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Mathematical Methods of Modern Physics - Problem Set 12

Summer Semester 2024

Due: The problem set will be discussed in the seminars on 04.07. and 05.07.

Internet: The problem sets can be downloaded from
https://home.uni-leipzig.de/stp/Mathematical_methods_2_ss24.html

The points for all exercises on this sheet are bonus points in the sense that they are not added to the total number of points required for admission to the exam. However, the content of the exercises is relevant to the exam as it is related to the content of the lecture.

40. Solving a real integral by contour integration *1+3 Bonus-Points*

We want to calculate the integral

$$\int_0^\infty \frac{x\sin(x)}{x^2+1}dx\ .$$

a) Show that $\int_0^\infty \frac{x\sin(x)}{x^2+1}dx = \frac{1}{2}\mathrm{Im}\left[\int_{-\infty}^\infty \frac{xe^{ix}}{x^2+1}dx\right]$.

b) Evaluate the integral using Jordan's Lemma and the Residue theorem.

41. Fourier transformation of even functions *2 Bonus-Points*

Let $\tilde{f}(q)$ be the Fourier transformation of a real function $f:\mathbb{R}\to\mathbb{R}$. Show that if f is an even function, i.e. $\forall x\in\mathbb{R}: f(x)=f(-x)$, this implies that $\tilde{f}(q)$ is also real and even.

42. Another representation of the delta function *4+2 Bonus-Points*

We want to consider another representation of the delta function, namely

$$\lim_{\epsilon\to 0}\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2} = \delta(x).$$

a) Show by contour integration that

$$\lim_{\epsilon\to 0}\int_{-\infty}^\infty dx\,\frac{1}{\pi}\frac{\epsilon}{x^2+\epsilon^2}f(x) = f(0),$$

for all functions $f(z)$ which are holomorphic in the upper half plane except for a finite number of poles, that are continuous on $\mathbb{R}$, and that fulfill $|f(z)| < |z|^{1-\delta}$ for $|z|\to\infty$ in the upper half-plane and some $\delta > 0$.

b) We can also consider an integral form of this representation. Show that

$$\delta(x) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{-\infty}^\infty dq\, e^{iqx}e^{-\epsilon|q|},$$

by solving the integral and identifying the representation given above.

43. The Heaviside function

3 Bonus-Points

Show by contour integration that

$$-\frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} dq\,\frac{e^{-iqx}}{q+i\epsilon} = \Theta(x)$$

is the integral representation of the Heaviside function $\Theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$.