsilon_1 - E_1)\, \delta(\varepsilon_2 - E_2)\, e^{-\frac{1}{2}(E_1^2 + E_2^2)/H_0^2} \cdot \frac{1}{\pi^{3/2} 2 H_0^3}$$

$$\rightarrow \quad \boxed{P(\varepsilon_1 \varepsilon_2) = \frac{1}{\sqrt{\pi} \; H_0^3} \, |\varepsilon_1 - \varepsilon_2| \, e^{-\frac{1}{2}(\varepsilon_1^2 + \varepsilon_2^2)/H_0^2}}$$

Rescale $s_1 = \varepsilon_1 / H_0$, $s_2 = \varepsilon_2 / H_0$ and calculate the distribution of level differences $s_2 - s_1$.

change of variables $\left. \begin{array}{l} s = s_2 - s_1 \\ u = s_1 + s_2 \end{array} \right\} \rightarrow \begin{array}{l} s_2 = \frac{1}{2}(s+u) \\ s_1 = \frac{1}{2}(u-s) \end{array}$

$$\left| \frac{\partial(s_1, s_2)}{\partial(s, u)} \right| = \begin{vmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{4}$$

$$\Rightarrow \quad P(s,u) = \frac{1}{4\sqrt{\pi}} \, s \, e^{-\frac{1}{4}s^2 - \frac{1}{4}u^2}$$

$$\Rightarrow \quad P(s) = \frac{1}{4\sqrt{\pi}} \, s \, e^{-\frac{1}{4}s^2} \underbrace{\int_{-\infty}^{\infty} du \, e^{-\frac{1}{4}u^2}}_{2\sqrt{\pi}} = \frac{1}{2} \, s \, e^{-\frac{1}{4}s^2}$$

The approximate distribution function for the distance between neighboring levels is (orthogonal ensemble)

$$\boxed{P_1(s) = \frac{\pi}{2} \, s \, e^{-\frac{\pi}{4}s^2}} \qquad \text{Wigner surmise}$$

What is the joint pdf for symmetric $N \times N$ matrices?

The exponential part is $e^{-\frac{1}{2H_0^2}\sum_{i=1}^{N} \epsilon_i^2}$, the measure is $\frac{\prod_{i=1}^{N} d\epsilon_i}{H_0^{N(N+1)/2}}$; to make the distribution dimensionless, a term with dimension $(\text{energy})^{N(N-1)/2}$ is needed. This term must be a polynomial in the ϵ_i, which vanishes as $|\epsilon_i - \epsilon_j|$ for $\epsilon_i \to \epsilon_j$, because in this situation the problem is effectively 2×2. The only polynomial satisfying these constraints is $\prod_{i<j} |\epsilon_i - \epsilon_j|$. Hence we conjecture

$$\boxed{P_N\{\epsilon_i\} = C_N \prod_{i>j} |\epsilon_i - \epsilon_j| \, e^{-\frac{1}{2H_0^2}\sum_i \epsilon_i^2}}$$

For the unitary ensemble ($\beta = 2$) and symplectic ensemble ($\beta = 4$), the factor $|\epsilon_i - \epsilon_j|$ is replaced by $|\epsilon_i - \epsilon_j|^{\beta}$.

In the following, we exploit F. J. Dyson's analogy with electromagnetism (J. Math. Phys. 3, 140, 157, 166 (1962)).

The joint pdf can be rewritten as

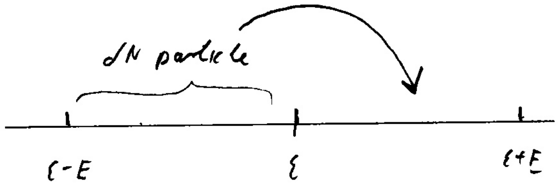
$$\boxed{P_N\{\varepsilon_i\} = C_N \exp\left\{ -\beta \sum_{i>j} \ln \frac{1}{|\varepsilon_i - \varepsilon_j|} - \frac{1}{2H_0^2} \sum_i \varepsilon_i^2 \right\}}$$

$P_N\{\varepsilon_i\}$ is the probability density for N particles at temperature β^{-1} to be at the points ε_i on the energy axis if the particles repel one another by the "two-dimensional Coulomb" law

$$\mathcal{U}(\varepsilon_i - \varepsilon_j) = \ln \frac{1}{|\varepsilon_i - \varepsilon_j|}$$

The particles are confined by a quadratic potential $\varepsilon_i^2/2H_0^2\beta$ near the point $\varepsilon = 0$. For fluctuations on a scale small compared to the confinement region, the influence of the confinement is unimportant, if the gas is in a state of mechanical equilibrium.

We are interested in fluctuations $\delta N(E)$ in an interval E → consider an arbitrary energy ε and a small neighborhood of length $2E$.



Let us calculate the energy W necessary to take δN particles from the left half of the intervall to the right half. We use a continuum approximation for that purpose, i.e. assume that the charges are equally distributed with a density

$$\rho(x) = \frac{\delta N}{E} \operatorname{sign}(x-\varepsilon)$$

If we remove δN charges from the l.h.s. of the interval, a negative background charge due to the confining potential, is left. This negative background charge is described by the negative charge density $-\frac{\delta N}{E}$ for $x < \varepsilon$

The potential energy W of this charge rearangement is

$$W = \frac{1}{2}\int_{-E}^{E} dx \int_{-E}^{E} dy \; \rho(x)\rho(y) \ln\frac{1}{|x-y|}$$

$$= \frac{(\delta N)^2}{E^2}\int_0^E dx \int_0^E dy \left(\ln\frac{1}{|x-y|} + \ln\frac{1}{x+y}\right)$$

Transform to new coordinates $u = \frac{1}{2}(y+x)$

$v = y - x$

$$\rightarrow \left.\begin{array}{l} y = u + \frac{1}{2}v \\ x = u - \frac{1}{2}v \end{array}\right\} \quad \begin{array}{l} \frac{1}{2}v \text{ from } -u \text{ to } u \text{ for } u < \frac{E}{2} \\ \frac{1}{2}v \text{ from } -(E-u) \text{ to } E-u \text{ for } u > \frac{E}{2} \end{array}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} 1 & -\frac{1}{2} \\ 1 & \frac{1}{2} \end{vmatrix} = 1$$

$$\rightarrow W = \frac{(\delta N)^2}{E^2}\left[\int_0^{E/2} du \int_{-2u}^{2u} dv + \int_{E/2}^{E} du \int_{-2(E-u)}^{2(E-u)} dv\right]\left(\ln\frac{1}{|v|} + \ln\frac{1}{2u}\right)$$

$$\rightarrow \quad W = -2\frac{(\delta N)^2}{E^2}\left[\int_0^{E/2} du \int_0^{2u} dv + \int_{E/2}^{E} du \int_0^{2(E-u)} dv\right](\ln v + \ln 2u)$$

$$= -2\frac{(\delta N)^2}{E^2}\left\{\int_0^{E/2} du\left[2u\ln 2u - 2u + 2u\ln 2u\right]\right.$$

$\int dx \ln x = x\ln x - x;$

$$\left.+\int_{E/2}^{E} du\left[2(E-u)\ln 2(E-u) - 2(E-u) + 2(E-u)\ln 2u\right]\right\}$$

$$= -2\frac{(\delta N)^2}{E^2}\left\{\int_0^{E} du\left[u\ln u - \frac{u}{2}\right] + \frac{1}{2}\int_0^{E} du\left[u\ln u - u\right] + \right.$$

$$\left. E\int_E^{2E} du \ln u - \frac{1}{2}\int_E^{2E} du\, u\ln u\right\}$$

$\int dx\, x\ln x = \frac{1}{2}x^2\ln x - \frac{x^2}{4}$

$\rightarrow \quad W = \delta N^2 \ln 4$ after some algebra.

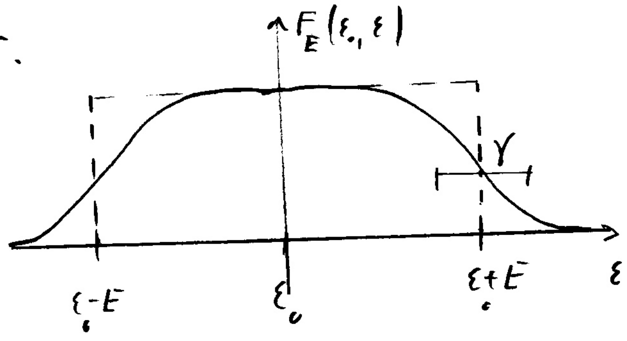
The strength of fluctuations is determined by the condition $W \simeq \frac{1}{\beta}$ ($\beta = 1$ for the orthogonal ensemble considered here).

$\rightarrow \quad (\delta N)^2 \simeq 1$

However, fluctuations in the number of levels in the segment $[\varepsilon - E, \varepsilon]$ arise not only from a scale E but also from scales $E/2, \ldots, E/2^n$, where the minimum energy $E/2^n = \Delta$

$\rightarrow \quad n = \ln \frac{E}{\Delta} / \ln 2 = \ln \langle N(E) \rangle / \ln 2$

Up to now we have considered the number of levels in an energy band with well defined boundaries. In physical applications, bands unsharp edges or alternatively broadened levels are important. The width of the broadening is denoted as γ.



$$N_{\varepsilon_0}(E) = \int_{-\infty}^{\infty} d\varepsilon \; F_E(\varepsilon - \varepsilon_0) \sum_i \delta(\varepsilon' - \varepsilon_i)$$

Here, fluctuations in $N(E)$ arise only when charge is transported over distances larger than γ $\rightarrow$ $\langle [\delta N(E)]^2 \rangle \approx \ln \frac{E}{\gamma}$

The Thouless approach, which analyzes how a level shifts when the boundary conditions change, implies that the boundaries of an energy band are unsharp on a scale E_c $\rightarrow$ $\langle [\delta N(E)]^2 \rangle \simeq 1$

So far we have only considered fluctuations in $N(E)$ and have assumed that E_c does not fluctuate. In reality, the diffusion constant D and the Thouless energy $E_c = \frac{\hbar}{L^2} D$ do fluctuate and make up $\frac{2}{3}$ of the mesoscopic fluctuations of G; $\frac{1}{3}$ of the mesoscopic fluctuations of G is due to fluctuations in $N(E)$. For details see B.L. Altshuler and B.I. Shklovskii, Sov. Phys. JETP 64, 127 (1986)