Mathematical Methods of Modern Physics SS 2024

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§1 Complex Functions

1.1 Complex Numbers

Def.: The set $\mathbb{C}$ of complex numbers is (as a set) equivalent to $\mathbb{R}^2$. Elements of $\mathbb{C}$ can be represented as tuples (x, y) with $x, y \in \mathbb{R}$.

In $\mathbb{C}$ we define addition and multiplication as follows:

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + x_2 y_1)$$

$(\mathbb{C}, +, \cdot)$ is a field (homework problem)

Difference between $\mathbb{R}$ and $\mathbb{C}$: there is no order relation in $\mathbb{C}$.

Consider now $(x, 0)$

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$$

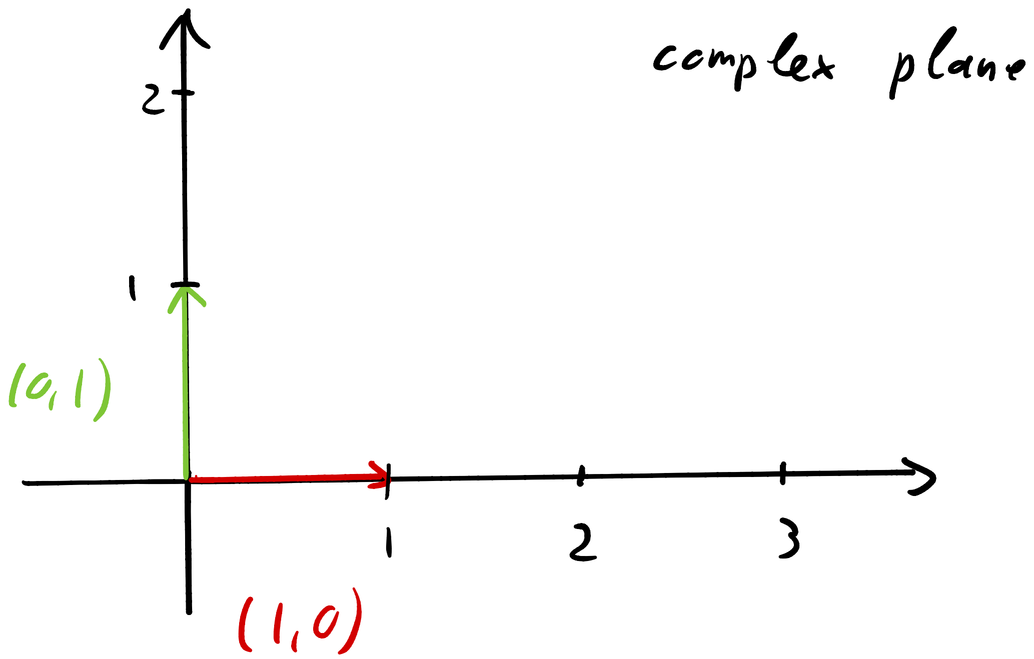
$$(x_1, 0) \cdot (x_2, 0) = (x_1 \cdot x_2, 0)$$

$\Rightarrow \quad \{(x,0) : x \in \mathbb{R}\} = \mathbb{R}$

Instead of $(x,0)$ we can simply write x.

Similarly, $(a,0)\cdot(x,y) = (ax, ay) = a(x,y)$

corresponds to scalar multiplication in the vector space $\mathbb{R}^2$.



Every complex number can be represented as

$$(x,y) = x\cdot(1,0) + y\,(0,1)$$

$$(1,0) \equiv 1 \qquad\qquad (0,1) \equiv i$$

$$(x,y) = x\cdot 1 + y\cdot i = x + iy$$

i is called imaginary unit.

$$i^2 = (0,1)\cdot(0,1) = (-1,0) = -1$$

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + x_1 i y_2 + i y_1 x_2 + i^2 y_1 y_2$$
$$= x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1)$$

This is the motivation for the definition of multiplication given above.

Def.: Let $z = (x,y) = x + iy \in \mathbb{C}$. Then we call $x = \mathrm{Re}(z)$ the real part of z and $y = \mathrm{Im}(z)$ the imaginary part, and $\bar{z} = x - iy$ the conjugate complex number.

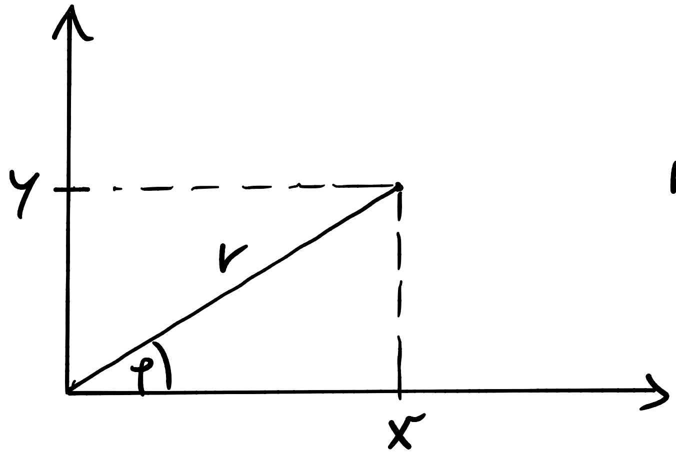
Rules: $\bar{\bar{z}} = z$

$$\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$$
$$\mathrm{Im}(z) = \frac{1}{2i}(z - \bar{z})$$

$$z\cdot\bar{z} = (x+iy)(x-iy) = x^2 + y$$

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{1}{x^2+y^2}\,\bar{z}$$

Polar coordinate representation of complex numbers



$$r = \sqrt{x^2+y^2} = \sqrt{z\bar{z}} =: |z|$$

$$\tan\varphi = \frac{y}{x} \Rightarrow \varphi = \arctan\frac{y}{x} =: \arg z$$

argument of z

$$z = x+iy = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\sin\varphi)$$

$\varphi = \arg z$ is unique only modulo 2π

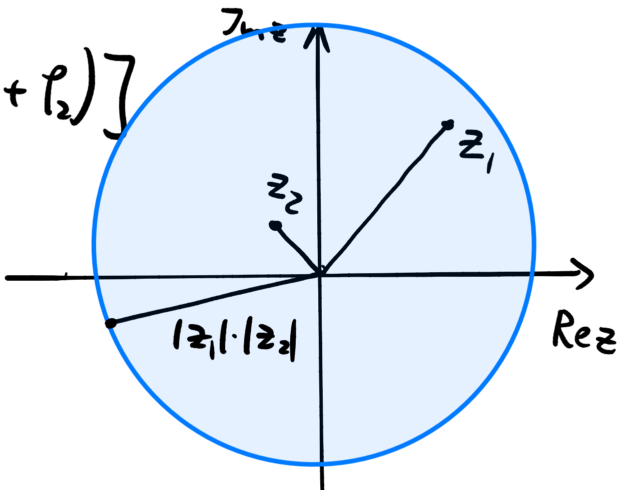
Example: polar coordinate representation of $z_1 \cdot z_2$

$$z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$$

$$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$$

$$z_1 \cdot z_2 = r_1 r_2\left[\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\cos\varphi_1\sin\varphi_2 + \cos\varphi_2\sin\varphi_1)\right]$$

$$= r_1 r_2\left[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right]$$



$$z = r(\cos\varphi + i\sin\varphi)$$

$$z^2 = r^2(\cos 2\varphi + i\sin 2\varphi)$$

$$\boxed{z^n = r^n(\cos n\varphi + i\sin n\varphi)} \qquad (*)$$

$$\cos y = 1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots + \dots = 1 + \frac{(iy)^2}{2!} + \frac{(iy)^4}{4!} + \dots$$

$$i\sin y = i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots + \dots\right) = iy + \frac{(iy)^3}{3!} + \frac{(iy)^5}{5!} + \dots$$

$$\Rightarrow \cos y + i\sin y = \sum_{n=0}^{\infty} \frac{(iy)^n}{n!} = e^{iy}$$

Using this relation, we can write $z = re^{i\varphi}$

Proof of $(*)$ $\quad z^n = (re^{i\varphi})^n = r^n e^{in\varphi} = r^n(\cos n\varphi + i\sin n\varphi)$

$\cos\varphi + i\sin\varphi$ lies on the <u>unit circle</u>

Example: quotient of two complex numbers in polar coordinates

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)} = \frac{r_1}{r_2}\left[\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2)\right]$$

Example: n-th root

Let $z = r(\cos\varphi + i\sin\varphi) = r[\cos(\varphi + 2\pi k) + i\sin(\varphi + 2\pi k)]$

with $k \in \mathbb{Z}$

Find complex number $w \equiv \sqrt[n]{z}$ with $w^n = z$

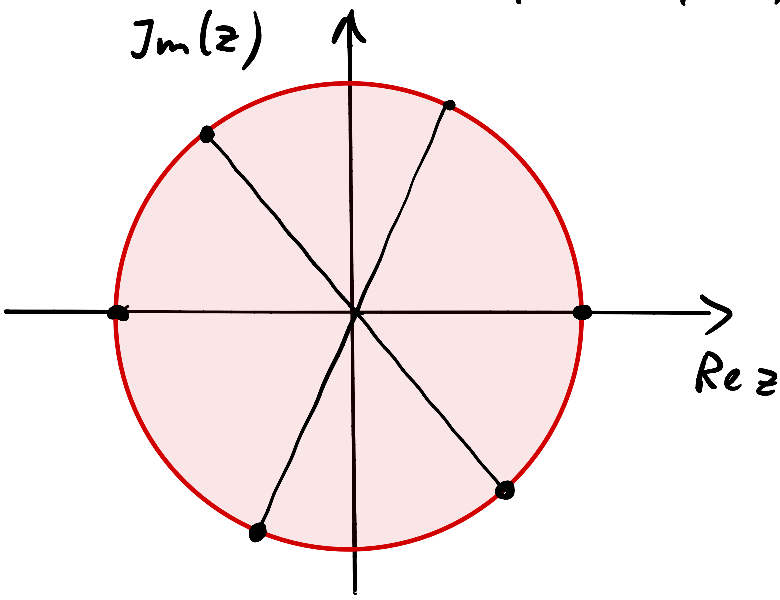
$$\sqrt[n]{z} = \left[r\, e^{i(\varphi + 2\pi k)}\right]^{\frac{1}{n}} = \sqrt[n]{r}\, e^{i\left(\frac{\varphi}{n} + \frac{k}{n}2\pi\right)}$$

There are exactly n different n-th roots of a complex number $z = re^{i\varphi}$ with $z \neq 0$

$$w_k = \sqrt[n]{r}\, e^{i\left(\frac{\varphi}{n} + \frac{k}{n}2\pi\right)}$$

with $k = 0, 1, \ldots, n-1$

6-th root of 1:



1.2 Complex Functions

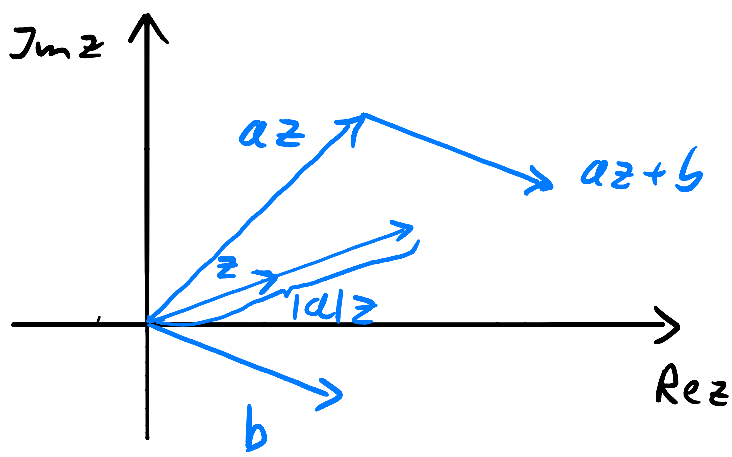
Let $A \subset \mathbb{C}$ $\quad f: A \to \mathbb{C}$ is a complex function

Example (1) $\quad A = \mathbb{C}$, $\quad f(z) = a \cdot z + b$; $\quad a, b \in \mathbb{C}$

$a \neq 0$

linear map

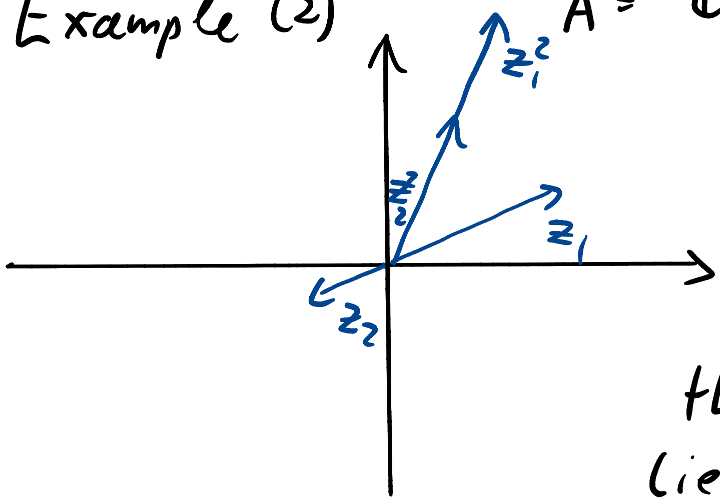
$$f: z \xmapsto{\text{stretching}} |a| z \xmapsto{\text{rotation}} a z \xmapsto{\text{translation}} az + b$$



Linear maps are bijective:

$$w = f(z) \qquad z = \frac{1}{a} w - \frac{b}{a}$$

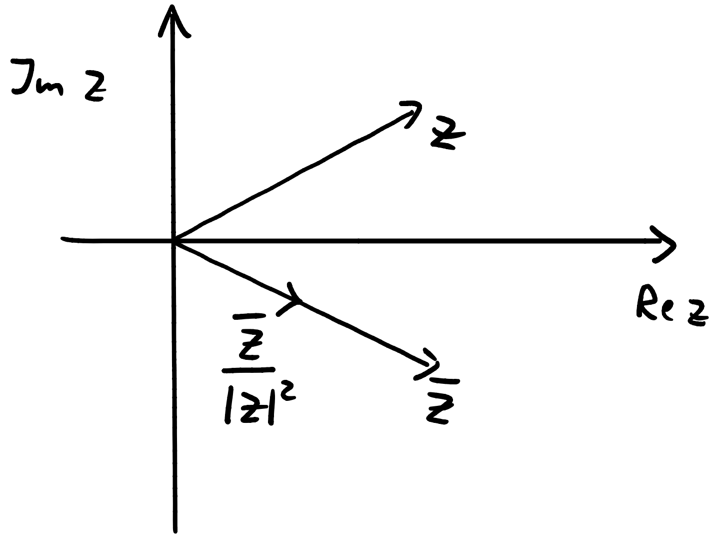
Example (2) $\quad A = \mathbb{C} \quad f(z) = z^2$



If z_1, z_2 lie on a straight line through the origin, then z_1^2 and z_2^2 lie on the same half line

Example (3) $A = \mathbb{C} \setminus \{0\}$ $\quad f(z) = \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$

$$= \frac{1}{|z|}\left(\frac{\bar{z}}{|z|}\right)$$



Claim: $f(z) = \frac{1}{z}$ maps "circles" onto "circles"

("circles" = circles + straight lines)

Proof: Consider $M = \{ z = x + iy : \alpha(x^2+y^2) + \beta x + \gamma y + \delta = 0 \}$

For $\alpha = 0$: straight lines

$\alpha \neq 0$: circles

$$x^2 + y^2 + \frac{\beta}{\alpha} x + \frac{\gamma}{\alpha} y + \frac{\delta}{\alpha} = 0$$

$$\left(x + \frac{\beta}{2\alpha}\right)^2 + \left(y + \frac{\gamma}{2\alpha}\right)^2 - \frac{1}{4}\frac{\beta^2}{\alpha^2} - \frac{1}{4}\frac{\gamma^2}{\alpha^2} + \frac{\delta}{\alpha} = 0$$

Circle centered at $\left(-\frac{\beta}{2\alpha}, -\frac{\gamma}{2\alpha}\right)$ and with

radius $\sqrt{\frac{\beta^2}{4\alpha^2} + \frac{\gamma^2}{4\alpha^2} - \frac{\delta}{\alpha}}$

Now let $0 \neq z \in M$

$$\Rightarrow \quad \alpha + \beta \frac{x}{x^2+y^2} + \gamma \frac{y}{x^2+y^2} + \frac{\delta}{x^2+y^2} = 0$$

$$\Rightarrow \quad \alpha + \beta \operatorname{Re} f(z) - \gamma \operatorname{Im} f(z) + \delta |f(z)|^2 = 0$$

For $z \in M$ satisfies $f(z) = u + iv$ the equation

$$\alpha + \beta u - \gamma v + \delta (u^2+v^2) = 0 \quad \text{circle}$$

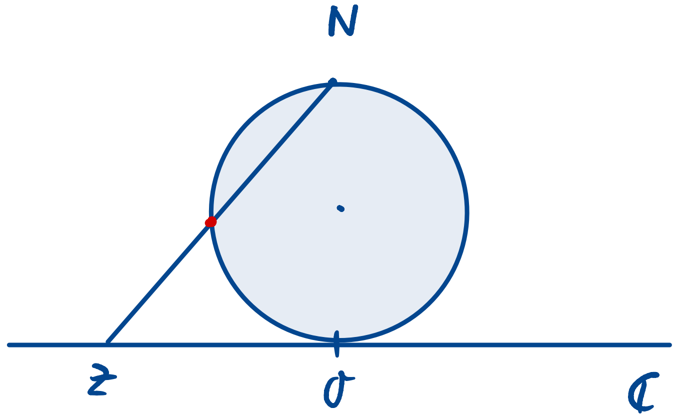
$f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ is bijective with the inverse map $f^{-1}(w) = \frac{1}{w}$

Def.: $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is called extended complex plane.

Then $\tilde{f} : \tilde{\mathbb{C}} \to \tilde{\mathbb{C}}$, $\tilde{f}(z) = \begin{cases} \frac{1}{z}, & z \in \tilde{\mathbb{C}} \setminus \{\infty, 0\} \\ \infty & z = 0 \\ 0 & z = \infty \end{cases}$

Stereographic projection and Riemann sphere:

Drawing a straight line from the north pole to the number z uniquely determines a point on the sphere.



Example 4: Linear fractional transformation

$$f(z) = \frac{az+b}{cz+d} \qquad a, b, c, d \in \mathbb{C}$$

$A = \mathbb{C} \setminus \{-\frac{d}{c}\}$, assume $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$

$c = 0$ Linear map

$c \neq 0$ Claim: $f = f_1 \circ f_2 \circ f_3$

where f_1 and f_3 are linear maps, and $f_2 = \frac{1}{z}$

Proof: $$f(z) = \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{a}{c}d + b}{cz+d}$$

$$= \frac{a}{c} + \frac{1}{c}\,\frac{bc-ad}{cz+d}$$

$f_3(z) = cz+d$, $f_2(z) = \frac{1}{z}$, $f_1(z) = \frac{bc-ad}{c} z + \frac{a}{c}$

With this $f(z) = f_1(f_2(f_3(z)))$

Every function $f(z) = \frac{az+b}{cz+d}$ with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$

can be extended to a bijective map $\hat{f}$ in $\hat{\mathbb{C}}$,

which maps "circles" into "circles".

i) $c = 0$ $\hat{f}(\infty) = \infty$

ii) $c \neq 0$ $\quad \tilde{f}(\infty) = \frac{a}{c}$, $\tilde{f}\left(-\frac{d}{c}\right) = \infty$

Example 5: complex exponential function

$$e^{iy} = \cos y + i \sin y$$

Def.: $z = x + iy$, then $e^z := e^x(\cos y + i \sin y)$

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(x+iy)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k (iy)^{n-k}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{\cancel{n!}\,1}{(n-k)!\,k!\,\cancel{n!}} x^k (iy)^{n-k}$$

$$= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{n=0}^{\infty} \frac{(iy)^n}{n!}\right) = e^x e^{iy} = e^x(\cos y + i \sin y)$$

The exponential function is periodic with period $2\pi i$

$$\boxed{\exp(z) = \exp(z + 2\pi i)}$$

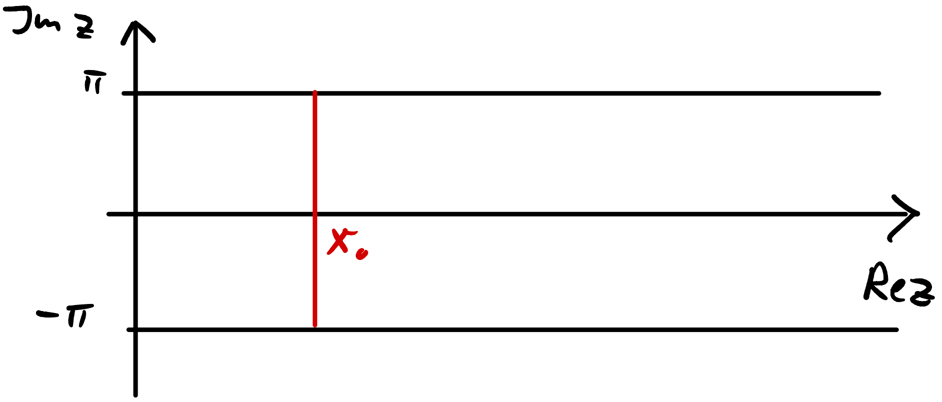
Proof: $\exp(z + 2\pi i) = \exp\left(x + i(y + 2\pi)\right)$

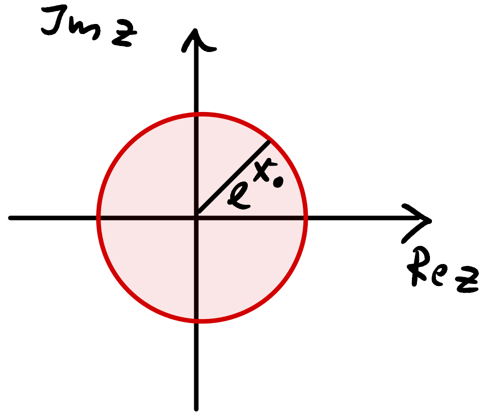
$$= e^x[\cos(y+2\pi) + i \sin(y + 2\pi)]$$

$$= \exp(z)$$

Consider $S = \{z : x + iy \mid -\pi < y \leq \pi\}$

$$e^{x_0}(\cos\varphi + i \sin\varphi) \qquad -\pi < \varphi \leq \pi$$





The straight line $x = x_0$ is mapped onto a circle with radius e^{x_0} $\Rightarrow$ $\exp: S \to \mathbb{C}\setminus\{0\}$ is bijective.

1.3 Convergence and Continuity

The norm $|z| = \sqrt{z\bar{z}} = \sqrt{x^2+y^2}$ is the Euclidean norm in $\mathbb{R}^2$.

Def.: a sequence $\{z_n\}$, $z_n \in \mathbb{C}$, $n \in \mathbb{N}$ is called convergent to z_0 $(z_n \to z_0)$, if

$$\forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n>N} \quad |z_n - z_0| < \varepsilon$$

$$\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \qquad z_n \to \infty \overset{\text{Def}}{\Longleftrightarrow} \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N} \; |z_n| > \frac{1}{\varepsilon}$$

$$\Longleftrightarrow \frac{1}{z_n} \to 0$$

Def.: $c \in \mathbb{C}$ is called limit value of a complex function f at z_0 ($\lim_{z \to z_0} f(z) = c$) if and onl if every sequence $\{z_n\}$ with $z_n \to z_0$ satisfies $f(z_n) \to c$.

Def.: Let $A \subseteq \tilde{\mathbb{C}}$, $f: A \to \mathbb{C}$

f is called continuous at $z_0 \in A$ $\Leftrightarrow$

$\lim_{z \to z_0} f(z) = f(z_0)$ $\Leftrightarrow$

$\{z_n\} \subset A,\ z_n \to z_0 \Rightarrow f(z_n) \to f(z_0)$

$\Leftrightarrow \forall_{\varepsilon > 0}\ \exists_{\delta > 0}\ \forall_{|z - z_0| < \delta}\ |f(z) - f(z_0)| < \varepsilon$

Example 6 Polynomials $f(z) = \sum_{j=0}^{n} a_j z^j$ are continuous in $\mathbb{C}$, $a_j \in \mathbb{C}$

Example 7 rational functions $f(z) = \frac{p(z)}{q(z)}$

p, q polynomials, $A = \{z \in \mathbb{C} : q(z) \neq 0\}$ are continuous in A.

Example ⑧ Logarithm

Def.: Let $A = \mathbb{C} \setminus \{0\}$ $\quad \ln z := \ln|z| + i \arg z$

1) For real z this definition agrees with the usual definition of the logarithm.

2) $\ln z$ is the reverse function of e^z

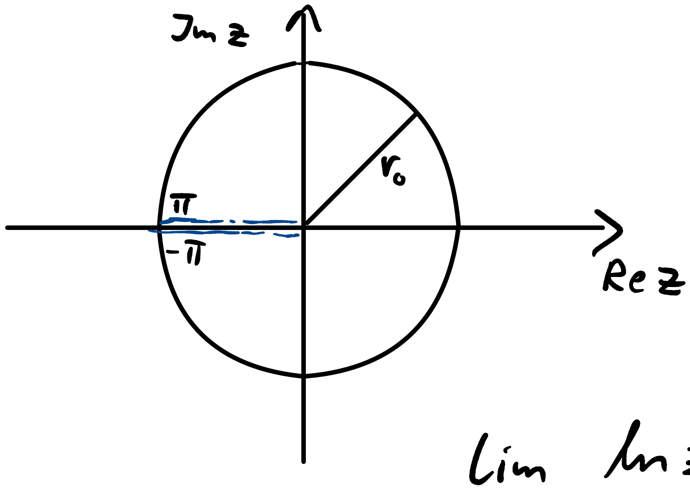
Proof: Let $z = x + iy$

$$w = e^z = e^x e^{iy}$$

$$\ln w = \ln|w| + i \underbrace{\arg w}_{y} = \ln e^x + iy = x + iy$$

Problem: $\arg z$ is not unique

$\to$ restrict $\arg z$ to $(-\pi, \pi)$



$$z = r_0 e^{i\varphi}$$

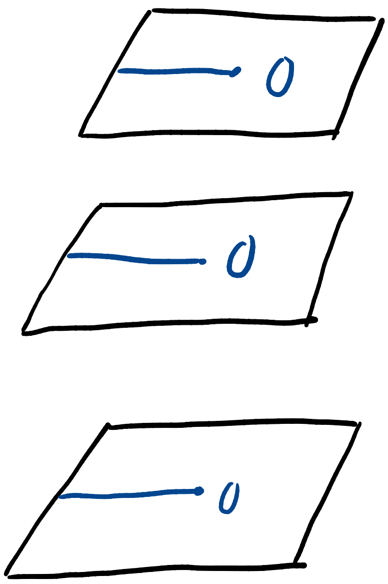
$$\lim_{\varphi \to \pi} \ln z = \lim_{\varphi \to \pi} (\ln r_0 + i\varphi) = \ln r_0 + i\pi$$

$$\lim_{\varphi \to -\pi} \ln z = \lim_{\varphi \to -\pi} (\ln r_0 + i\varphi) = \ln r_0 - i\pi$$

but $\lim\limits_{\varphi \to \pi} z = -r_0 \qquad \lim\limits_{\varphi \to -\pi} z = -r_0$

Riemann sheets

Expand the definition of arg z such that the logarithm is continuous



$\pi < \arg z \leq 3\pi$

$-\pi < \arg z \leq \pi$

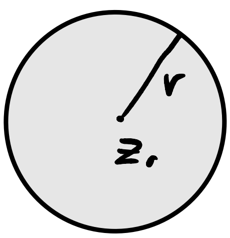
$-3\pi < \arg z \leq -\pi$

move from one sheet to the other when approaching the "cut"

Other example for discontinuity: $f(z) = \sqrt[n]{r}\, e^{i\varphi/n}$

1.4 Complex Differentiation

Def.: $K(z_0, r) := \{ z \in \mathbb{C} : |z - z_0| < r \}$
is the "open ball" around z_0 with radius r.

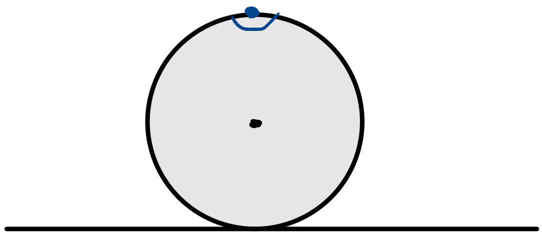


area of the circle without the boundary.

Def.: Let $A \subset \mathbb{C}$. A $z \in \mathbb{C}$ is called inner point of A if there exists an ε such that $K(z, \varepsilon) \subset A$.

Def.: $\Omega \subset \mathbb{C}$ is called open if all its points are inner points. $A \subset \mathbb{C}$ is called closed if $\mathbb{C} \setminus A$ is open.

Extension to $\tilde{\mathbb{C}}$



$$K(\infty, \varepsilon) := \{ z \in \mathbb{C} : |z| > \tfrac{1}{\varepsilon} \} \cup \{\infty\}$$

Def.: Let $\Omega \subset \mathbb{C}$ be open, $z_0 \in \Omega$ and $f: \Omega \to \mathbb{C}$.

f is called partially differentiable w.r.t. x or y in z_0, if the following limits exist:

$$f_x(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(x_0 + h + iy_0) - f(x_0 + iy_0) \right]$$

$$f_y(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h} \left[f(x_0 + i(h + y_0)) - f(x_0 + iy_0) \right]$$

f is called continuously partially differentiable in Ω

if $f_x(z)$ and $f_y(z)$ exist for all $z \in \Omega$ and are continuous.

Def.: Let $\Omega \in \mathbb{C}$ be open, $f: \Omega \to \mathbb{C}$, $z_0 \in \Omega$

f is called complex differentiable at z_0 if the limit $f'(z_0) = \lim\limits_{\substack{z \to z_0 \\ z \neq z_0}} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

Theorem: f is complex differentiable in z_0

$\Rightarrow f$ is continuous at z_0

Proof.: f complex differentiable $\Rightarrow f'(z_0)$ exists.

$$\Rightarrow \varphi(z) := \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \xrightarrow[z \to z_0]{} 0$$

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \varphi(z)(z-z_0)$$

$$\xrightarrow[z \to z_0]{} f(z_0)$$

Theorem: Let f be complex differentiable at z_0. Then the partial derivatives of f exist in z_0, and the following holds

$$f'(z_0) = f_x(z_0) = \frac{1}{i} f_y(z_0)$$

Remark: When letting $f = u + iv$ with u, v real-valued functions, then the <u>Cauchy-Riemann differential equations</u>

$$u_x = v_y \quad , \quad v_x = -u_y$$

are equivalent to $f_x = \frac{1}{i} f_y$

Proof: Consider two special sequences

1) $z := z_0 + h$ with $h \in \mathbb{R}$, $h \to 0$

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h}\left[f(z_0+h) - f(z_0)\right]$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h}\left[f(x_0 + h + iy_0) - f(x_0 + iy_0)\right]$$

$$= f_x(z_0)$$

2) $z := z_0 + ih \quad h \in \mathbb{R},\ h \to 0$

$$f'(z_0) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{ih}\left[f(z_0 + ih) - f(z_0) \right]$$

$$= \frac{1}{i} \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{1}{h}\left[f(x_0 + i(h + y_0)) - f(x_0 + iy_0) \right]$$

$$= \frac{1}{i} f_y(z_0)$$

Proof of the remark: $f(z_0) = u(z_0) + iv(z_0)$

$$\Rightarrow \quad f_x(z_0) = u_x(z_0) + iv_x(z_0)$$

$$f_y(z_0) = u_y(z_0) + iv_y(z_0)$$

$$\frac{1}{i} f_y(z_0) = v_y(z_0) - iu_y(z_0)$$

Hence, $\quad f_x = \frac{1}{i} f_y \iff u_x = v_y,\ v_x = -u_y$

Theorem: Let $\Omega \subset \mathbb{C}$ be open, $f: \Omega \to \mathbb{C}$ be continuously partially differentiable at $z_0 \in \Omega$, and let $f_x(z_0) = \frac{1}{i} f_y(z_0)$. Then f is complex differentiable at z_0.

Remark: Since $f'(z_0) = f_x(z_0)$, f is even continuously differentiable at z_0.

Proof: Use a relation from real analysis:

Let $\Omega \in \mathbb{R}^n$ be open, $f: \Omega \to \mathbb{R}^m$ be continuously partially differentiable at $z_0 \in \Omega$. Then there exists a function $\varphi: \Omega \to \mathbb{R}^m$ with $\varphi(z_0) = 0$, such that the following relation hold ($(Df)(z_0)$ denotes the Jacobian matrix of partial derivatives)

$$f(z) = f(z_0) + (Df)(z_0) \cdot (z - z_0) + (z - z_0) \cdot \varphi(z)$$

Identify the complex function $f: \Omega \to \mathbb{C}$ with

$$f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2, \quad z = \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u(z) \\ v(z) \end{pmatrix} = f(z)$$

There exists φ such that

$$f(z) - f(z_0) = (Df)(z - z_0) + (z - z_0)\varphi(z) \quad (*)$$

$$(Df)(z_0)(z - z_0) = \begin{bmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

$$= \begin{bmatrix} u_x(x - x_0) + u_y(y - y_0) \\ v_x(x - x_0) + v_y(y - y_0) \end{bmatrix}$$

$$\stackrel{CR}{=} \begin{bmatrix} u_x(x-x_0) - v_x(y-y_0) \\ v_x(x-x_0) + u_x(y-y_0) \end{bmatrix}$$

$$= u_x(z_0)\begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix} + v_x(z_0)\begin{bmatrix} -(y-y_0) \\ x-x_0 \end{bmatrix}$$

$$= [u_x(z_0) + i v_x(z_0)](z-z_0)$$

Divide $(*)$ by $(z-z_0)$

$$\frac{f(z)-f(z_0)}{z-z_0} = u_x(z_0) + i v_x(z_0) + \underbrace{\frac{z-z_0}{z-z_0} f(z)}_{\xrightarrow[z\to z]{} 0}$$

$$\Big\downarrow z \to z_0$$

$$f'(z_0) = u_x(z_0) + i v_y(z_0) = f_x(z_0)$$

$\Rightarrow$ f is complex differentiable at z_0

Def.: Let $\Omega \subset \mathbb{C}$ be open. $f: \Omega \to \mathbb{C}$ is called holomorphic if f is complex differentiable at all $z \in \Omega$ and f' is continuous in Ω.

f is holomorphic at ∞ if $g(z) := f(\frac{1}{z})$ is holomorphic at 0.

Remark: If f is holomorphic in a disc around z_0, then f is called holomorphic at z_0.

Example ①: $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z$ (identity)

$$f_x(z) = 1, \quad f_y(z) = i \quad \forall_{z \in \mathbb{C}}$$

f_x and f_y are continuous and we have $f_x = \frac{1}{i} f_y$
$\Rightarrow$ f is holomorphic in $\mathbb{C}$.

② $f(z) = \frac{1}{z}$ is holomorphic at ∞, since
$g(z) := f\left(\frac{1}{z}\right) = z$ is holomorphic at 0.

③ $f: \mathbb{C} \to \mathbb{C}$ $\quad f(z) = Re(z)$

f is partially differentiable $f_x(z) = 1$, $f_y(z) = 0$
$\Rightarrow f_x \neq \frac{1}{i} f_y \Rightarrow f(z)$ is not complex differentiable

Theorem: Let $\Omega \subset \mathbb{C}$ be open, $u: \Omega \to \mathbb{R}$ two times continuously partial differentiable. Then the following holds:

$\Delta u := u_{xx} + u_{yy} = 0 \iff \exists$ a function f holomorphic in Ω with $Re(f) = u$

Proof: "$\Leftarrow$" Let $f = u + iv$ be holomorphic

$\Rightarrow u_x = v_y, \quad u_y = -v_x$

$\Rightarrow u_{xx} = v_{yx}, \quad u_{yy} = -v_{xy}$

$$\left.\begin{array}{l} u_{xx} \text{ continuous} \Rightarrow v_{yx} \text{ continuous} \\ u_{yy} \text{ continuous} \Rightarrow v_{xy} \text{ continuous} \end{array}\right\}$$

$\Rightarrow v_{yx} = v_{xy}$

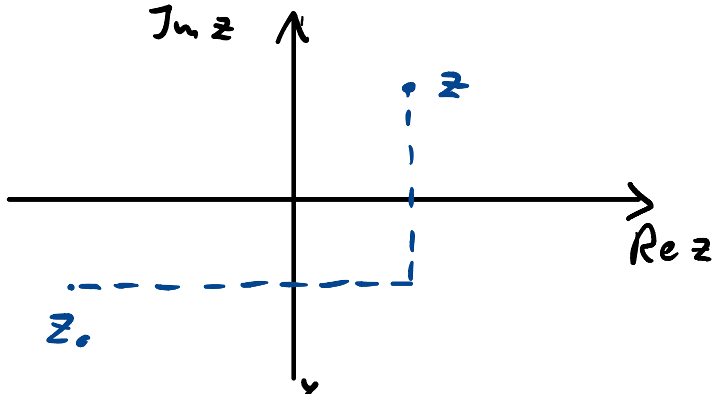
$\Rightarrow \Delta u = u_{xx} + u_{yy} = v_{yx} - v_{yx} = 0$

"$\Rightarrow$" Let $u : \Omega \to \mathbb{R}$ be given, with $\Delta u = 0$

We need to find $v : \Omega \to \mathbb{R}$, such that $f = u + iv$ is holomorphic, i.e. $v_x = -u_y$, $v_y = u_x$

Choose a $z_0 \in \Omega$ and let $v(z_0) = 0$

Determine $v(z)$ via integration over the following path



$$v(z) := \int_{x_0}^{x} v_x(t, y_0)\,dt + \int_{y_0}^{y} v_y(x, s)\,ds$$

$$= -\int_{x_0}^{x} u_y(t, y_0)\,dt + \int_{y_0}^{y} u_x(x, s)\,ds$$

Check that the Cauchy-Riemann differential eqs. are satisfied:

$$v_x(x,y) = -u_y(x,y_0) + \int_{y_0}^{y} u_{xx}(x,s)\,ds$$

$$= -u_y(x,y_0) - \int_{y_0}^{y} u_{yy}(x,s)\,ds$$

$$= -u_y(x,y_0) - u_y(x,y) + u_y(x,y_0) = -u_y(x,y)$$

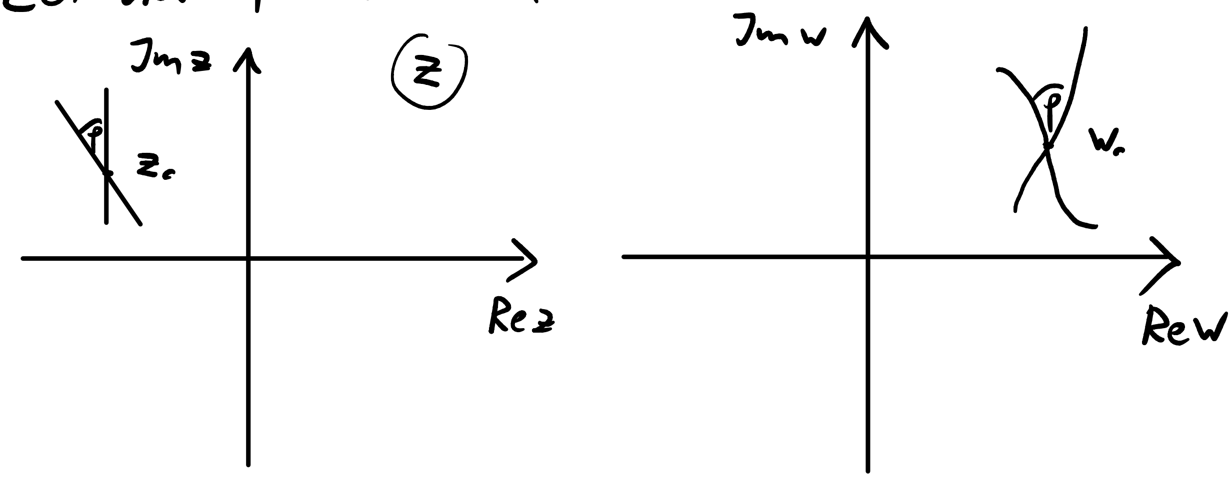
Geometrical interpretation of complex differentiability

Let f be holomorphic at z_0.

$$\varrho(z) := \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \xrightarrow[z\to z_0]{} 0$$

$$\Rightarrow f(z) = f(z_0) + f'(z_0)(z-z_0) + \varrho(z)(z-z_0)$$

Let now $f'(z_0) \neq 0$, $w = f(z)$, $w_0 = f(z_0)$



Straight line through z_0 $\quad z(t) = z_0 + t e^{i\alpha} \quad t \in \mathbb{R}$

$$f'(z_0) = a = |a|\, e^{i \arg a}$$

$$w(t) = f(z(t)) = f(z_0) + f'(z_0)te^{i\alpha} + \underbrace{\rho(z_0 + te^{i\alpha})te^{i\alpha}}_{\to 0 \text{ for } t \to 0}$$

$$\approx w_0 + a t e^{i\alpha} = w_0 + |a| t \, e^{i(\alpha + \arg a)}$$

Consider two curves, which intersect each other at a point z_0. These curves are mapped onto two curves in the w-plane, which intersect each other at w_0 with the same angle as the original curves in the z-plane.

Rules for differentiation:

1. f, g are holomorphic at z_0, and $a, b \in \mathbb{C}$

$\Rightarrow$ $af + bg$, $f \cdot g$, $\frac{f}{g}$ $(g(z_0) \neq 0)$ are holomorphic as well and one has

$$(af + bg)' = af' + bg'$$

$$(f \cdot g)' = f'g + fg'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

2. Chain rule

f holomorphic at z_0, F holomorphic at $f(z_0)$

$\Rightarrow$ $g = F \circ f$ holomorphic at z_0, and one has

$$g'(z_0) = F'(f(z_0)) \cdot f'(z_0)$$

Examples: ① $f(z) = z^n$ proof by induction

f holomorphic for all $n \in \mathbb{N}$, and $f'(z) = n z^{n-1}$

② Polynomials: $f(z) = a_0 + a_1 z + \dots + a_n z^n$

holomorphic in $\mathbb{C}$ and

$f'(z) = a_1 + 2a_2 z + \dots + n a_n z^{n-1}$

③ rational functions $f(z) = \dfrac{p(z)}{q(z)}$ are holomorphic

in $A = \{ z \in \mathbb{C} : q(z) \neq 0 \}$

Let $p(z) = \sum_{j=0}^{m} a_j z^j$, $q(z) = \sum_{j=0}^{n} b_j z^j$, $m \leq n$

Extension of the domain:

$$\tilde{A} = \{ z \in \tilde{\mathbb{C}} : q(z) \neq 0 \} \text{ and let}$$

$$f(\infty) = \begin{cases} \dfrac{a_m}{b_m} & m = n \\ 0 & m < n \end{cases}$$

$$f(z) = \frac{a_0 + a_1 z + \dots + a_m z^m}{b_0 + b_1 z + \dots + b_n z^n} = z^{m-n} \frac{\frac{a_0}{z^m} + \frac{a_1}{z^{m-1}} + \dots + a_m}{\frac{b_0}{z^n} + \frac{b_1}{z^{n-1}} + \dots + b_n}$$

$$\xrightarrow[z \to \infty]{} \begin{cases} \dfrac{a_m}{b_n} & m = n \\ 0 & m < n \end{cases}$$

Claim: f is holomorphic at ∞ (for $n \geq m$)

Proof: we need to show that $g(z) := f(\frac{1}{z})$ is holomorphic at 0.

$$g(z) = f\left(\frac{1}{z}\right) = z^{n-m} \frac{a_0 z^m + a_1 z^{m-1} + \dots + a_m}{b_0 z^n + b_1 z^{n-1} + \dots + b_n}$$

The denominator is different from zero for $z = 0$

$\Rightarrow$ g holomorphic at 0, since z^{n-m} holomorphic at 0 due to $n \geq m$ $\Rightarrow$ fraction is holomorphic as well.

④ Exponential function

Claim: $f(z) = \exp(z)$ is holomorphic for $\forall_{z \in \mathbb{C}}$, and $(e^z)' = e^z$

Proof: $\exp(x + iy) = e^x (\cos y + i \sin y)$

$$f_x = e^x (\cos y + i \sin y), \quad \frac{1}{i} f_y = \frac{1}{i} e^x (-\sin y + i \cos y)$$
$$= e^x (\cos y + i \sin y)$$

$\Rightarrow$ $f_x = \frac{1}{i} f_y$, f_x is continuous everywhere

$\Rightarrow$ f is holomorphic.

$$f'(z) = f_x(z) = \exp(z)$$

(5) $f(z) = \bar{z} = x - iy$

$$f_x = 1 \neq \frac{1}{i} f_y = \frac{-i}{i} = -1 \qquad \forall_{z \in \mathbb{C}}$$

(6) The real valued function $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & : x \leq 0 \\ x^2 & : x > 0 \end{cases}$$ continuously differentiable on the real axis

Interpret f as a complex function

$$f: \mathbb{R} =: \Omega \to \mathbb{C}$$

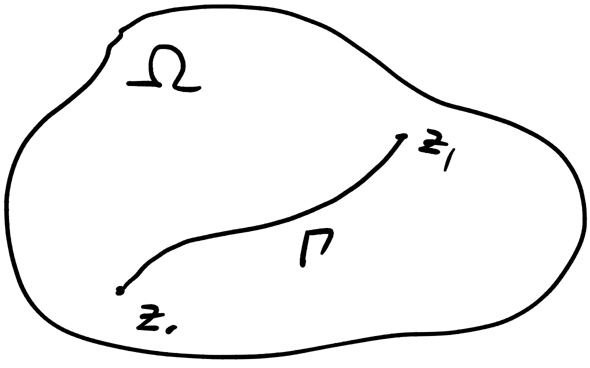
$$f_x = 2x, \quad \frac{1}{i} f_y = \frac{1}{i} \cdot 0 = 0$$

$f_x \neq \frac{1}{i} f_y$ for $x > 0$, f is not complex differentiable

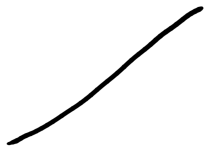
Since $\mathbb{R}$ is not open in $\mathbb{C}$, the definition of differentiability is difficult, holomorphy is impossible to define.

§ 2 Integral Theorems

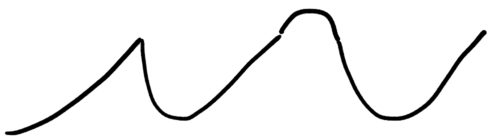
2.1 Line Integrals



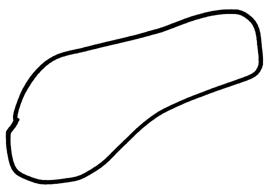
"rectifiable curve"



path segment



path as a combination of segments



curve

Def.: $\gamma : [a,b] \to \mathbb{C}$ continuously differentiable

($i.e.$ $\gamma = \gamma_1 + i\gamma_2$, $\gamma_j : [a,b] \to \mathbb{R}$ continuously differentiable)

and the following holds true:

1) $\varphi'(t) = \varphi_1'(t) + i\,\varphi_2'(t) \neq 0 \quad \forall_{t \in [a,b]}$

2) $\varphi(t) \neq \varphi(\tilde{t})$ with $t \neq \tilde{t}$

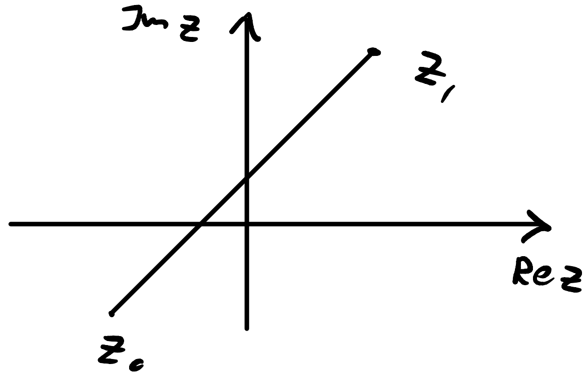
Then $\Gamma(\varphi(a), \varphi(b)) = \{ z \in \mathbb{C} : z = \varphi(t),\ a \leq t \leq b \}$ is called path segment in $\mathbb{C}$ with initial point $z_0 = \varphi(a)$ and final point $z_1 = \varphi(b)$. $\varphi(t)$ is called parametrization of the path segment.

Def.: $-\Gamma = \{ z \in \mathbb{C} : z = \varphi(-t),\ -b \leq t \leq -a \}$

is called the inverse path segment.

$\psi : [-b, -a] \to \mathbb{C}$, $\psi(t) = \varphi(-t)$ is the parametrization of $-\Gamma$

Example:



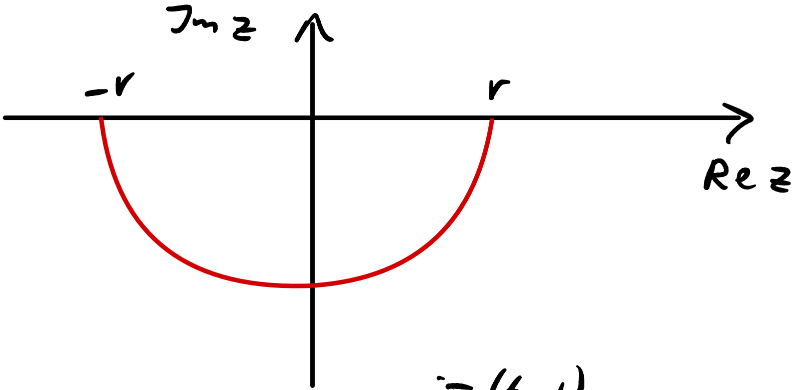
$$\varphi_1(t) = (1-t)\,z_0 + t\,z_1 \,, \quad \varphi_1 : [0,1] \to \mathbb{C}$$

$$\varphi_2(t) = z_0 + \frac{t}{|z_1 - z_2|}(z_1 - z_0)\,, \quad \varphi_2 : [0, |z_1 - z_2|] \to \mathbb{C}$$

$$\varphi_3(t) = z_0 \left(\cos \tfrac{\pi}{2} t\right)^2 + z_1 \left(\sin \tfrac{\pi}{2} t\right)^2, \quad \varphi_3 : [0,1] \to \mathbb{C}$$

$$-\Gamma : \psi(t) = (1-t)z_1 + t z_0 \qquad \psi : [0,1] \to \mathbb{C}$$

Example semi-circle



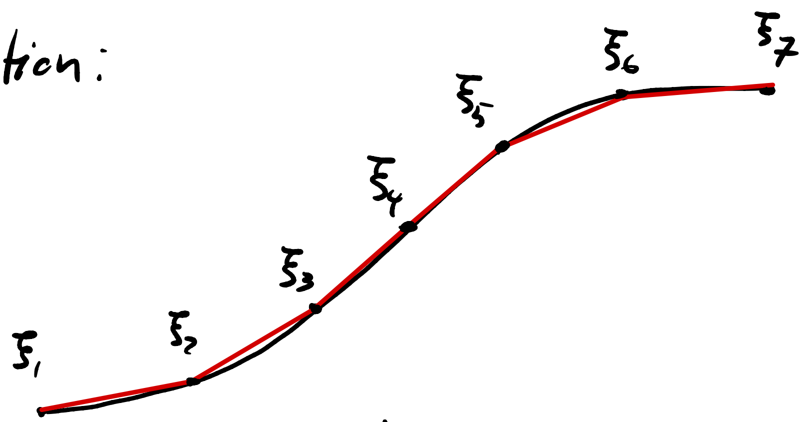
parametrization $\varphi(t) = r e^{i\pi(t-1)}$, $\quad \varphi : [0,1] \to \mathbb{C}$

inverse path segment : $\psi(t) = r e^{i\pi(-t)}$, $\quad \psi : [0,1] \to \mathbb{C}$

Let $\varphi : [a,b] \to \mathbb{C}$ be a parametrization of Γ,

then $|\Gamma| = \int_a^b |\varphi'(t)|\,dt$ is the length of Γ.

Motivation:



$$|\Gamma| \approx \sum_{j=2}^{N} |\xi_j - \xi_{j-1}| \qquad \xi_j = \varphi(t_j)$$

$$= \sum_{j=2}^{N} \frac{|\varphi(t_j) - \varphi(t_{j-1})|}{t_j - t_{j-1}} (t_j - t_{j-1})$$

$$\approx \sum_{j=2}^{N} |\varphi'(t_j)| (t_j - t_{j-1})$$

$$\xrightarrow[N\to\infty]{} \int_a^b |\varphi'(t)| \, dt$$

Lemma: Let $\varphi : [a,b] \to \mathbb{C}$, $\psi : [c,d] \to \mathbb{C}$

be parametrizations of the same path segment Γ.

Then there exists a continuously diff. function

$$\tau : [c,d] \to [a,b]$$

with $\tau(c) = a$, $\tau(d) = b$ and $\tau'(t) > 0$ for $t \in (c,d)$,

such that $\psi(t) = \varphi(\tau(t))$

Proof: $\varphi : [a,b] \to \Gamma$ is bijective

Def.: $\tau := \varphi^{-1} \circ \psi$

$$\Rightarrow \tau(c) = \varphi^{-1}(\psi(c)) = \varphi^{-1}(\varphi(a)) = a$$

$$\tau(d) = \varphi^{-1}(\psi(d)) = \varphi^{-1}(\varphi(b)) = b$$

φ, ψ are continuous and bijective $\Rightarrow$ τ is continuous, bijective

$\Rightarrow \tau'(t) \neq 0$ for $t \in (c,d)$

in addition $\tau'(t) > 0$, since $\tau(c) = a < b = \tau(d)$

chain rule, differentiation of the inverse function

$$\tau'(t) = \frac{d}{dt}\left[\varphi^{-1}(\psi(t))\right] \qquad \left[f^{-1}(y)\right]' = \frac{1}{f(x)}$$

$$= \frac{d}{d\psi}\varphi^{-1}(\psi) \quad \frac{d\psi}{dt}$$

$$= \frac{1}{\varphi'(\tau(t))} \quad \psi'(t)$$

Theorem: Let $\varphi : [a,b] \to \mathbb{C}$ and $\psi : [c,d] \to \mathbb{C}$ be parametrizations of a path segment Γ.

Then $|\Gamma| = \int_a^b |\varphi'(t)|\,dt = \int_c^d |\psi'(t)|\,dt$

Proof: $\psi(t) := \varphi(\tau(t))$ with $\tau(t)$ from above.

$$\Rightarrow \psi'(t) = \varphi'(\tau(t))\,\tau'(t) \quad , \tau'(t) > 0$$

$$\Rightarrow \int_c^d |\psi'(t)|\,dt = \int_c^d |\varphi'(\tau(t))|\,\tau'(t)\,dt$$

substitute $s := \tau(t)$

$$= \int_{\tau(c)}^{\tau(d)} |\varphi'(s)|\,ds = \int_a^b |\varphi'(s)|\,ds \qquad \#$$

Def.: Let Γ be a path segment and $f: \Gamma \to \mathbb{C}$ continuous, and φ is a parametrization of Γ.

Then the integral of f along Γ is defined as

$$\boxed{\int_\Gamma f(z)\,dz := \int_a^b f(\varphi(t))\,\varphi'(t)\,dt}$$