Querying Circumscribed Description Logic Knowledge Bases

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Abstract

Circumscription is one of the main approaches for defining non-monotonic description logics (DLs). While the decidability and complexity of traditional reasoning tasks such as satisfiability of circumscribed DL knowledge bases (KBs) is well understood, for evaluating conjunctive queries (CQs) and unions thereof (UCQs), not even decidability had been established. In this paper, we prove decidability of (U)CQ evaluation on circumscribed DL KBs and obtain a rather complete picture of both the combined complexity and the data complexity, for DLs ranging from $\mathcal{ALCHIO}$ via $\mathcal{EL}$ to various versions of DL-Lite. We also study the much simpler atomic queries (AQs).

1 Introduction

While standard description logics (DLs) such as those underlying the OWL 2 ontology language do not support non-monotonic reasoning, it is generally acknowledged that extending DLs with non-monotonic features is very useful. Concrete examples of applications include ontological modeling in the biomedical domain (Rector 2004; Stevens et al. 2007) and the formulation of access control policies (Bonatti and Samarati 2003). Circumscription is one of the traditional AI approaches to non-monotonicity, and it provides an important way to define non-monotonic DLs. In contrast to other approaches such as default rules, it does not require the adoption of strong syntactic restrictions to preserve decidability. DLs with circumscription are closely related to several other approaches to non-monotonic DLs, in particular to DLs with defeasible inclusions and typicality operators (Bonatti, Faella, and Sauro 2011; Casini and Straccia 2013; Giordano et al. 2013; Bonatti et al. 2015b; Pensel and Turhan 2018).

The main feature of circumscription is that selected predicate symbols can be minimized, that is, the extension of these predicates is required to be minimal regarding set inclusion. Other predicates may vary freely or be declared fixed. In addition, a preference order can be declared on the minimized predicates. In DLs, minimizing or fixing role names causes undecidability of reasoning, and consequently only concept names may be minimized or fixed (Bonatti, Lutz, and Wolter 2009). The traditional AI use of circumscription is to introduce and minimize abnormality predicates such as $\mathsf{AbnormalBird}$, which makes it possible to formulate defeasible implications such as 'birds fly, unless they are abnormal, which shouldn't be assumed unless there is a reason to do so'. Circumscription is also closely related to the closure of predicates symbols as studied for instance in (Lutz, Seylan, and Wolter 2013; Ngo, Ortiz, and Simkus 2016; Lutz, Seylan, and Wolter 2019); in fact, this observation goes back to (Reiter 1977; Lifschitz 1985). While DLs normally assume an open world semantics and represent incomplete knowledge, such closed predicates are interpreted under a closed world assumption, reflecting the fact that complete knowledge is available regarding those predicates. Circumscription may then be viewed as a soft form of closing concept names: there are no other instances of a minimized concept name except the explicitly asserted ones, unless we are forced to introduce (a minimal set of) additional instances to avoid inconsistency.

A primary application of DLs is in ontology-mediated querying, where an ontology is used to enrich data with domain knowledge. Somewhat surprisingly, rather little is known about ontology-mediated querying with DLs that support circumscription. The most popular choice of queries are conjunctive queries (CQs) and unions thereof (UCQs), and to the best of our knowledge in this case not even decidability is known. The aim of this paper is to close this gap and study the decidability and precise complexity of ontology-mediated querying for DLs with circumscription, both w.r.t. combined complexity and data complexity. We consider the expressive DL $\mathcal{ALCHIO}$, the tractable (without circumscription) DL $\mathcal{EL}$, and several members of the DL-Lite family that has been tailored specifically towards ontology-mediated querying. These may be viewed as logical cores of the profiles OWL 2 DL, OWL 2 EL, and OWL 2 QL of the OWL 2 ontology language (Motik et al. 2009).

One of our main results is that UCQ evaluation is decidable in all these DLs when circumscription is added. It is 2EXP-complete in $\mathcal{ALCHIO}$ w.r.t. combined complexity, and thus not harder than query evaluation without circumscription. W.r.t. data complexity, however, there is a significant increase from coNP- to Π^p_2-completeness. For $\mathcal{EL}$, both combined and data complexity turns out to be identical to that of $\mathcal{ALCHIO}$, which improves lower bounds from (Bonatti, Faella, and Sauro 2011). All these lower bounds already hold for CQs. Remarkably, the Π^p_2 lower bound for data complexity already holds when there is only a single minimized concept name (and thus no preference relation), and without fixed predicates. The complexities for DL-Lite are lower, though still high. A summary can be found in Table 1. Evaluation is 'only' coNP-complete w.r.t. data complexity in all cases, except when also role disjointness constraints are added (this case is not in the table). The combined complexity remains at 2EXP when role inclusions are present, and drops to coNEXP without them. The lower bounds already apply to very basic positive versions of DL-Lite that do not even provide concept disjointness constraints, and the upper bounds apply to expressive versions that include all Boolean operators.

We also study evaluation of the basic yet important atomic queries (AQs), conjunctive queries of the form $A(x)$ with A a concept name. Also here, we obtain a rather complete picture of the complexity landscape. It is known from (Bonatti, Lutz, and Wolter 2009) that AQ evaluation in $\mathcal{ALCHIO}$ is $\text{coNExp}^{\text{NP}}$-complete w.r.t. combined complexity. We show that the lower bound holds already for $\mathcal{EL}$. Moreover, our Π_2^p-lower bound for the data complexity of (U)CQ-evaluation in $\mathcal{EL}$ mentioned above only requires an AQ, and thus AQ evaluation in both $\mathcal{ALCHIO}$ and $\mathcal{EL}$ are Π_2^p-complete w.r.t. data complexity. For DL-Lite, the data complexity drops to PTIME in all considered versions, and the combined complexity ranges from coNEXP-complete to Π_2^p-complete depending on which Boolean operators are admitted. A summary can be found in Table 2.

Proofs are in the appendix, see supplemental material.

Related Work. A foundational paper on description Logics with circumscription is (Bonatti, Lutz, and Wolter 2009) where concept satisfiability and knowledge base consistency is studied in the $\mathcal{ALCHIO}$ family of DLs; these problems are interreducible with AQ evaluation in polynomial time (up to complementation). The same problems have been considered in (Bonatti, Faella, and Sauro 2011) for $\mathcal{EL}$ and DL-Lite and in (Bonatti et al. 2015a) for DLs without the finite model property, including a version of DL-Lite. The recent (Bonatti 2021) is the only work we are aware of that considers ontology-mediated querying in the context of circumscription. It provides lower bounds for $\mathcal{EL}$ and DL-Lite, which are both imnproved in the current paper, but no decidability results / upper bounds. A relaxed version of circumscription that admits lower complexity has recently been studied in (Stefano, Ortiz, and Simkus 2022). We have already mentioned connections to DLs with defeasible inclusions and typicality operators in the introduction, including concrete references. A connection between circumscription and the complexity class Π_2^p was first observed in (Eiter and Gottlob 1993) and this complexity shows up in our data complexity results. Our proofs, however, are quite different.

2 Preliminaries

Let $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ be countably infinite sets of *concept names*, *role names*, and *individual names*. An *inverse role* takes the form r^- with r a role name, and a *role* is a role name or an inverse role. If $r = s^-$ is an inverse role, then r^- denotes s. An $\mathcal{ALCIO}$ *concept* C is built according to the rule $C, D ::= \top \mid A \mid \{a\} \mid \neg C \mid C \sqcap D \mid \exists r.D$ where A ranges over concept names, a over individual names, and r over roles. A concept of the form $\{a\}$ is called a *nominal*. We write $\bot$ as abbreviation for $\neg\top$, $C \sqcup D$ for $\neg(\neg C \sqcap \neg D)$, and $\forall r.C$ for $\neg\exists r.\neg C$. An $\mathcal{ALCI}$ *concept* is a nominal-free $\mathcal{ALCIO}$ concept. An $\mathcal{EL}$ *concept* is an $\mathcal{ALCI}$ concept that uses neither negation nor inverse roles.

An $\mathcal{ALCHIO}$ *TBox* is a finite set of *concept inclusions (CIs)* $C \sqsubseteq D$, where C, D are $\mathcal{ALCIO}$ concepts, and *role inclusions (RIs)* $r \sqsubseteq s$, where r, s are roles. In an $\mathcal{EL}$ *TBox*, only $\mathcal{EL}$ concepts may be used in CIs, and RIs are disallowed. An *ABox* is a finite set of *concept assertions* $A(a)$ and *role assertions* $r(a, b)$ where A is a concept name, r a role name, and a, b are individual names. We use $\mathsf{ind}(\mathcal{A})$ to denote the set of individual names used in $\mathcal{A}$. An $\mathcal{ALCHIO}$ *knowledge base (KB)* takes the form $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with $\mathcal{T}$ an $\mathcal{ALCHIO}$ TBox and $\mathcal{A}$ an ABox. $\mathcal{ALCHI}$ TBoxes and KBs are defined analogously, but may not use nominals.

The semantics is defined as usual in terms of interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ with $\Delta^\mathcal{I}$ the (non-empty) *domain* and $\cdot^\mathcal{I}$ the *interpretation function*, we refer to (Baader et al. 2017) for full details. An interpretation satisfies a CI $C \sqsubseteq D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ and likewise for RIs. It satisfies an assertion $A(a)$ if $a \in A^\mathcal{I}$ and $r(a, b)$ if $(a, b) \in r^\mathcal{I}$; we thus make the *standard names assumption*. An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, written $\mathcal{I} \models \mathcal{T}$, if it satisfies all inclusions in it. Models of ABoxes and KBs are defined likewise. For an interpretation $\mathcal{I}$ and $\Delta \subseteq \Delta^\mathcal{I}$, we use $\mathcal{I}|_\Delta$ to denote the restriction of $\mathcal{I}$ to subdomain Δ.

A *signature* is a set of concept and role names referred to as *symbols*. For any syntactic object O such as a TBox or an ABox, we use $\mathsf{sig}(O)$ to denote the symbols used in O and $|O|$ to denote the *size* of O, meaning the encoding of O as a word over a suitable alphabet.

We next introduce several more restricted DLs of the DL-Lite family. A *basic concept* is of the form A or $\exists r.\top$. A DL-Lite$^\mathcal{H}_\mathsf{core}$ TBox is a finite set that contains concept inclusions $C \sqsubseteq D$ and *(concept) disjointness assertions* $C \sqsubseteq \neg D$ where in both cases C, D are basic concepts, as well as role inclusions $r \sqsubseteq s$ with r, s roles. We drop superscript $\cdot^\mathcal{H}$ if no role inclusions are admitted, use subscript $\cdot_\mathsf{horn}$ to indicate that the concepts C, D in concept inclusions may be conjunctions of basic concepts, and subscript $\cdot_\mathsf{bool}$ to indicate that C, D may be Boolean combinations of basic concepts, that is, built from basic concepts using $\neg$, $\sqcap$, $\sqcup$.

A *circumscription pattern* is a tuple $\mathsf{CP} = (\prec, \mathsf{M}, \mathsf{F}, \mathsf{V})$, where $\prec$ is a strict partial order on M called the *preference relation*, and M, F and V are a partition of $\mathsf{N_C}$. The elements of M, F and V are the *minimized*, *fixed* and *varying* concept names. Role names always vary to avoid undecidability (Bonatti, Lutz, and Wolter 2009). The preference relation $\prec$ on M induces a preference relation $<_\mathsf{CP}$ on interpretations by setting $\mathcal{J} <_\mathsf{CP} \mathcal{I}$ if the following conditions hold:

1. $\Delta^\mathcal{J} = \Delta^\mathcal{I}$,
2. for all $A \in \mathsf{F}$, $A^\mathcal{J} = A^\mathcal{I}$,
3. for all $A \in \mathsf{M}$ with $A^\mathcal{J} \not\subseteq A^\mathcal{I}$, there is a $B \in \mathsf{M}$, $B \prec A$, such that $B^\mathcal{J} \subsetneq B^\mathcal{I}$,
4. there exists an $A \in \mathsf{M}$ such that $A^\mathcal{J} \subsetneq A^\mathcal{I}$ and for all $B \in \mathsf{M}$, $B \prec A$ implies $B^\mathcal{J} = B^\mathcal{I}$.

	$\mathcal{EL}$, $\mathcal{ALCHIO}$	DL-Lite$^{\mathcal{H}}_{\text{core}}$, DL-Lite$^{\mathcal{H}}_{\text{bool}}$	DL-Lite$_{\text{bool}}$	DL-Lite$_{\text{core}}$, DL-Lite$_{\text{horn}}$
Combined complexity	2EXP-c.(Thm. 1, 2)	2EXP-c.(†) (Thm. 1, 5)	CONEXP-c.(Thm. 6, 13)	CONEXP-c.(†) (Thm. 6, 7)
Data complexity	Π_2^{P}-c.(Thm. 3, 4)	CONP-c.(Thm. 8, 9)	CONP-c.(Thm. 8, 9)	CONP-c.(Thm. 8, 9)

Table 1: Complexity of (U)CQ evaluation on circumscribed KBs. .(†) indicates that UCQs are needed for lower bound.

A *circumscribed KB (cKB)* takes the form $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ where $\mathcal{K}$ is a KB and CP a circumscription pattern. A model $\mathcal{I}$ of $\mathcal{K}$ is a *model* of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, denoted $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, if no $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ is a model of $\mathcal{K}$. A cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ is *satisfiable* if it has a model.

A *conjunctive query (CQ)* takes the form $q(\bar{x}) = \exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ where $\bar{x}$ and $\bar{y}$ are tuples of variables and φ is a conjunction of *atoms* $A(z)$ and $r(z, z')$, with $A \in \mathsf{N_C}$, $r \in \mathsf{N_R}$, and z, z' variables from $\bar{x} \cup \bar{y}$. The variables in $\bar{x}$ are the *answer variables*, and $\mathsf{var}(q)$ denotes $\bar{x} \cup \bar{y}$. We take the liberty to view q as a set of atoms, writting e.g. $\alpha \in q$ to indicate that α is an atom in q. We may also write $r^-(x, y) \in q$ in place of $r(y, x) \in q$. A CQ q gives rise to an interpretation $\mathcal{I}_q$ with $\Delta^{\mathcal{I}_q} = \mathsf{var}(q)$, $A^{\mathcal{I}_q} = \{x \mid A(x) \in q\}$, and $r^{\mathcal{I}_q} = \{(x, y) \mid r(x, y) \in q\}$ for all $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. A *union of conjunctive queries (UCQ)* $q(\bar{x})$ is a disjunction of CQs that all have the same answer variables $\bar{x}$. The *arity* of q is the length of $\bar{x}$ and q is *Boolean* if it is of arity zero. An *atomic query (AQ)* is a CQ of the simple form $A(x)$, with A a concept name.

With a homomorphism from a CQ q to an interpretation $\mathcal{I}$, we mean a homomorphism from $\mathcal{I}_q$ to $\mathcal{I}$ (defined as usual). A tuple $\bar{d} \in (\Delta^{\mathcal{I}})^{|\bar{x}|}$ is an *answer* to a UCQ q on an interpretation $\mathcal{I}$, written $\mathcal{I} \models q(\bar{d})$, if there is a homomorphism h from a CQ p in q to $\mathcal{I}$ with $h(\bar{x}) = \bar{d}$. A tuple $\bar{a} \in \mathsf{ind}(\mathcal{A})$ is an *answer* to a UCQ q on a cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, written $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$, if $\mathcal{I} \models q(\bar{a})$ for all models $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$.

Example 1. *Consider a database about universities. The TBox contains domain knowledge such as*

$$\begin{aligned} \mathsf{University} &\sqsubseteq \mathsf{Organization} \\ \mathsf{Organization} &\sqsubseteq \mathsf{Public} \sqcup \mathsf{Private} \\ \mathsf{Public} \sqcap \mathsf{Private} &\sqsubseteq \bot. \end{aligned}$$

Circumscription can be used to express defeasible inclusions. For example, from a European perspective, universities are usually public:

$$\mathsf{University} \sqsubseteq \mathsf{Public} \sqcup \mathsf{Ab}_U$$

where Ab_U *is a fresh 'abnormality' concept name that is minimized. If the ABox contains*

$$\mathsf{University(leipzigu)}, \mathsf{University(mit)}, \mathsf{Private(mit)}$$

and we pose the CQ $q(x) = \mathsf{Organization}(x) \wedge \mathsf{Public}(x)$, *then the answer is* leipzigu.

We may also use circumscription to implement a soft closed world assumption, similar in spirit to soft constraints in constraint satisfaction. Assume that the ABox additionally contains a database of nonprofit corporations:

$$\mathsf{NPC(greenpeace)} \quad \mathsf{NPC(wwf)}$$

and that this database is essentially complete, *expressed by minimizing* NPC. *If we also know that*

$$\begin{aligned} \mathsf{IvyLeagueU} &\sqsubseteq \exists \mathsf{ownedBy}.(\mathsf{NPC} \sqcap \mathsf{Rich}) \\ \mathsf{DonationBased} &\sqsubseteq \neg \mathsf{Rich} \end{aligned}$$

$$\mathsf{DonationBased(greenpeace)} \quad \mathsf{DonationBased(wwf)}$$
$$\mathsf{IvyLeagueU(harvard)}$$

then we are forced to infer that our list of NPCs was not actually complete as all explicitly known NPCs are not rich, but a rich NPC must exist. A strict closed world assumption would instead result in an inconsistency.

Let $\mathcal{L}$ be a description logic such as $\mathcal{ALCHIO}$ or $\mathcal{EL}$ and let $\mathcal{Q}$ be a query language such as UCQ, CQ, or AQ. With $\mathcal{Q}$ *evaluation on circumscribed* $\mathcal{L}$ *KBs*, we mean the problem to decide, given an $\mathcal{L}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a query $q(\bar{x})$ from $\mathcal{Q}$, and a tuple $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$, whether $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$. When studying the combined complexity of this problem, all of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, q, and $\bar{a}$ are treated as inputs. For data complexity, in contrast, we assume q, $\mathcal{T}$, and CP to be fixed and thus of constant size.

3 Between $\mathcal{ALCHIO}$ and $\mathcal{EL}$

We study the complexity of query evaluation on circumscribed KBs for DLs between $\mathcal{ALCHIO}$ and $\mathcal{EL}$. In fact, we prove 2EXP-completeness in combined complexity and Π_2^{P}-completeness in data complexity for all these DLs.

3.1 Fundamental Observations

We start with some fundamental observations that underlie the subsequent proofs, first observing a reduction from UCQ evaluation on circumscribed $\mathcal{ALCHIO}$ KBs to UCQ evaluation on circumscribed $\mathcal{ALCHI}$ KBs. Note that a nominal may be viewed as a (strictly) closed concept name with a single instance. In fact, this reduction is a simple version of a reduction from query evaluation with closed concept names to query evaluation on cKBs in the proof of Theorem 2 below.

Proposition 1. *UCQ evaluation on circumscribed* $\mathcal{ALCHIO}$ *KBs can be reduced in polynomial time to UCQ evaluation on circumscribed* $\mathcal{ALCHI}$ *KBs.*

Proof. Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be an $\mathcal{ALCHIO}$ cKB, with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, and let $q(\bar{x})$ be a UCQ. Let N be the set of individual names a such that the nominal $\{a\}$ is used in $\mathcal{T}$. Introduce fresh concept names A_a, B_a, D_a for every $a \in N$. We obtain $\mathcal{T}'$ from $\mathcal{T}$ by replacing every $a \in N$ with A_a and adding the CI $A_a \sqcap \neg B_a \sqsubseteq D_a$, $\mathcal{A}'$ from $\mathcal{A}$ by adding $A_a(a)$

and $B_a(a)$ for every $a \in N$, and q' from q by adding the disjunct $\exists y\, D_a(y)$ for every $a \in N$.[1] Set $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$. The circumscription pattern CP' is obtained from CP by minimizing the concept names B_a with higher preference than any other concept name (and with no preferences between them). We show in the appendix that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$ iff $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}') \models q'(\bar{a})$ for all $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. □

We are thus left with $\mathcal{ALCHI}$ KBs. We generally assume that TBoxes are in *normal form*, meaning that every concept inclusion in $\mathcal{T}$ has one of the following shapes:

$$\top \sqsubseteq A \qquad A \sqsubseteq \exists r.B \qquad \exists r.B \sqsubseteq A$$
$$A_1 \sqcap A_2 \sqsubseteq A \qquad A \sqsubseteq \neg B \qquad \neg B \sqsubseteq A$$

where A, A_1, A_2, B range over $\mathsf{N_C}$ and r ranges over roles. The set of concept names in $\mathcal{T}$ is denoted $\mathsf{N_C}(\mathcal{T})$.

Lemma 1. *Every $\mathcal{ALCHI}$ TBox $\mathcal{T}$ can be transformed in linear time into an $\mathcal{ALCHI}$ TBox $\mathcal{T}'$ in normal form such that for all cKBs $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$, UCQs $q(\bar{x})$ that do not use symbols from $\mathsf{sig}(\mathcal{T}') \setminus \mathsf{sig}(\mathcal{T})$, and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$: $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A}) \models q(\bar{a})$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}) \models q(\bar{a})$.*

Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. A *type for $\mathcal{T}$* is a set of concept names $t \subseteq \mathsf{N_C}(\mathcal{T})$ such that for some model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and $d \in \Delta^{\mathcal{I}}$,

$$t = \mathsf{tp}_{\mathcal{I}}(d) \coloneqq \{A \in \mathsf{N_C}(\mathcal{T}) \mid d \in A^{\mathcal{I}}\}.$$

The set of all types for $\mathcal{T}$ is denoted $\mathsf{TP}(\mathcal{T})$. For an interpretation $\mathcal{I}$ and $\Delta \subseteq \Delta^{\mathcal{I}}$, we set $\mathsf{tp}_{\mathcal{I}}(\Delta) = \{\mathsf{tp}_{\mathcal{I}}(d) \mid d \in \Delta\}$ and write $\mathsf{TP}(\mathcal{I})$ for $\mathsf{tp}_{\mathcal{I}}(\Delta^{\mathcal{I}})$. For a role r, we further write $t \rightsquigarrow_r t'$ if for all $A, B \in \mathsf{N_C}$:

- $B \in t'$ and $\mathcal{T} \models \exists r.B \sqsubseteq A$ implies $A \in t$ and
- $B \in t$ and $\mathcal{T} \models \exists r^-.B \sqsubseteq A$ implies $A \in t'$.

We next show how to identify a 'core' part of an interpretation $\mathcal{I}$. These cores will play an important role in dealing with circumscription in our upper bound proofs.

Definition 1. *Let $\mathcal{I}$ be an interpretation. We use $\mathsf{TP}_{\mathsf{core}}(\mathcal{I})$ to denote the set of all types $t \in \mathsf{TP}(\mathcal{I})$ such that*

$$|\{d \in \Delta^{\mathcal{I}} \setminus \mathsf{ind}(\mathcal{A}) \mid \mathsf{tp}_{\mathcal{I}}(d) = t\}| < |\mathsf{TP}(\mathcal{T})|$$

and set $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I}) = \mathsf{TP}(\mathcal{I}) \setminus \mathsf{TP}_{\mathsf{core}}(\mathcal{I})$ and

$$\Delta^{\mathcal{I}}_{\mathsf{core}} = \{d \in \Delta^{\mathcal{I}} \mid \mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}_{\mathsf{core}}(\mathcal{I})\}.$$

So the core consists of all elements whose types are realized not too often, except possibly in the ABox. A good way of thinking about cores is that if a model $\mathcal{I}$ of $\mathcal{K}$ is minimal w.r.t. $<_{\mathsf{CP}}$, then all instances of minimized concept names are in the core. This is not strictly true, however, since we may have $A \sqsubseteq B$ where A is $\top$ or fixed, and B is minimized.

The following crucial lemma provides a sufficient condition for a model $\mathcal{J}$ of $\mathcal{K}$ to be minimal w.r.t. $<_{\mathsf{CP}}$. This is relative to a model $\mathcal{I}$ of $\mathcal{K}$ that is known to be minimal w.r.t. $<_{\mathsf{CP}}$ and based on satisfying the same types as $\mathcal{I}$, both inside and outside the core.

[1]Strictly speaking, we need to adjust $\exists y\, D_a(y)$ so that it has the same answer variables as the other CQs in q. This is easy by adding to $\mathcal{T}'$ a CI $\top \sqsubseteq T$ for a fresh concept name T and extending $\exists y\, D_a(y)$ with atom $T(x)$ for every answer variable x.

Lemma 2 (Core Lemma). *Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and let $\mathcal{J}$ be a model of $\mathcal{K}$ with $\Delta^{\mathcal{I}}_{\mathsf{core}} \subseteq \Delta^{\mathcal{J}}$. If*

1. *$\mathsf{tp}_{\mathcal{I}}(d) = \mathsf{tp}_{\mathcal{J}}(d)$ for all $d \in \Delta^{\mathcal{I}}_{\mathsf{core}}$ and*
2. *$\mathsf{tp}_{\mathcal{J}}(\Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\mathsf{core}}) = \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$,*

then $\mathcal{J}$ is a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$.

We give a sketch of the proof of (the contrapositive of) Lemma 2. Assume that $\mathcal{J}$ is not a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Then there must be a model $\mathcal{J}'$ of $\mathcal{K}$ with $\mathcal{J}' <_{\mathsf{CP}} \mathcal{J}$ and to obtain a contradiction it suffices to construct from $\mathcal{J}'$ a model $\mathcal{I}'$ of $\mathcal{K}$ with $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$. Note that $\mathcal{I}$ and $\mathcal{J}$ may have domains of different size. The elements of $\Delta^{\mathcal{I}}_{\mathsf{core}}$ receive the same type in $\mathcal{I}'$ as in $\mathcal{J}'$. Each non-core type t is realized in $\mathcal{I}$ very often, meaning at least $|\mathsf{TP}(\mathcal{T})|$ many times. We consider the set T of types in $\mathcal{J}'$ of those elements that had type t in $\mathcal{J}$. Since t is realized in $\mathcal{I}$ very often, we have enough room to realize in $\mathcal{I}'$ exactly the types from T among those elements that had type t in $\mathcal{I}$.

We next use the core lemma to show that if $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$ for any CQ q and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$, then this is witnessed by a countermodel $\mathcal{I}$ that has a regular shape. Here and in what follows, a *countermodel* is a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{I} \not\models q(\bar{a})$. With regular shape, we mean that there is a 'base part' that contains the ABox, the core of $\mathcal{I}$, as well as some additional elements; all other parts of $\mathcal{I}$ are tree-shaped with their root in the base part, and potentially with edges that go back to the core (but not to other parts of the base!). We next make this precise.

Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Set $\Omega = \{rA \mid B \sqsubseteq \exists r.A \in \mathcal{T}\}$ and fix a function f that chooses, for every $d \in \Delta^{\mathcal{I}}$ and $rA \in \Omega$ with $d \in (\exists r.A)^{\mathcal{I}}$, an element $f(d, rA) = e \in A^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$. Further choose, for every $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$, a representative $e_t \in \Delta^{\mathcal{I}}$ with $\mathsf{tp}_{\mathcal{I}}(e_t) = t$. We inductively define the set $\mathcal{P}$ of *paths through $\mathcal{I}$* along with a mapping h assigning to each $p \in \mathcal{P}$ an element of $\Delta^{\mathcal{I}}$:

- each element d of the set
$$\Delta^{\mathcal{I}}_{\mathsf{base}} := \mathsf{ind}(\mathcal{A}) \cup \Delta^{\mathcal{I}}_{\mathsf{core}} \cup \{e_t \mid t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})\}$$
is a path in $\mathcal{P}$ and $h(d) = d$;
- if $p \in \mathcal{P}$ with $h(p) = d$ and $rA \in \Omega$ with $f(d, rA)$ defined and not from $\Delta^{\mathcal{I}}_{\mathsf{core}}$, then $p' = prA$ is a path in $\mathcal{P}$ and $h(p') = f(d, rA)$.

For every role r, define

$$\begin{aligned} R_r = \{&(a,b) \mid a, b \in \mathsf{ind}(\mathcal{A}), \mathcal{K} \models r(a,b)\} \cup \\ \{&(d,e) \mid d, e \in \Delta^{\mathcal{I}}_{\mathsf{core}}, (d,e) \in r^{\mathcal{I}}\} \cup \\ \{&(p,p') \mid p' = prA \in \mathcal{P}\} \cup \\ \{&(p,e) \mid e = f(h(p), rA) \in \Delta^{\mathcal{I}}_{\mathsf{core}}\}. \end{aligned}$$

Now the *unraveling* of $\mathcal{I}$ is defined by setting

$$\Delta^{\mathcal{I}'} = \mathcal{P} \qquad A^{\mathcal{I}'} = \{p \in \mathcal{P} \mid h(p) \in A^{\mathcal{I}}\}$$
$$r^{\mathcal{I}'} = \bigcup_{\mathcal{T} \models s \sqsubseteq r} (R_s \cup \{(e,d) \mid (d,e) \in R_{s^-}\})$$

for all concept names A and role names r. It is easy to verify that h is a homomorphism from $\mathcal{I}'$ to $\mathcal{I}$. This version of unraveling is inspired by constructions from (Manière 2022) where the resulting model is called an *interlacing*.

Lemma 3. *Let $q(\bar{x})$ be a UCQ and $\bar{a} \in \mathsf{ind}(\mathcal{A})$. If $\mathcal{I}$ is a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$, then so is its unraveling $\mathcal{I}'$.*

For some of our upper bound proofs, it will be important that the reference model $\mathcal{I}$ in the core lemma is finite and sufficiently small. The following lemma shows that any reference model $\mathcal{I}$ can be turned into a reference model $\mathcal{J}$ with these properties. We construct the desired $\mathcal{J}$ by starting from $\mathcal{I}|_{\mathcal{A} \cup \Delta^{\mathcal{I}}_{\mathsf{core}}}$ and adding exactly $m = |\mathsf{TP}(\mathcal{T})|$ instances of each type from $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$.

Lemma 4. *Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. There exists a model $\mathcal{J}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ such that $|\Delta^{\mathcal{J}}| \leq |\mathcal{A}| + 2^{2|\mathcal{T}|}$, $\mathcal{I}|_{\Delta^{\mathcal{I}}_{\mathsf{core}}} = \mathcal{J}|_{\Delta^{\mathcal{J}}_{\mathsf{core}}}$, and $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I}) = \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{J})$.*

3.2 Combined complexity

We show that in any DL between $\mathcal{EL}$ and $\mathcal{ALCHIO}$, the evaluation of CQs and of UCQs on cKBs is 2EXP-complete w.r.t. combined complexity, starting with the upper bound.

Theorem 1. *UCQ evaluation on circumscribed $\mathcal{ALCHIO}$ KBs is in* 2EXP *w.r.t. combined complexity.*

By Proposition 1, it suffices to consider $\mathcal{ALCHI}$. Assume that we are given as an input an $\mathcal{ALCHI}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, a UCQ $q(\bar{x})$, and a tuple $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. We have to decide whether or not there is a countermodel $\mathcal{I}$ against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$.

Fix a set Δ of size $|\mathsf{ind}(\mathcal{A})| + 2^{2|\mathcal{T}|+1}$ s.t. $\mathsf{ind}(\mathcal{A}) \cup E \subseteq \Delta$, where $E = \{e_t \mid t \in \mathsf{TP}(\mathcal{T})\}$. Note that we may assume the base domain $\Delta^{\mathcal{I}}_{\mathsf{base}}$ of unraveled interpretations to be a subset of Δ. In an outer loop, our algorithm iterates over all pairs $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$ such that the following conditions are satisfied:

- $\mathcal{I}_{\mathsf{base}}$ is a model of $\mathcal{A}$ with $\Delta^{\mathcal{I}_{\mathsf{base}}} \subseteq \Delta$ and $\mathcal{I}_{\mathsf{base}} \not\models q(\bar{a})$; we set $\Delta_{\mathsf{core}} = \Delta^{\mathcal{I}_{\mathsf{base}}} \setminus (\mathsf{ind}(\mathcal{A}) \cup E)$;
- $\mathsf{tp}_{\mathcal{I}_{\mathsf{base}}}(\Delta^{\mathcal{I}_{\mathsf{base}}} \setminus \Delta_{\mathsf{core}}) = T_{\overline{\mathsf{core}}}$;
- $\mathsf{tp}_{\mathcal{I}_{\mathsf{base}}}(\Delta_{\mathsf{core}}) \cap T_{\overline{\mathsf{core}}} = \emptyset$.

For each pair $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$, we then check whether the following additional conditions are satisfied:

1. $\mathcal{I}_{\mathsf{base}}$ can be extended to a model $\mathcal{I}$ of $\mathcal{T}$ such that
   (a) $\mathcal{I}|_{\Delta^{\mathcal{I}_{\mathsf{base}}}} = \mathcal{I}_{\mathsf{base}}$,
   (b) $\mathsf{tp}_{\mathcal{I}}(\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}) \subseteq T_{\overline{\mathsf{core}}}$, and
   (c) $\mathcal{I} \not\models q(\bar{a})$;
2. there exists a model $\mathcal{J}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ such that $\mathcal{J}|_{\Delta^{\mathcal{J}}_{\mathsf{core}}} = \mathcal{I}_{\mathsf{base}}|_{\Delta_{\mathsf{core}}}$ and $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{J}) = T_{\overline{\mathsf{core}}}$.

We return 'yes' if the check fails for all pairs, and 'no' otherwise.

If the checks succeed, then the model $\mathcal{I}$ of $\mathcal{K}$ from Point 1 is a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$. In particular, we may apply Lemma 2, using the model $\mathcal{J}$ from Point 2 as the reference model, to show that $\mathcal{I}$ is minimal w.r.t. $<_{\mathsf{CP}}$. Conversely, any countermodel $\mathcal{I}_0$ against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$ can be unraveled into a countermodel $\mathcal{I}$ from which we can read off a pair $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$ in the obvious way, and then $\mathcal{I}$ witnesses Point 1 and choosing $\mathcal{J} = \mathcal{I}$ witnesses Point 2.

We next detail how Points 1 and 2 can be checked in 2EXP. Point 2 is simple: by Lemma 4, it suffices to consider models $\mathcal{J}$ of size at most $|\mathcal{A}| + 2^{2|\mathcal{T}|}$ and thus we can simply iterate over all candidate interpretations $\mathcal{J}$ up to this size, then check whether $\mathcal{J}$ is a model of $\mathcal{K}$ with $\mathcal{J}|_{\Delta^{\mathcal{J}}_{\mathsf{core}}} = \mathcal{I}_{\mathsf{base}}|_{\Delta_{\mathsf{core}}}$ and $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{J}) = T_{\overline{\mathsf{core}}}$, and then iterate over all models $\mathcal{J}'$ of $\mathcal{K}$ with $\Delta^{\mathcal{J}'} = \Delta^{\mathcal{J}}$ to check in a brute-force way that $\mathcal{J}$ is minimal w.r.t. $<_{\mathsf{CP}}$.

For Point 1, we use a mosaic approach inspired by (Manière 2022), that is, we assemble the interpretation $\mathcal{I}$ from Point 1 by combining small pieces called *mosaics*. Fix a pair $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$. A *match triple* for an interpretation $\mathcal{J}$ takes the form $(p, \widehat{p}, h)$ such that p is a CQ in q, $\widehat{p} \subseteq p$, and h is a partial map from $\mathsf{var}(\widehat{p})$ to $\Delta^{\mathcal{J}}$ that is a homomorphism from $\widehat{p}|_{\mathsf{dom}(h)}$ to $\mathcal{J}$. Intuitively, $\mathcal{J}$ is one of the pieces from which we assemble $\mathcal{I}$ and the triple $(p, \widehat{p}, h)$ expresses that a homomorphism from $\widehat{p}$ to $\mathcal{I}$ exists, with the variables in $\mathsf{dom}(h)$ being mapped to the current piece $\mathcal{J}$. A match triple is called *complete* if $\widehat{p} = p$ and *incomplete otherwise*. A *specification* for $\mathcal{J}$ is a set S of match triples for $\mathcal{J}$ and we call S *saturated* if the following conditions are satisfied:

- if p is a CQ in q, $\widehat{p} \subseteq p$, and h is a homomorphism from $\widehat{p}$ to $\mathcal{J}$, then $(p, \widehat{p}, h) \in S$;
- if $(p, \widehat{p}, h), (p, \widehat{p}', h') \in S$ and $h(x) = h'(x)$ is defined for all $x \in \mathsf{var}(\widehat{p}) \cap \mathsf{var}(\widehat{p}')$, then $(p, \widehat{p} \cup \widehat{p}', h \cup h') \in S$.

Definition 2. *A* mosaic *for $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$ is a tuple $M = (\mathcal{J}, S)$ where*

- *$\mathcal{J}$ is an interpretation such that*
  1. *$\Delta^{\mathcal{I}_{\mathsf{base}}} \subseteq \Delta^{\mathcal{J}} \subseteq \Delta^{\mathcal{I}_{\mathsf{base}}} \uplus \{e^*_1, e^*_2\}$;*
  2. *$\mathcal{J}|_{\Delta^{\mathcal{I}_{\mathsf{base}}}} = \mathcal{I}_{\mathsf{base}}$;*
  3. *$\mathsf{tp}_{\mathcal{J}}(e^*_i) \in T_{\overline{\mathsf{core}}}$ if $e^*_i \in \Delta^{\mathcal{J}}$, for $i \in \{1, 2\}$;*
  4. *$\mathcal{J}$ satisfies all $\exists r.A \sqsubseteq B \in \mathcal{T}$ and all $r \sqsubseteq s \in \mathcal{T}$.*
- *S is a saturated specification for $\mathcal{J}$ that contains only incomplete match triples.*

We use $\mathcal{J}_M$ to refer to $\mathcal{J}$ and S_M to refer to S.

Let $\mathcal{M}$ be a set of mosaics for $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$. We say that $M \in \mathcal{M}$ is *good in* $\mathcal{M}$ if for every $e \in \Delta^{\mathcal{J}_M}$ and every $A \sqsubseteq \exists r.B \in \mathcal{T}$ with $e \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_M}$, we find a mosaic $M' \in \mathcal{M}$ such that the following conditions are satisfied:

1. $\mathsf{tp}_{\mathcal{J}_M}(e) = \mathsf{tp}_{\mathcal{J}_{M'}}(e)$;
2. $e \in (\exists r.B)^{\mathcal{J}_{M'}}$;
3. if $(p, \widehat{p}, h') \in S_{M'}$, then $(p, \widehat{p}, h) \in S_M$ where h is the restriction of h' to range $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e\}$.

If M is not good in $\mathcal{M}$, then it is *bad*. We now start with the set of all mosaics for $(\mathcal{I}_{\mathsf{base}}, T_{\overline{\mathsf{core}}})$ and then repeatedly and exhaustively eliminate bad mosaics.

Lemma 5. *$\mathcal{I}_{\mathsf{base}}$ can be extended to a model $\mathcal{I}$ of $\mathcal{T}$ that satisfies Conditions (a) to (c) iff at least one mosaic survives the elimination process.*

We provide a matching lower bound for Theorem 1. It is rather strong in that it already applies to CQs, to $\mathcal{EL}$ KBs, and uses a single minimized concept name (and consequently no preferences) and no fixed concept names. It is proved by a subtle reduction from CQ evaluation on $\mathcal{EL}$ KBs with closed concept names, that is, with KBs $(\mathcal{T}, \mathcal{A})$ enriched with a set Σ of close concept names A that in all models $\mathcal{I}$, have to be interpreted as $A^{\mathcal{I}} = \{a \mid A(a) \in \mathcal{A}\}$. This problem was proved to be 2EXP-hard in (Ngo, Ortiz, and Simkus 2016). The reduction also sheds some light on the connection between circumscription and closing concept names. In a sense, circumscription generalizes the latter, although the reduction is not entirely straightforward.

Theorem 2. *CQ evaluation on circumscribed $\mathcal{EL}$ KBs is* 2EXP*-hard w.r.t. combined complexity. This holds even with a single minimized concept name and no fixed concept names.*

3.3 Data complexity

We show that in any DL between $\mathcal{EL}$ and $\mathcal{ALCHIO}$, the evaluation of CQs and of UCQs on cKBs is Π_2^{P}-complete w.r.t. data complexity. We start with the upper bound.

Theorem 3. *UCQ evaluation on circumscribed $\mathcal{ALCHIO}$ KBs is in Π_2^P w.r.t. data complexity.*

We may again limit our attention to $\mathcal{ALCHI}$. To prove Theorem 3, the main step is to show that if a countermodel exists, then there is one of polynomial size with the size of the TBox and query assumed to be a constant. Note that this is not a consequence of Lemma 4 since there is no guarantee that, if the model $\mathcal{I}$ from that lemma is a countermodel, then so is $\mathcal{J}$. Once the size bound is in place, we obtain the Π_2^{P} upper bound by a straightforward guess-and-check procedure.

Lemma 6. *Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be an $\mathcal{ALCHI}$ cKB, $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $q(\bar{x})$ a UCQ, and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. If $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$, then there exists a countermodel $\mathcal{I}$ with $|\Delta^{\mathcal{I}}| \leq (|\mathcal{A}|+2^{|\mathcal{T}|})^{|\mathcal{T}|^{|q|}}$.*

To prove Lemma 6, we start from the unraveling $\mathcal{I}'$ of a countermodel $\mathcal{I}$ and apply a quotient construction. Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. We may view an interpretation $\mathcal{I}$ as an undirected graph

$$G_{\mathcal{I}} = (\Delta^{\mathcal{I}}, \{\{d, e\} \mid (d, e) \in r^{\mathcal{I}} \text{ for some } r \in \mathsf{N_R}\}).$$

For $n \geq 0$ and $\Delta \subseteq \Delta^{\mathcal{I}}$, we use $\mathcal{N}_n^{\mathcal{I},\Delta}(d)$ to denote the *n-neighborhood of d in $\mathcal{I}$ up to Δ*, that is, the set of all elements $e \in \Delta^{\mathcal{I}}$ such that $G_{\mathcal{I}}$ contains a path $d_0, \ldots, d_n$ with $d_0 = d$, $d_0, \ldots, d_{n-1} \notin \Delta$, and $d_n = e$.

Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and $\mathcal{I}'$ the unraveling of $\mathcal{I}$. We consider neighborhoods in $\mathcal{I}'$ and use $\Delta := \Delta^{\mathcal{I}}_{\mathsf{base}}$. Recall that the elements of $\Delta^{\mathcal{I}'}$ are paths through $\mathcal{I}$, that is, sequences $p = d_0 r_1 A_1 \cdots r_n A_n$ with $d_0 \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $r_i A_i \in \Omega$; the *length* of p, denoted by $|p|$, is n. By definition of $\mathcal{I}'$ and of neighborhoods, for every $d \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $n \geq 0$, there is a unique path $p_{d,n} \in \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d)$ that is a prefix of all paths in $\mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d)$, that is, all $e \in \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ take the form $p_{d,n} r_1 A_1 \cdots r_n A_n$.

We now define an equivalence relation $\sim_n$ on $\Delta^{\mathcal{I}'}$, for every $n \geq 0$, by setting $d_1 \sim_n d_2$ for $d_1, d_2 \in \Delta^{\mathcal{I}'}$ if $d_1 = d_2 \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ or $d_1, d_2 \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$ and the following conditions are satisfied:

1. $d_1 = p_{d_1,n} w$ and $d_2 = p_{d_2,n} w$ for some $w \in \Omega^*$;
2. for every $w = r_1 A_1 \cdots r_n A_n \in \Omega^*$, either

$$p_{d_1,n} w \notin \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d_1) \text{ and } p_{d_2,n} w \notin \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d_2)$$

   or the following conditions are satisfied:

   (a) $p_{d_1,n} w \in \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d_1)$ and $p_{d_2,n} w \in \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(d_2)$;

   (b) $\mathsf{tp}_{\mathcal{I}'}(p_{d_1,n} w) = \mathsf{tp}_{\mathcal{I}'}(p_{d_2,n} w)$;

   (c) $(p_{d_1,n} w, e) \in r^{\mathcal{I}'}$ iff $(p_{d_2,n} w, e) \in r^{\mathcal{I}'}$ for all roles r and $e \in \Delta^{\mathcal{I}}_{\mathsf{base}}$.
3. $|d_1| \equiv |d_2| \mod 2|q| + 3$.

We use $\overline{d}$ to denote the equivalence class w.r.t. $\sim_{|q|+1}$ of an element $d \in \Delta^{\mathcal{I}'}$. The *quotient* $\mathcal{I}'/{\sim_{|q|+1}}$ of $\mathcal{I}'$ is the interpretation whose domain is the set of all equivalence classes of $\sim_{|q|+1}$ and where

$$\begin{aligned} A^{\mathcal{I}'/\sim_{|q|+1}} &= \{\overline{d} \mid d \in A^{\mathcal{I}'}\} \\ r^{\mathcal{I}'/\sim_{|q|+1}} &= \{(\overline{d}, \overline{e}) \mid (d, e) \in r^{\mathcal{I}'}\} \end{aligned}$$

for all concept names A and role names r. It can be verified that $|\Delta^{\mathcal{I}'/\sim_{|q|+1}}| \leq (|\mathcal{A}| + 2^{|\mathcal{T}|})^{|\mathcal{T}|^{|q|}}$, as desired.

Lemma 7. *Let $q(\bar{x})$ be a UCQ and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. If $\mathcal{I}$ is a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$, then so is $\mathcal{I}'/{\sim_{|q|+1}}$.*

To prove Lemma 7, it is routine work to show that $\mathcal{I}'/{\sim_{|q|+1}}$ is a model of $\mathcal{K}$. Minimality w.r.t. $<_{\mathsf{CP}}$ is proved using Lemma 2. The most subtle part of the proof is showing that $\mathcal{I}'/{\sim_{|q|+1}} \not\models q(\bar{a})$. This is done by using an assumed homomorphism h from a CQ p in q to $\mathcal{I}'/{\sim_{|q|+1}}$ with $h(\bar{x}) = \bar{a}$ to construct a homomorphism from p to $\mathcal{I}$ with the same property.

This finishes the proof of Lemma 6 and Theorem 3.

We next provide a matching lower bound for Theorem 3. It already applies to AQs and to $\mathcal{EL}$ KB and when there is a single minimized concept name and no fixed concept name. It is proved by a subtle reduction from the validity of $\forall\exists$3SAT sentences. Several non-obvious technical tricks are needed to indeed make the reduction work with a single minimized concept name. Our result improves upon a known coNP lower bound from (Bonatti 2021).

Theorem 4. *AQ evaluation on circumscribed $\mathcal{EL}$ KBs is Π_2^P-hard. This holds even with a single minimized concept name and no fixed concept names.*

4 DL-Lite

We next consider the DL-Lite family of DLs. Without circumscription, these enjoy low complexity of query evaluation, typically NP-complete in combined complexity and in AC^0 in data complexity (depending on the dialect). With circumscription, the complexity tends to be still very high, though in some relevant cases it is lower than in $\mathcal{ALCHIO}$.

4.1 Combined complexity

Our first result shows that, when role inclusions are admitted, nothing is gained from transitioning from $\mathcal{ALCHIO}$ to DL-Lite. The proof is a variation of that of Theorem 2, but technically much simpler.

Theorem 5. *UCQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{core}}$ KBs is* 2EXP*-hard w.r.t. combined complexity. This holds even with a single minimized concept name, no fixed concept names and no disjointness constraints.*

We now move to DL-Lite$_{\mathsf{bool}}$ as a very expressive DL-Lite dialect without role inclusions and observe that the combined complexity decreases.

Theorem 6. *UCQ evaluation on circumscribed DL-Lite$_{\mathsf{bool}}$ KBs is in* coNEXP *w.r.t. combined complexity.*

To prove Theorem 6, we first observe that we can refine the unraveling of countermodels $\mathcal{I}$ from Section 3.1 such that each element outside of the base part $\Delta^{\mathcal{I}}_{\mathsf{base}}$ has at most one successor per role. This property allows us to simplify the notion of a neighborhood in the quotient construction in Section 3.3. This, in turn, yields the following result.

Lemma 8. *Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a DL-Lite$_{\mathsf{bool}}$ cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $q(\bar{x})$ a UCQ, and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. If $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$, then there exists a countermodel $\mathcal{I}$ with $|\Delta^{\mathcal{I}}| \leq (|\mathcal{A}| + 2^{|\mathcal{T}|})^{|q|^2(|\mathcal{T}|+1)}$.*

Lemma 8 yields a coNEXP$^{\text{NP}}$ upper bound by a straightforward guess-and-check procedure: guess a model $\mathcal{I}$ of $\mathcal{K}$ with $\mathcal{I} \not\models q(\bar{a})$ and $|\Delta^{\mathcal{I}}|$ bounded as in Lemma 8, and then use the oracle to check that $\mathcal{I}$ is minimal w.r.t. $<_{\mathsf{CP}}$. More precisely, the oracle decides, given a cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, whether $\mathcal{I}$ is non-minimal w.r.t. $<_{\mathsf{CP}}$ by guessing a model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$; c.f. (Bonatti, Lutz, and Wolter 2009).

To arrive at a coNEXP upper bound as desired, we need to replace the oracle with a more efficient method to check whether a given model $\mathcal{I}$ of $\mathcal{K}$ is minimal w.r.t. $<_{\mathsf{CP}}$. The crucial observation is that instead of guessing a model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$, it suffices to consider certain interpretations $\mathcal{I}'$ of polynomial size, which essentially are sub-interpretations of $\mathcal{I}$, and guess models $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}'$ of $\mathcal{K}$. Intuitively, we replace an expensive 'global' test by exponentially many inexpensive 'local' tests. This even holds in the presence of role inclusions, that is, in DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$, which shall be useful in Section 5.

Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and $\mathcal{I}$ a model of $\mathcal{K}$. For each role r used in $\mathcal{T}$ such that $r^{\mathcal{I}} \neq \emptyset$, we choose a witness $w_r \in (\exists r^-)^{\mathcal{I}}$. Every $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ gives rise to an interpretation $\mathcal{I}_{\mathcal{P}}$ as follows:

$$
\begin{aligned}
\Delta^{\mathcal{I}_{\mathcal{P}}} &= \mathcal{P} \cup \mathsf{ind}(\mathcal{A}) \cup \{w_r \mid r \text{ used in } \mathcal{T}, r^{\mathcal{I}} \neq \emptyset\} \\
A^{\mathcal{I}_{\mathcal{P}}} &= A^{\mathcal{I}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}} \\
r^{\mathcal{I}_{\mathcal{P}}} &= r^{\mathcal{I}} \cap (\mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A})) \\
&\quad \cup \{(e, w_s) \mid e \in (\exists s)^{\mathcal{I}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}}, \mathcal{T} \models s \sqsubseteq r\} \\
&\quad \cup \{(w_s, e) \mid e \in (\exists s)^{\mathcal{I}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}}, \mathcal{T} \models s \sqsubseteq r^-\}.
\end{aligned}
$$

The interpretations $\mathcal{I}_{\mathcal{P}}$ are the sub-interpretations of $\mathcal{I}$ mentioned above. Note, however, that they are not truly sub-interpretations on $\mathcal{I}$ since some r-edges have been rerouted to w_r. Also note that $|\mathcal{I}_{\mathcal{P}}| \leq |\mathcal{A}| + |\mathcal{T}| + |\mathcal{P}|$. It is not difficult to show that $\mathcal{I}_{\mathcal{P}}$ is a model of $\mathcal{K}$, for all $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$. The next lemma characterizes the (non)-minimality of $\mathcal{I}$ in terms of the (non)-minimality of the interpretations $\mathcal{I}_{\mathcal{P}}$.

Lemma 9. *The following are equivalent:*

1. *There exists a model $\mathcal{I}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$;*
2. *There exist $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$ and a family $(\mathcal{J}_e)_{e \in \Delta^{\mathcal{I}}}$ of models of $\mathcal{K}$ such that $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P} \cup \{e\}}$ and $\mathcal{J}_e|_{\Delta^{\mathcal{I}_{\mathcal{P}}}} = \mathcal{J}_{e'}|_{\Delta^{\mathcal{I}_{\mathcal{P}}}}$ for all $e, e' \in \Delta^{\mathcal{I}}$.*

It should now be clear that we have established Theorem 5. After guessing the model $\mathcal{I}$ of $\mathcal{K}$ with $\mathcal{I} \not\models q(\bar{a})$, we check the complement of Point 2 of Lemma 9 in a brute-force way. More precisely, we use several nested loops: first iterate over all $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$, then over all interpretations $\mathcal{J}_0$ with $\Delta^{\mathcal{J}_0} = \Delta^{\mathcal{I}_{\mathcal{P}}}$, then over all $e \in \Delta^{\mathcal{I}}$, and finally over all models $\mathcal{J}_e$ of $\mathcal{K}$ with $\mathcal{J}_e|_{\Delta^{\mathcal{I}_{\mathcal{P}}}} = \mathcal{J}_0|_{\Delta^{\mathcal{I}_{\mathcal{P}}}}$, and test whether $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P} \cup \{e\}}$. We accept if for every $\mathcal{P}$ and $\mathcal{J}_0$, there is an e such that for all $\mathcal{J}_e$ the final check fails. Overall, we obtain a coNEXP algorithm.

We provide a matching lower bound that holds even for DL-Lite$_{\mathsf{core}}$ cKBs. It is proved by reduction from the complement of Succinct3COL, which is known to be NEXP-complete (Papadimitriou and Yannakakis 1986).

Theorem 7. *UCQ answering on circumscribed DL-Lite$_{\mathsf{core}}$ KBs is* coNEXP*-hard w.r.t. combined complexity. This holds even with a single minimized concept name, no fixed concept names and no disjointness constraints.*

When DL-Lite$_{\mathsf{core}}$ is replaced with DL-Lite$_{\mathsf{bool}}$, then it suffices to use AQs in place of UCQs, see Theorem 13 below.

4.2 Data complexity

We now consider data complexity, where the landscape is less diverse. Indeed, we obtain coNP-completeteness for all DLs between DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ and DL-Lite$_{\mathsf{core}}$, and both for UCQs and CQs.

Theorem 8. *UCQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ KBs is in* coNP *w.r.t. data complexity.*

To prove Theorem 8, we again guess a model $\mathcal{I}$ of $\mathcal{K}$ with $\mathcal{I} \not\models q(\bar{a})$ and $|\Delta^{\mathcal{I}}|$ bounded as in Theorem 8, and then verify that $\mathcal{I}$ is minimal w.r.t. $<_{\mathsf{CP}}$. For the latter, we introduce a variation of Lemma 9. The original version of Lemma 9 is not helpful because its Point 2 involves deciding whether, given an interpretation $\mathcal{I}_{\mathcal{P} \cup \{e\}}$, there is a model $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P} \cup \{e\}}$ of $\mathcal{K}$, and there is no reason to believe that this can be done in polynomial time in data complexity. We actually conjecture this problem to be coNP-complete. The problem stems from the fact that $\mathcal{I}_{\mathcal{P} \cup \{e\}}$ contains all individuals from $\mathcal{A}$ and so, intuitively, we have to define the sub-interpretations $\mathcal{I}_{\mathcal{P}}$ so that their size is independent of that of the ABox.

Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and $\mathcal{I}$ a model of $\mathcal{K}$. We assume $\mathcal{T}$ to be in normal form.

For an element $e \in \Delta^\mathcal{I}$, we define its *ABox type* to be

$$\mathsf{tp}_\mathcal{A}(e) = \{A \mid A \in \mathsf{N_C}, \mathcal{K} \models A(e)\}.$$

Note that $\mathsf{tp}_\mathcal{A}(e)$ actually needs not be a proper type as defined in Section 3 due to the presence of disjunction in DL-Lite$^\mathcal{H}_\mathsf{bool}$. If $e \notin \mathsf{ind}(\mathcal{A})$, which is permitted, then $\mathsf{tp}_\mathcal{A}(e)$ is simply the set of all concepts names A with $\mathcal{K} \models \top \sqsubseteq A$.

For each pair (t_1, t_2) such that $\mathsf{tp}_\mathcal{A}(e) = t_1$ and $\mathsf{tp}_\mathcal{I}(e) = t_2$ for some $e \in \Delta^\mathcal{I}$, we choose such an element e_{t_1,t_2}; we use E to denote the set of all elements chosen in this way. For each role r used in $\mathcal{T}$ such that $r^\mathcal{I} \neq \emptyset$, we choose an element $w_r \in (\exists r^-)^\mathcal{I}$. Now define, for every $\mathcal{P} \subseteq \Delta^\mathcal{I}$, the interpretation $\mathcal{I}_\mathcal{P}$ as follows:

$$\Delta^{\mathcal{I}_\mathcal{P}} = \mathcal{P} \cup E \cup \{w_r \mid r \text{ used in } \mathcal{T}, r^\mathcal{I} \neq \emptyset\}$$
$$A^{\mathcal{I}_\mathcal{P}} = A^\mathcal{I} \cap \Delta^{\mathcal{I}_\mathcal{P}}$$
$$r^{\mathcal{I}_\mathcal{P}} = \{(e, w_s) \mid e \in (\exists s)^\mathcal{I} \cap \Delta^{\mathcal{I}_\mathcal{P}}, \mathcal{T} \models s \sqsubseteq r\}$$
$$\cup \{(w_s, e) \mid e \in (\exists s)^\mathcal{I} \cap \Delta^{\mathcal{I}_\mathcal{P}}, \mathcal{T} \models s \sqsubseteq r^-\}$$

Note that $|\mathcal{I}_\mathcal{P}| \leq 4^{|\mathcal{T}|} + |\mathcal{T}| + |\mathcal{P}|$. We also define an ABox

$$\mathcal{A}_\mathcal{P} = \{A(a) \mid a \in \mathsf{ind}(\mathcal{A}) \cap \Delta^{\mathcal{I}_\mathcal{P}}, A \in \mathsf{tp}_\mathcal{A}(a)\}$$

and set $\mathcal{K}_\mathcal{P} = (\mathcal{T}, \mathcal{A}_\mathcal{P})$. The ABoxes $\mathcal{A}_\mathcal{P}$ will act as a decomposition of the ABox $\mathcal{A}$, in a similar way in which the interpretations $\mathcal{I}_\mathcal{P}$ acted as a decomposition of the interpretation $\mathcal{I}$ in the previous section. Note that $\bigcup_\mathcal{P} \mathcal{A}_\mathcal{P}$ does not contain role assertions. This is compensated by the use of $\mathsf{tp}_\mathcal{A}(a)$ in the definition of $\mathcal{A}_\mathcal{P}$ and the fact that, since $\mathcal{T}$ is in normal form, all endpoints of role assertions are 'visible' in the ABox types. It notably contrasts with the $\mathcal{EL}$ case, in which forgetting role assertions could result in axioms $\exists r.B \sqsubseteq A$ to be left unsatisfied (this constraint underlies the proof of Theorem 4).

The following is the announced variation of Lemma 9.

Lemma 10. *The following are equivalent:*

1. $\mathcal{I} \models \mathsf{Circ_{CP}}(\mathcal{K})$;
2. $\mathcal{I}_\mathcal{P} \models \mathsf{Circ_{CP}}(\mathcal{K}_\mathcal{P})$ *for all* $\mathcal{P} \subseteq \Delta^\mathcal{I}$ *with* $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$.

In the "1 $\Rightarrow$ 2" direction we use the witnesses e_{t_1,t_2} to extend a potential model $\mathcal{J}' <_\mathsf{CP} \mathcal{I}_\mathcal{P}$ of $\mathsf{Circ_{CP}}(\mathcal{K}_\mathcal{P})$ to a model $\mathcal{J} <_\mathsf{CP} \mathcal{I}$ of $\mathcal{K}$, obtaining a contradiction. In the "2 $\Rightarrow$ 1" direction, for a potential model $\mathcal{J} <_\mathsf{CP} \mathcal{I}$ of $\mathcal{K}$, we can find a $\mathcal{P}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$ so that we can construct from $\mathcal{J}$ a model $\mathcal{J}' <_\mathsf{CP} \mathcal{I}_\mathcal{P}$ of $\mathsf{Circ_{CP}}(\mathcal{K}_\mathcal{P})$.

To establish Theorem 8, it thus suffices to argue that checking Point 2 of Lemma 10 can be implemented in time polynomial in $|\mathcal{A}|$. We iterate over all (polynomially many) sets $\mathcal{P}$ and check whether $\mathcal{I}_\mathcal{P} \models \mathsf{Circ_{CP}}(\mathcal{K}_\mathcal{P})$ in a brute force way. More precisely, we iterate over all models $\mathcal{J}$ of $\mathcal{K}_\mathcal{P}$ with $\Delta^\mathcal{J} = \Delta^\mathcal{I}$ (of which there are only polynomially many, thanks to the modified definition of $\Delta^{\mathcal{I}_\mathcal{P}}$), and make sure that $\mathcal{J} \not<_\mathsf{CP} \mathcal{I}_\mathcal{P}$ for any such $\mathcal{J}$. Overall, we clearly obtain a coNP algorithm.

The coNP upper bound turns out to be to be tight, even for DL-Lite$_\mathsf{core}$ cKBs and CQs, and with very restricted circumscription patterns. We reduce from 3-colorability.

Theorem 9. *CQ evaluation on circumscribed DL-Lite$_\mathsf{core}$ KBs is* coNP*-hard w.r.t. data complexity. This holds even with a single minimized concept name, no fixed concept names, and no disjointness constraints.*

5 Atomic Queries

We study the evaluation of atomic queries on circumscribed KBs, which is closely related to concept satisfiability w.r.t. such KBs. In fact, the two problems are mutually reducible in polynomial time. Our results from this section, summarized in Table 2, can thus also be viewed as completing the complexity landscape for concept satisfiability, first studied in (Bonatti, Lutz, and Wolter 2006; Bonatti, Faella, and Sauro 2011).

5.1 Between $\mathcal{ALCHIO}$ and $\mathcal{EL}$

Concept satisfiability w.r.t. circumscribed $\mathcal{ALCIO}$ KBs was proved to be coNExp$^\text{NP}$-complete in (Bonatti, Lutz, and Wolter 2009). The proof of the upper bound can easily be extended to cover also role inclusions. We thus obtain:

Theorem 10. (Bonatti, Lutz, and Wolter 2009) *AQ evaluation on circumscribed $\mathcal{ALCHIO}$ KBs is in* coNExp$^\text{NP}$ *w.r.t. combined complexity.*

An alternative way to obtain Theorem 10 for $\mathcal{ALCHI}$ is to use Lemma 6, which yields the existence of single exponentially large countermodels in the special case where the query q is of constant size (here $|q| = 1$), and a straightforward guess-and check procedure as sketched in Section 4.1. This can be lifted to $\mathcal{ALCHIO}$ by a variation of (the proof of) Proposition 1 that tailors to AQs.

We next prove a matching lower bound for $\mathcal{EL}$, improving on an Exp lower bound from (Bonatti, Faella, and Sauro 2011).

Theorem 11. *AQ evaluation on circumscribed $\mathcal{EL}$ KBs is* coNExp$^\text{NP}$*-hard w.r.t. combined complexity. This holds even without fixed concept names and with an empty preference order.*

This is proved by a reduction from AQ evaluation on $\mathcal{ALC}$ cKBs, which is known to be coNExp$^\text{NP}$-hard (Bonatti, Lutz, and Wolter 2009).

Regarding data complexity, it suffices to recall that Theorem 4 applies even to AQs and $\mathcal{EL}$ cKBs.

5.2 DL-Lite

Recall that in Section 4.1, we have proved that UCQ evaluation over DL-Lite$_\mathsf{bool}$ cKBs is in coNExp w.r.t. combined complexity. We actually started with a guess-and-check procedure that only gives a coNExp$^\text{NP}$ upper bound, relying on countermodels of single exponential size as per Lemma 8, and then improve to coNExp using Lemma 9. Here, we use exactly the same algorithm. The only difference in the correctness proof is that Lemma 8 is replaced with Lemma 6 since the former does not support role inclusions while the latter delivers a single exponential upper bound for queries of constant size.

Theorem 12. *AQ evaluation on circumscribed DL-Lite$^\mathcal{H}_\mathsf{bool}$ KBs is in* coNExp *w.r.t. combined complexity.*

We match this upper bound even in absence of role inclusions, which notably demonstrate that evaluating AQs and UCQs over DL-Lite$_\mathsf{bool}$ cKBs is equally difficult.

	$\mathcal{EL}$, $\mathcal{ALCHIO}$	DL-Lite$_{\mathsf{bool}}$, DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$	DL-Lite$_{\mathsf{core}}$, DL-Lite$^{\mathcal{H}}_{\mathsf{horn}}$
Combined complexity	coNExp$^{\text{NP}}$-c.$^{(\dagger)}$ (Thm. 10, 11)	coNExp-c.(Thm. 12, 13)	Π^{p}_2-c.$^{(\ddagger)}$ (Thm. 14)
Data complexity	Π^{p}_2-c.(Thm. 3, 4)	in PTime (Thm. 15)	in PTime (Thm. 15)

Table 2: Complexity of AQ evaluation on circumscribed KBs. $^{(\dagger)}$: completeness already known for $\mathcal{ALC}(\mathcal{IO})$. $^{(\ddagger)}$: hardness already known.

Theorem 13. *AQ evaluation on circumscribed DL-Lite$_{\mathsf{bool}}$ KBs is* coNExp*-hard w.r.t. combined complexity.*

The proof is by reduction from the complement of the NExp-complete Succinct-3COL problem and relies on fixed concept names.

The situation is more favorable if we restrict our attention to DL-Lite$^{\mathcal{H}}_{\mathsf{horn}}$ cKBs, which is still a very expressive dialect of the DL-Lite family. For DL-Lite$^{\mathcal{H}}_{\mathsf{horn}}$, we prove that if there is a countermodel, then there is one of linear size, in drastic contrast to the DL-Lite$_{\mathsf{bool}}$ case where the proof of Theorem 13 shows that exponentially large countermodels cannot be avoided.

Theorem 14. *AQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{horn}}$ KBs is in Π^{P}_2 w.r.t. combined complexity.*

A matching lower bound for DL-Lite$_{\mathsf{core}}$ cKBs can be found in (Bonatti, Faella, and Sauro 2011).

We now move to data complexity, where we obtain tractability even for the maximally expressive DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ dialect of DL-Lite.

Theorem 15. *AQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ KBs is in* PTime *w.r.t. data complexity.*

We sketch the proof of Theorem 15. Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $A_0(x)$ an AQ, and $a_0 \in \mathsf{Ind}(\mathcal{A})$. We construct an ABox $\mathcal{A}'$ whose size is independent of that of $\mathcal{A}$ and which can replace $\mathcal{A}$ when deciding whether a_0 is an answer to A_0. The construction of $\mathcal{A}'$ may be viewed as a variation of the constructions that we have used in the proofs of Theorems 6 and 8 to avoid an NP oracle. In particular, it reuses the notion of ABox types from the latter. The construction employed here is also different, however, as it works directly with ABoxes rather than with countermodels.

Let $\mathsf{TP}(\mathcal{A})$ denote the set of all ABox types t realized in $\mathcal{A}$, that is, all t such that $\mathsf{tp}_{\mathcal{A}}(a) = t$ for some $a \in \mathsf{Ind}(\mathcal{A})$. For every $t \in \mathsf{TP}(\mathcal{A})$, set

$$m_t = \min(|\{a \in \mathsf{ind}(\mathcal{A}) \mid \mathsf{tp}_{\mathcal{A}}(a) = t\}|,\ 4^{|\mathcal{T}|} + 1)$$

and choose m_t individuals $a_{t,1}, \ldots, a_{t,m_t} \in \mathsf{ind}(\mathcal{A})$ such that $\mathsf{tp}_{\mathcal{A}}(a_{t,i}) = t$ for $1 \leq i \leq m_t$. We assume that the individual a_0 of interest is among the chosen ones. Now define a set of witnesses and an ABox

$$\begin{array}{rcl} W & = & \{a_{t,i} \mid t \in \mathsf{TP}(\mathcal{A}),\ 1 \leq i \leq m_t\} \\ \mathcal{A}' & = & \{A(a) \mid a \in W, A \in \mathsf{tp}_{\mathcal{A}}(a)\}, \end{array}$$

and set $\mathcal{K}' = (\mathcal{T}, \mathcal{A}')$. Note that $|\mathsf{ind}(\mathcal{A}')| \leq 2^{|\mathcal{T}|}(4^{|\mathcal{T}|} + 1)$, which is independent of the original ABox $\mathcal{A}$. Also note that, just like the ABoxes $\mathcal{A}_{\mathcal{P}}$ from the proof of Theorem 8, $\mathcal{A}'$ no longer contains role assertions. Keeping in W at least $4^{|\mathcal{T}|} + 1$ copies of each ABox type t (if existent) ensures that in any model $\mathcal{I}'$ of $\mathcal{K}'$, we can find a type t' such that the combination of an ABox type and a regular type (t, t') is realized at least $2^{|\mathcal{T}|}$ many times in $\mathcal{I}'$ among individuals from W. This allows us to extend $\mathcal{I}'$ into a model of $\mathcal{K}$ while preserving $<_{\mathsf{CP}}$-minimality. The key observation is now as follows:

Lemma 11. $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ *iff* $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0(a_0)$.

Lemma 11 now easily gives PTime membership as the size of $\mathcal{A}'$ is bounded by a constant. We compute $\mathcal{K}'$ in polynomial time and check whether $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0(a_0)$, which is can be decided in 2Exp by Theorem 1, that is, in constant time w.r.t. data complexity. The correctness of this procedure immediately follows from Lemma 11.

5.3 Negative role inclusions

DL-Lite is often defined to additionally include negative role inclusions of the form $r \sqsubseteq \neg s$, with the obvious semantics. It is known that these sometimes lead to an increase in computational complexity, see for example (Manière 2022). We close with observing that this is also the case for circumscription. While querying circumscribed DL-Lite KBs (in all considered dialects) is coNP-complete w.r.t. data complexity for (U)CQs and in PTime for AQs, adding negative role inclusions result in a jump back to Π^{p}_2. We prove this by reduction from $\forall\exists$3SAT. Some ideas are shared with the proof of Theorem 4, but the general strategy of the reduction is a different one.

Theorem 16. *AQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{core}}$ KBs with negative role inclusions is Π^{P}_2-hard. This holds even without fixed concept names and with a single negative role inclusion.*

6 Conclusion

While we provide a fairly complete picture of the complexity of query evaluation on circumscribed KBs, some cases remain open. For example, the lower bounds for UCQ evaluation on DL-Lite cKBs given in Theorems 5 and 7 cannot be improved in an obvious way to CQs, for which the complexity remains open. Also, the lower bounds provided in Theorems 13, 16 and the Π^{p}_2 one from (Bonatti, Faella, and Sauro 2011) rely on the preference relation in circumscription patterns and it remains open whether the complexity decreases when the preference relation is forced to be empty. Finally, it would be interesting to study query evaluation under ontologies that are sets of existential rules or formulated in the guarded (negation) fragment of first-order logic, extended with circumscription. We believe that these problems are still decidable.

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A Proofs for Section 3.1

To complete the proof of Proposition 1, it suffices to show the following.

Lemma 12. $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$ *iff* $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}') \models q'(\bar{a})$ *for all* $\bar{a} \in \mathsf{ind}(\mathcal{A})$.

Proof. First assume that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$. Then there is a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ such that $\mathcal{I} \not\models q(\bar{a})$. Let $\mathcal{I}'$ be defined like $\mathcal{I}$ except that $A_a^{\mathcal{I}'} = B_a^{\mathcal{I}'} = \{a\}$ and $D_a^{\mathcal{I}'} = \emptyset$ for all $a \in N$. It is readily checked that $\mathcal{I}'$ is a model of $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}')$ and that $\mathcal{I}' \not\models q'(\bar{a})$.

Now assume that $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}') \not\models q'(\bar{a})$. Then there is a model $\mathcal{I}'$ of $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}')$ with $\mathcal{I}' \not\models q'(\bar{a})$. Since $\mathcal{I}'$ is a model of $\mathcal{A}'$, we have $a \in B_a^{\mathcal{I}'}$ for all $a \in N$. Moreover, if $\{a\} \subsetneq B_a^{\mathcal{I}'}$, then we can find a model $\mathcal{J}'$ of $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}')$ with $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ by setting $B_a^{\mathcal{J}'} = \{a\}$ and $D_a^{\mathcal{J}'} = A_a^{\mathcal{I}'} \setminus \{a\}$. This contradicts the minimality of $\mathcal{I}'$, and consequently we must have $B_a^{\mathcal{I}'} = \{a\}$. But then also $A_a^{\mathcal{I}'} = \{a\}$ since $A(a) \in \mathcal{A}'$, $A_a \sqcap \neg B_a \sqsubseteq D_a \in \mathcal{T}'$, and the disjunct $\exists y\, D_a(y)$ is not satisfied in $\mathcal{I}'$. It can be verified that, consequently, $\mathcal{I}'$ is also a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and we are done. □

Lemma 1. *Every $\mathcal{ALCHI}$ TBox $\mathcal{T}$ can be transformed in linear time into an $\mathcal{ALCHI}$ TBox $\mathcal{T}'$ in normal form such that for all cKBs $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A})$, UCQs $q(\bar{x})$ that do not use symbols from $\mathsf{sig}(\mathcal{T}') \setminus \mathsf{sig}(\mathcal{T})$, and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$: $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}, \mathcal{A}) \models q(\bar{a})$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}', \mathcal{A}) \models q(\bar{a})$.*

We omit a proof of the above lemma, which is entirely standard. The same normal form was used for $\mathcal{ALCHI}$, e.g. in (Manière 2022).

Lemma 2 (Core Lemma)**.** *Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and let $\mathcal{J}$ be a model of $\mathcal{K}$ with $\Delta^{\mathcal{I}}_{\mathsf{core}} \subseteq \Delta^{\mathcal{J}}$. If*

1. *$\mathsf{tp}_{\mathcal{I}}(d) = \mathsf{tp}_{\mathcal{J}}(d)$ for all $d \in \Delta^{\mathcal{I}}_{\mathsf{core}}$ and*
2. *$\mathsf{tp}_{\mathcal{J}}(\Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\mathsf{core}}) = \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$,*

then $\mathcal{J}$ is a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$.

Proof. Assume to the contrary that $\mathcal{J}$ is not a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. As $\mathcal{J}$ is a model of $\mathcal{K}$ by prerequisite, there is thus a model $\mathcal{J}'$ of $\mathcal{K}$ with $\mathcal{J}' <_{\mathsf{CP}} \mathcal{J}$. To derive a contradiction, we construct a model $\mathcal{I}'$ of $\mathcal{K}$ with $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}}$ and show that $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$. For each type $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$, let

$$\begin{aligned} D_t &= \{d \in \Delta^{\mathcal{I}} \setminus \mathsf{ind}(\mathcal{A}) \mid \mathsf{tp}_{\mathcal{I}}(d) = t\} \\ S_t &= \{\mathsf{tp}_{\mathcal{J}'}(d) \mid d \in \Delta^{\mathcal{J}} : \mathsf{tp}_{\mathcal{J}}(d) = t\}. \end{aligned}$$

We have

- $|D_t| \geq |\mathsf{TP}(\mathcal{T})|$ by definition of $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ and
- $|S_t| \leq |\mathsf{TP}(\mathcal{T})|$, by definition of S_t.

Moreover, the '$\supseteq$' direction of Point 2 implies $S_t \neq \emptyset$. We can thus find a surjective function $\pi_t : D_t \to S_t$.

Let π be the function that is obtained as the union of the (domain disjoint) functions π_t, for all $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. By definition of π and of $\Delta^{\mathcal{I}}_{\mathsf{core}}$, the domain of π is

$$\Delta^{\mathcal{I}} \setminus \big(\Delta^{\mathcal{I}}_{\mathsf{core}} \cup \mathsf{ind}(\mathcal{A})\big).$$

Now consider the range of π. By Point 1 and the '$\subseteq$' direction of Point 2, $\mathsf{tp}_{\mathcal{J}}(d) \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ iff $d \notin \Delta^{\mathcal{I}}_{\mathsf{core}}$. By definition, the range of π is thus

$$\{\mathsf{tp}_{\mathcal{J}'}(d) \mid d \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\mathsf{core}}\}.$$

Extend π to domain $\Delta^{\mathcal{I}}$ by setting $\pi(d) = \mathsf{tp}_{\mathcal{J}'}(d)$ for all $d \in \Delta^{\mathcal{I}}_{\mathsf{core}} \cup \mathsf{ind}(\mathcal{A})$. It is easy to see that now, π is a surjective function from $\Delta^{\mathcal{I}}$ to $\mathsf{TP}(\mathcal{J}')$. We construct an interpretation $\mathcal{I}'$ as follows:

$$\begin{aligned} \Delta^{\mathcal{I}'} &= \Delta^{\mathcal{I}} \\ A^{\mathcal{I}'} &= \{d \in \Delta^{\mathcal{I}'} \mid A \in \pi(d)\} \\ r^{\mathcal{I}'} &= \{(d,e) \in \Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}'} \mid \pi(d) \rightsquigarrow_r \pi(e)\} \end{aligned}$$

for all concept names A and role names r. Notice that $\mathsf{tp}_{\mathcal{I}'}(d) = \pi(d)$, hence $\mathsf{TP}(\mathcal{I}') \subseteq \mathsf{TP}(\mathcal{J}')$. We first show that $\mathcal{I}'$ is a model of $\mathcal{K}$ by considering each possible shape of assertions and inclusion (recall that $\mathcal{T}$ is in normal form):

- $A(a)$. Since $\mathcal{J}'$ is a model of $\mathcal{K}$, we have $a \in A^{\mathcal{J}'}$, hence $A \in \mathsf{tp}_{\mathcal{J}'}(a)$. By definition of π, we have $\pi(a) = \mathsf{tp}_{\mathcal{J}'}(a)$, hence $A \in \pi(a)$. The definition of $A^{\mathcal{I}'}$ yields $a \in A^{\mathcal{I}'}$.
- $r(a,b)$. Since $\mathcal{J}'$ is a model of $\mathcal{K}$, we have $(a,b) \in r^{\mathcal{J}'}$, hence $\mathsf{tp}_{\mathcal{J}'}(a) \rightsquigarrow_r \mathsf{tp}_{\mathcal{J}'}(b)$. By definition of π, we have $\pi(a) = \mathsf{tp}_{\mathcal{J}'}(a)$ and $\pi(b) = \mathsf{tp}_{\mathcal{J}'}(b)$. The definition of $r^{\mathcal{I}'}$ yields $(a,b) \in r^{\mathcal{I}'}$.
- Satisfaction of CIs of shape $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ and $\neg B \sqsubseteq A$ immediately follows from $\mathsf{TP}(\mathcal{I}') \subseteq \mathsf{TP}(\mathcal{J}')$ and $\mathcal{J}'$ being a model of $\mathcal{T}$.
- $A \sqsubseteq \exists r.B$. Let $d \in A^{\mathcal{I}'}$. Then $A \in \pi(d)$. Since $\pi(d) \in \mathsf{TP}(\mathcal{J}')$, there exists $d' \in \Delta^{\mathcal{J}'}$ such that $\mathsf{tp}_{\mathcal{J}'}(d') = \pi(d)$. Since $\mathcal{J}'$ is a model of $\mathcal{T}$, there exists $e' \in B^{\mathcal{J}'}$ such that $(d',e') \in r^{\mathcal{J}'}$. In particular, $B \in \mathsf{tp}_{\mathcal{J}'}(e')$ and $\mathsf{tp}_{\mathcal{J}'}(d') \rightsquigarrow_r \mathsf{tp}_{\mathcal{J}'}(e')$. It follows from π being surjective that there exists $e \in \Delta^{\mathcal{I}'}$ with $\pi(e) = \mathsf{tp}_{\mathcal{J}'}(e')$. Then $e \in B^{\mathcal{I}'}$ and $(d,e) \in r^{\mathcal{I}'}$ by definition of $\mathcal{I}'$, hence $d \in (\exists r.B)^{\mathcal{I}'}$.
- $\exists r.B \sqsubseteq A$. Let $d \in (\exists r.B)^{\mathcal{I}'}$. Then there exists $e \in B^{\mathcal{I}'}$ such that $(d,e) \in r^{\mathcal{I}'}$. By definition of $r^{\mathcal{I}'}$, we have $\pi(d) \rightsquigarrow_r \pi(e)$. Since $B \in \mathsf{tp}_{\mathcal{I}'}(e) = \pi(e)$ and $\exists r.B \sqsubseteq A$, the definition of $\rightsquigarrow_r$ yields $A \in \pi(d) = \mathsf{tp}_{\mathcal{I}'}(d)$. Thus $d \in A^{\mathcal{I}'}$.
- $r \sqsubseteq s$. Let $(d,e) \in r^{\mathcal{I}'}$. Then $\pi(d) \rightsquigarrow_r \pi(e)$. Since $\mathcal{T} \models r \sqsubseteq s$, it is immediate that $\pi(d) \rightsquigarrow_s \pi(e)$. Therefore the definition of $s^{\mathcal{I}'}$ yields $(d,e) \in s^{\mathcal{I}'}$.

It remains to show that $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$. We make use of the following claim.

Claim 1. *$A^{\mathcal{I}'} \odot A^{\mathcal{I}}$ iff $A^{\mathcal{J}'} \odot A^{\mathcal{J}}$, for each concept name A and $\odot \in \{\subseteq, \supseteq\}$.*

Let A be a concept name. We prove the claim for $\odot = \subseteq$ only; for $\odot = \supseteq$, the arguments are similar.

"⇒". Let $A^{\mathcal{I}'} \subseteq A^{\mathcal{I}}$ and $d \in A^{\mathcal{J}'}$; we need to show that $d \in A^{\mathcal{J}}$. First assume that $d \in \Delta^{\mathcal{I}}_{\mathsf{core}}$. Then $\mathsf{tp}_{\mathcal{I}}(d) = \mathsf{tp}_{\mathcal{J}}(d)$ and $\mathsf{tp}_{\mathcal{I}'}(d) = \mathsf{tp}_{\mathcal{J}'}(d)$. So $A^{\mathcal{I}'} \subseteq A^{\mathcal{I}}$ clearly implies $d \in A^{\mathcal{J}}$, as required. Now assume that $d \notin \Delta^{\mathcal{I}}_{\mathsf{core}}$. Let $t = \mathsf{tp}_{\mathcal{J}}(d)$. Then $\mathsf{tp}_{\mathcal{J}'}(d) \in S_t$ and consequently we find a $d' \in D_t$ with $\pi(d') = \mathsf{tp}_{\mathcal{J}'}(d)$. This implies $\mathsf{tp}_{\mathcal{I}'}(d') = \mathsf{tp}_{\mathcal{J}'}(d)$. Since $d' \in D_t$, we have $\mathsf{tp}_{\mathcal{I}}(d) = t = \mathsf{tp}_{\mathcal{J}}(d)$. So again, $A^{\mathcal{I}'} \subseteq A^{\mathcal{I}}$ implies $d \in A^{\mathcal{J}}$, as required.

"⇐". Let $A^{\mathcal{J}'} \subseteq A^{\mathcal{J}}$ and $d \in A^{\mathcal{I}'}$; we need to show $d \in A^{\mathcal{I}}$. First assume that $d \in \Delta^{\mathcal{I}}_{\mathsf{core}}$. Then $\mathsf{tp}_{\mathcal{I}}(d) = \mathsf{tp}_{\mathcal{J}}(d)$ and $\mathsf{tp}_{\mathcal{I}'}(d) = \mathsf{tp}_{\mathcal{J}'}(d)$. So $A^{\mathcal{J}'} \subseteq A^{\mathcal{J}}$ clearly implies $d \in A^{\mathcal{I}}$, as required. Now assume that $d \notin \Delta^{\mathcal{I}}_{\mathsf{core}}$ and let $t' = \mathsf{tp}_{\mathcal{I}'}(d)$. By definition of $\mathcal{I}'$, we have $t' = \pi(d)$. By definition of π, there is a $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ such that $t' = \pi_t(d)$. Then $d \in D_t$ and $t' \in S_t$. The former yields $\mathsf{tp}_{\mathcal{I}}(d) = t$ and due to the latter, there is a d' such that $\mathsf{tp}_{\mathcal{J}}(d) = t = \mathsf{tp}_{\mathcal{I}}(d)$ and $\mathsf{tp}_{\mathcal{J}'}(d) = t' = \mathsf{tp}_{\mathcal{I}'}(d)$. So again, $A^{\mathcal{J}'} \subseteq A^{\mathcal{J}}$ implies $d \in A^{\mathcal{I}}$, as required.

It is easy to see that since $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$, the claim implies $\mathcal{J}' <_{\mathsf{CP}} \mathcal{J}$. We have derived a contradiction and conclude that $\mathcal{J}$ is a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, as desired. □

Lemma 3. *Let $q(\bar{x})$ be a UCQ and $\bar{a} \in \mathsf{ind}(\mathcal{A})$. If $\mathcal{I}$ is a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$, then so is its unraveling $\mathcal{I}'$.*

Proof. Let $\mathcal{I}$ be a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$. It is clear that $\mathcal{I}' \not\models q[\bar{a}]$ since any homomorphism g from q to $\mathcal{I}'$ with $g(\bar{x}) = \bar{a}$ could be composed with the homomorphism h from $\mathcal{I}'$ to $\mathcal{I}$ to show that $\mathcal{I} \models q[\bar{a}]$. It is also straightforward to verify that

$(*)$ $\mathsf{tp}_{\mathcal{I}'}(d) = \mathsf{tp}_{\mathcal{I}}(h(d))$ for all $d \in \Delta^{\mathcal{I}'}$.

Using this and the definition of role extensions in $\mathcal{I}'$, it can be verified that $\mathcal{I}'$ is a model of $\mathcal{K}$. Let us consider each possible shape of assertions and axioms in our normal form:

- Assertions from $\mathcal{A}$ are clearly satisfied as $\mathcal{I}$ is a model of $\mathcal{A}$ and $\mathcal{I}'$ preserves $\mathcal{I}|_{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ (recall $\mathsf{ind}(\mathcal{A}) \subseteq \Delta^{\mathcal{I}}_{\mathsf{base}}$).
- Satisfaction of CIs with shape $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ and $\neg B \sqsubseteq A$ follows from remark $(*)$ on types.
- $A \sqsubseteq \exists r.B$. Let $d \in A^{\mathcal{I}'}$. Thus $h(d) \in A^{\mathcal{I}}$, and since $\mathcal{I}$ models $\mathcal{K}$, $d' = f(h(d), r.B)$ is defined. If $d' \in \Delta^{\mathcal{I}}_{\mathsf{core}}$, then definition of $\mathcal{I}'$ yields $(d, d') \in r^{\mathcal{I}'}$ and $d' \in B^{\mathcal{I}'}$. Otherwise, $dr.B$ is a path through $\mathcal{I}$ and by definition of $\mathcal{I}'$ we have in this case $(d, dr.B) \in r^{\mathcal{I}'}$ and $dr.B \in B^{\mathcal{I}'}$. In both cases, $d \in (\exists r.B)^{\mathcal{I}'}$.
- $\exists r.B \sqsubseteq A$. Let $d \in (\exists r.B)^{\mathcal{I}'}$. That is, there exists $e \in B^{\mathcal{I}'}$ such that $(d, e) \in r^{\mathcal{I}'}$. Therefore $h(e) \in B^{\mathcal{I}}$ and $(h(d), h(e)) \in r^{\mathcal{I}}$, ie $h(d) \in (\exists r.B)^{\mathcal{I}}$. From $\mathcal{I}$ being a model of $\mathcal{K}$, it follows $h(d) \in A^{\mathcal{I}}$, thus $d \in A^{\mathcal{I}'}$.
- Axioms with shape $r \sqsubseteq s$ are clearly satisfied from the definition of $s^{\mathcal{I}'}$.

It remains to prove that $\mathcal{I}'$ is minimal w.r.t. $<_{\mathsf{CP}}$. We use Lemma 2 with $\mathcal{I}$ as reference model, and it suffices to show that the preconditions of that lemma are satisfied. Clearly, $\Delta^{\mathcal{I}}_{\mathsf{core}} \subseteq \Delta^{\mathcal{I}'}$. Moreover, $(*)$ and the fact that $h(d) = d$ for all $d \in \Delta^{\mathcal{I}}_{\mathsf{core}}$ implies that Condition 1 of Lemma 2 is satisfied. We next verify that Condition 2 is also satisfied. First let $d \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\mathsf{core}}$. We have to show that $\mathsf{tp}_{\mathcal{J}}(d) \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. If $d \in \Delta^{\mathcal{I}}_{\mathsf{base}}$, then $h(d) = d$. From $d \notin \Delta^{\mathcal{I}}_{\mathsf{core}}$, we get $\mathsf{tp}_{\mathcal{I}}(d) \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ and it remains to apply $(*)$. If $d \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$, then $h(d) \notin \Delta^{\mathcal{I}}_{\mathsf{core}}$ and again we may use $(*)$. Conversely, let $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. We have to show that there is a $d \in \Delta^{\mathcal{J}} \setminus \Delta^{\mathcal{I}}_{\mathsf{core}}$ with $\mathsf{tp}_{\mathcal{J}}(d) = t$. But that d is e_t since $h(e_t) = e_t$ and by $(*)$. □

Lemma 4. *Let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. There exists a model $\mathcal{J}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ such that $|\Delta^{\mathcal{J}}| \leq |\mathcal{A}| + 2^{2|\mathcal{T}|}$, $\mathcal{I}|_{\Delta^{\mathcal{I}}_{\mathsf{core}}} = \mathcal{J}|_{\Delta^{\mathcal{J}}_{\mathsf{core}}}$, and $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I}) = \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{J})$.*

Proof. Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Assume that $t_i \notin \Delta^{\mathcal{I}}$, for every $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ and $1 \leq i \leq m$. Define $\mathcal{J}$ by setting

$$\Delta^{\mathcal{J}} = \mathsf{ind}(\mathcal{A}) \cup \Delta^{\mathcal{I}}_{\mathsf{core}} \cup \{t_i \mid t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I}), 1 \leq i \leq m\}$$

$$A^{\mathcal{J}} = (A^{\mathcal{I}} \cap \Delta^{\mathcal{J}}) \cup \{t_i \mid A \in t, 1 \leq i \leq m\}$$

$$\begin{aligned} r^{\mathcal{J}} = {} & (r^{\mathcal{I}} \cap (\Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}})) & \text{(i)} \\ & \cup \{(e, t_i) \mid e \in \Delta^{\mathcal{I}}, \mathsf{tp}_{\mathcal{I}}(e) \rightsquigarrow_r t, 1 \leq i \leq m\} & \text{(ii)} \\ & \cup \{(t_i, e) \mid e \in \Delta^{\mathcal{I}}, t \rightsquigarrow_r \mathsf{tp}_{\mathcal{I}}(e), 1 \leq i \leq m\} & \text{(iii)} \\ & \cup \{(t_i, t'_j) \mid t \rightsquigarrow_r t', 1 \leq i, j \leq m\} & \text{(iv)} \end{aligned}$$

for all concept names A and role names r. It is easy to see that $\mathsf{tp}_{\mathcal{J}}(e) = \mathsf{tp}_{\mathcal{I}}(e)$ for all $e \in \Delta^{\mathcal{I}} \cap \Delta^{\mathcal{J}}$ and $\mathsf{tp}_{\mathcal{J}}(t_i) = t$ for all $t \in \mathsf{TP}(\mathcal{T})$ and $1 \leq i \leq m$. This implies that $\mathcal{I}_{|\Delta^{\mathcal{I}}_{\mathsf{core}}} = \mathcal{J}_{|\Delta^{\mathcal{J}}_{\mathsf{core}}}$ and $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I}) = \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{J})$ as desired. It also implies that Conditions 1 and 2 from Lemma 2 are satisfied. To show that $\mathcal{J}$ is a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, it thus suffices to show that $\mathcal{J}$ is a model of $\mathcal{K}$. It is clear by construction that $\mathcal{J}$ is a model of $\mathcal{A}$. For $\mathcal{T}$, we consider all different forms of axioms:

- Satisfaction of CIs of the form $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ and $\neg B \sqsubseteq A$ immediately follows from $\mathsf{TP}(\mathcal{J}) \subseteq \mathsf{TP}(\mathcal{I})$ and $\mathcal{I}$ being a model of $\mathcal{T}$. We distinguish two cases:
- $A \sqsubseteq \exists r.B$. Let $d \in A^{\mathcal{J}}$.
  - If $d \in \Delta^{\mathcal{I}}$, then we have $d \in A^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, there exists $e \in B^{\mathcal{I}}$ such that $(d, e) \in r^{\mathcal{I}}$. If $e \in \Delta^{\mathcal{I}}$, then $(d, e) \in r^{\mathcal{J}}$ due to Case (i) in the definition of $r^{\mathcal{J}}$. Otherwise $t := \mathsf{tp}_{\mathcal{I}}(e) \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. In particular, $B \in t$ and $\mathsf{tp}_{\mathcal{I}}(d) \rightsquigarrow_r t$. Therefore $t_1 \in B^{\mathcal{J}}$ and $(d, t_1) \in r^{\mathcal{J}}$ due to Case (ii), proving $d \in (\exists r.B)^{\mathcal{J}}$.
  - If $d \notin \Delta^{\mathcal{I}}$, then $d = t_i$ for some $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$ and $1 \leq i \leq m$. By definition of $\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$, there exists $d' \in \Delta^{\mathcal{I}}$ s.t. $\mathsf{tp}_{\mathcal{I}}(d') = t = \mathsf{tp}_{\mathcal{J}}(d)$. From $d \in A^{\mathcal{J}}$, we get $A \in t$. Therefore $d' \in A^{\mathcal{I}}$, and since $\mathcal{I}$ is a model of $\mathcal{T}$, there exists $e \in B^{\mathcal{I}}$ such that $(d', e) \in r^{\mathcal{I}}$. In particular, $B \in \mathsf{tp}_{\mathcal{I}}(e)$ and $t \rightsquigarrow_r \mathsf{tp}_{\mathcal{I}}(e)$. If $e \in \Delta^{\mathcal{J}}$, then $e \in B^{\mathcal{J}}$

and Case (iii) in the definition of $r^{\mathcal{J}}$ ensures $(d,e) \in r^{\mathcal{J}}$, which gives $d \in (\exists r.B)^{\mathcal{J}}$. If $e \notin \Delta^{\mathcal{J}}$, then $t' = \mathsf{tp}_{\mathcal{I}}(e) \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. Let $e' = t'_j$ for some j with $1 \leq j \leq m$. Case (iv) in the definition of $r^{\mathcal{J}}$ yields $(d,e') \in r^{\mathcal{J}}$, which again gives $d \in (\exists r.B)^{\mathcal{J}}$.

- $\exists r.B \sqsubseteq A$. Let $d \in (\exists r.B)^{\mathcal{J}}$. Then there exists $e \in B^{\mathcal{J}}$ such that $(d,e) \in r^{\mathcal{J}}$. From each case in definition of $r^{\mathcal{J}}$, we easily get $\mathsf{tp}_{\mathcal{J}}(d) \rightsquigarrow_r \mathsf{tp}_{\mathcal{J}}(e)$. Since $e \in B^{\mathcal{J}}$ we have in particular $B \in \mathsf{tp}_{\mathcal{J}}(e)$. Combined with $\mathcal{T} \models \exists r.B \sqsubseteq A$ and the definition of $\mathsf{tp}_{\mathcal{J}}(d) \rightsquigarrow_r \mathsf{tp}_{\mathcal{J}}(e)$, we obtain $A \in \mathsf{tp}_{\mathcal{J}}(d)$, that is $d \in A^{\mathcal{J}}$.
- $r \sqsubseteq s$. Let $(d,e) \in r^{\mathcal{J}}$. If this is due to Case (i) in the definition of $r^{\mathcal{J}}$, then $(d,e) \in s^{\mathcal{J}}$ since $\mathcal{I}$ is a model of $\mathcal{T}$. For all other three cases, it suffices to note that, due to $r \sqsubseteq s \in \mathcal{T}$, $t_1 \rightsquigarrow_r t_2$ implies $t_1 \rightsquigarrow_s t_2$ for all types t_1, t_2.

Finally, we remark that the size of $\Delta^{\mathcal{J}}$ is bounded by $|\mathsf{ind}(\mathcal{A})| + \left|\Delta^{\mathcal{I}}_{\mathsf{core}}\right| + m \cdot |\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})|$. From $\left|\Delta^{\mathcal{I}}_{\mathsf{core}}\right| \leq m \cdot |\mathsf{TP}_{\mathsf{core}}(\mathcal{I})|$ and $|\mathsf{TP}_{\mathsf{core}}(\mathcal{I})| + |\mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})| = |\mathsf{TP}(\mathcal{I})| \leq m$ we obtain $\left|\Delta^{\mathcal{J}}\right| \leq |\mathcal{A}| + m^2$. Recalling $m = |\mathsf{TP}(\mathcal{T})| \leq 2^{|\mathcal{T}|}$, we obtain $|\Delta^{\mathcal{J}}| \leq |\mathcal{A}| + 2^{2|\mathcal{T}|}$ as required. □

B Proofs for Section 3.2

Lemma 5. *$\mathcal{I}_{\mathsf{base}}$ can be extended to a model $\mathcal{I}$ of $\mathcal{T}$ that satisfies Conditions (a) to (c) iff at least one mosaic survives the elimination process.*

To prove the "⇐" direction of Lemma 5, assume we are given a good mosaic M_0. We aim to extend $\mathcal{J}_{M_0}$, hence $\mathcal{I}_{\mathsf{base}}$, into a model $\mathcal{I}$ of $\mathcal{T}$ that satisfies Conditions (a) to (c). For each good mosaic M, each $e \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_M}$ for some A such that $A \sqsubseteq \exists r.B \in \mathcal{T}$, we choose a good mosaic M' such that:

1. $\mathsf{tp}_{\mathcal{J}_M}(e) = \mathsf{tp}_{\mathcal{J}_{M'}}(e)$;
2. $e \in (\exists r.B)^{\mathcal{J}_{M'}}$;
3. if $(p, \widehat{p}, h) \in S_M$, then $(p, \widehat{p}, h') \in S_{M'}$ where h' is the restriction of h to range $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e\}$.

We denote $\mathsf{ch}^{r.B}_{M,e}$ this chosen M'. Then, starting from M_0, we build a tree-shaped set of words which witnesses the acceptance of M_0.

Definition 3. *The mosaic tree M_0^+ is the smallest set of words such that:*

- *$(M_0, \emptyset, \emptyset) \in M_0^+$;*
- *If $w \cdot (M, \_, \_) \in M_0^+$ and $e \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_M}$ for some A such that $A \sqsubseteq \exists r.B \in \mathcal{T}$, then $w \cdot (M, \_, \_) \cdot (\mathsf{ch}^{r.B}_{M,e}, r.B, e) \in M_0^+$.*

It remains to 'glue' together the interpretations $\mathcal{J}_M$ according to the structure of M_0^+. Since a mosaic M may occur more than once, we create a copy of $\mathcal{J}_M$ for each node in M_0^+ of the form $w \cdot (M, \_, \_)$. We do not duplicate however elements from $\mathcal{I}_{\mathsf{base}}$ as they precisely are those we want to reuse. Hence only the two special elements e_1^* and e_2^* may be duplicated. When a node $w \cdot (M_1, r_1.B_1, e_1) \cdot (M_2, r_2.B_2, e_2)$ is encountered, we merge element e_2 from M_2 with the already-existing e_2 from M_1, which is indicated with a superscript $w \cdot (M_1, r_1.B_1, e_1)$ as follows $e_2^{w \cdot (M_1, r_1.B_1, e_1)}$. Other elements from $\Delta^{\mathcal{J}_{M_2}} \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}$ are considered freshly introduced by this node, which is indicated with a matching superscript, that is: $e^{w \cdot (M_1, r_1.B_1, e_1) \cdot (M_2, r_2.B_2, e_2)}$. Formally, the copying and merging of elements is achieved by the following duplicating functions, defined for each $w \cdot (M, r.B, e) \in M_0^+$.

$$\lambda_{w \cdot (M, r.B, e)} : \Delta^{\mathcal{J}_M} \to \Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{(e_i^*)^w, (e_i^*)^{w \cdot (M, r.B, e)}\}$$
$$d \mapsto \begin{cases} d & \text{if } d \in \Delta^{\mathcal{I}_{\mathsf{base}}} \\ d^w & \text{if } d = e \notin \Delta^{\mathcal{I}_{\mathsf{base}}} \\ d^{w \cdot (M, r.B, e)} & \text{otherwise} \end{cases}$$

The desired model $\mathcal{I}$ can then be defined as:

$$\mathcal{I} = \bigcup_{w \cdot (M, r.B, e) \in M_0^+} \lambda_{w \cdot (M, r.B, e)}(\mathcal{J}_M),$$

that is the domain (resp. the interpretation of each concept name and each role name) of $\mathcal{I}$ is the union across all $w \cdot (M, r.B, e) \in M_0^+$ of the image by $\lambda_{w \cdot (M, r.B, e)}$ of the domain (resp. the interpretation of each concept name and each role name) of $\mathcal{J}_M$.

By definition, each $\lambda_{w \cdot (M, r.B, e)}$ is a homomorphism from $\mathcal{J}_M$ to $\mathcal{I}$. Due to Condition 1 in the definition of good mosaics, if the range of two duplicating functions overlap, then the common element satisfies the same concepts, so concept membership in $\mathcal{I}$ transfers back to $\mathcal{J}_M$. More formally:

Lemma 13. *For all $w \cdot (M, r.B, e) \in M_0^+$, all $d \in \Delta^{\mathcal{J}_M}$ and all $A \in \mathsf{N_C}$, if $\lambda_{w \cdot (M, r.B, e)}(d) \in A^{\mathcal{I}}$, then $d \in A^{\mathcal{J}_M}$.*

Proof. Let $u_1 = w_1 \cdot (M_1, r_1.B_1, e_1) \in M_0^+$ be a node from the mosaic tree, d_1 an element from $\Delta^{\mathcal{J}_{M_1}}$ and A a concept name. Assume $\lambda_{w_1 \cdot M_1}(e_1) \in A^{\mathcal{I}}$. By definition of $A^{\mathcal{I}}$ there exists a node $u_2 = w_2 \cdot (M_2, r_2.B_2, e_2)$ from the mosaic tree, and an element $d_2 \in A^{\mathcal{J}_{M_2}}$ such that $\lambda_{u_1}(d_1) = \lambda_{u_2}(d_2)$. We refer to this equality as $(*)$ and distinguish 5 cases.

1. $d_1 \in \Delta^{\mathcal{I}_{\mathsf{base}}}$ or $d_2 \in \Delta^{\mathcal{I}_{\mathsf{base}}}$.
   $(*)$ yields $d_1 = d_2$. Interpretation $\mathcal{J}_{M_2}$ preserves $\mathcal{I}_{\mathsf{base}}$, hence $d_2 \in A^{\mathcal{I}_{\mathsf{base}}}$. Interpretation $\mathcal{J}_{M_1}$ preserves $\mathcal{I}_{\mathsf{base}}$, hence $d_1 \in A^{\mathcal{J}_{M_1}}$.
   In the remaining cases, we thus assume $d_1, d_2 \notin \Delta^{\mathcal{I}_{\mathsf{base}}}$.
2. $d_1 = e_1$ and $d_2 = e_2$.
   $(*)$ yields $M_1 = M_2$ and $d_1 = d_2$.
3. $d_1 \neq e_1$ and $d_2 = e_2$.
   $(*)$ yields $w_1 = w_2 \cdot (M_2, r_2.B_2, e_2)$ and $d_1 = d_2$. In particular, $M_1 = \mathsf{ch}^{r_2, B_2}_{M_2, e_2}$. From Condition 1 in the definition of good mosaics, we obtain $d_1 \in A^{\mathcal{J}_{M_1}}$.
4. $d_1 = e_1$ and $d_2 \neq e_2$.
   Same arguments as for Case 3 but this time with $M_2 = \mathsf{ch}^{r_1, B_1}_{M_1, e_1}$.
5. $d_1 \neq e_1$ and $d_2 \neq e_2$.
   $(*)$ yields the existence of $w \cdot (M, r.B, e)$ such that $w_1 = w_2 = w \cdot (M, r.B, e)$. It follows that $M_1 = \mathsf{ch}^{r,B}_{M,e}$ and $M_2 = \mathsf{ch}^{r,B}_{M,e}$. Using same arguments as in Cases 3 and 4, we obtain $d_1 \in A^{\mathcal{J}_{M_1}}$. □

With Lemma 13 in hand, we are ready to show that $\mathcal{I}$ is a model of $\mathcal{K}$.

Lemma 14. *$\mathcal{I}$ is a model of $\mathcal{K}$.*

Proof. We consider each possible shape of assertion and axiom in $\mathcal{K}$ w.r.t. the normal form from Lemma 1.

- Assertions $A(a)$ and $p(a,b)$ from the ABox are satisfied from $\mathcal{I}_{\mathsf{base}}$ being a model of $\mathcal{A}$ that is preserved in $\mathcal{J}_{M_0}$ (Condition 2 in the definition of a mosaic), hence transferred in $\mathcal{I}$.
- Axioms with shapes $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ and $\neg B \sqsubseteq A$ are satisfied from Lemma 13 and the fact every mosaic is only allowed to give proper types to its element.
- Axioms with shapes $\exists r.A_1 \sqsubseteq A_2$ and $p \sqsubseteq r$ are satisfied from Lemma 13 and Condition 4 in the definition of mosaics, precisely requiring those to hold.
- Axioms with shape $A_1 \sqsubseteq \exists r.A_2$ are those that really involve the mosaics. Let $d_1 \in a_1^{\mathcal{I}}$. By definition of ${a_1}^{\mathcal{I}}$, there exist a node $u = w \cdot (M, \_, \_) \in M_0^+$ and an element $d_1' \in \Delta^{\mathcal{J}_M}$ such that $d_1' \in {a_1}^{\mathcal{J}_M}$ and $\lambda_u(d_1') = d_1$. If $d_1' \in (\exists r.A_2)^{\mathcal{J}_M}$, then we pick $d_2' \in A_2^{\mathcal{J}_M}$ and $(d_1', d_2') \in r^{\mathcal{J}_M}$. It then follows immediately that $\lambda_u(d_2') \in A_2^{\mathcal{I}}$ and $(\lambda_u(d_1'), \lambda_u(d_2')) \in r^{\mathcal{I}}$, ensuring $d_1 \in (\exists r.A_2)^{\mathcal{I}}$. Otherwise $d_1' \notin (\exists r.A_2)^{\mathcal{J}_M}$, thus we have $u_0 = u \cdot (M_0, r.A_2, d_1') \in M_0^+$ with $M_0 = \mathsf{ch}^{r.A_2}_{M,d_1'}$. Notice $\lambda_u(d_1') = \lambda_{u_0}(d_1')$. From Condition 2 in the definition of good mosaics, we get $d_1' \in (\exists r.A_2)^{\mathcal{J}_{M_0}}$ hence we pick $d_2' \in A_2^{\mathcal{J}_{M_0}}$ and $(d_1', d_2') \in r^{\mathcal{J}_{M_0}}$. It then follows immediately that $\lambda_{u_0}(d_2') \in A_2^{\mathcal{I}}$ and $(\lambda_{u_0}(d_1'), \lambda_{u_0}(d_2')) \in r^{\mathcal{I}}$, ensuring $d_1 \in (\exists r.A_2)^{\mathcal{I}}$.

□

This proves $\mathcal{I}_{\mathsf{base}}$ can indeed by extended into a model of $\mathcal{T}$. Notice Condition (a) is clearly satisfied from Condition 2 in the definition of mosaics. Similarly, Condition (b) is clearly satisfied from Condition 3 in the definition of mosaics joint with Lemma 13. It remains to verify Condition (c), that is for all $p \in q$, there is no homomorphism from p in $\mathcal{I}$.

By contradiction, assume there exists $p \in q$ and a homomorphism $\pi : p \to \mathcal{I}$. Notice that for each $v \in \mathsf{var}(p)$ s.t. $\pi(v) \notin \Delta^{\mathcal{I}_{\mathsf{base}}}$, we thus have w such that $\pi(v) = d^w$ for some $d \in \{e_1^*, e_2^*\}$. To identify from which nodes the facts of $\pi(p)$ come involving d^w, we notice the following remark, issuing from the definition of functions λ_w for $w \in M_0^+$.

Remark 1. *Let $w \in M_0^+$ and $d \in \{e_1^*, e_2^*\}$. For all $w' \cdot (M, r.B, e) \in M_0^+$, we have that $d^w \in \lambda_{w' \cdot (M,r.B,e)}(\Delta^{\mathcal{J}_M})$ iff one of the two following conditions holds:*

- *$w = w' \cdot (M, r.B, e)$;*
- *$w = w'$ and $d = e$.*

It follows that, for each $v \in \mathsf{var}(p)$ s.t. $\pi(v) \notin \Delta^{\mathcal{I}_{\mathsf{base}}}$, the set of nodes $W_v = \{w' \cdot (M, r.B, e) \in M_0^+ \mid \pi(v) \in \lambda_{w' \cdot (M,r.B,e)}(\Delta^{\mathcal{J}_M})\}$ is finite. On M_0^+, we consider the order given by the prefix relation, that for all $w_1, w_2 \in M_0^+$, we define $w_1 \leq w_2$ iff w_1 is a prefix of w_2. We now denote W_p the prefix closure of $\bigcup_v W_v$, where v ranges over $v \in \mathsf{var}(p)$ s.t. $\pi(v) \notin \Delta^{\mathcal{I}_{\mathsf{base}}}$. In particular, $(M_0, \emptyset, \emptyset) \in W_p$. For every $w \in M_0^+$, we define the interpretation:

$$\mathcal{I}_w = \bigcup_{\substack{w' \cdot (M,r.B,e) \in M_0^+ \\ w \leq w' \cdot (M,r.B,e)}} \lambda_{w' \cdot (M,r.B,e)}(\mathcal{J}_M).$$

We further set $p_w \subseteq p$ consisting of those atoms from p that are mapped by π in $\mathcal{I}_w$, and π_w the subsequent homomorphism, that is the restriction of π that maps to $\mathcal{I}_w$, i.e.:

$$\pi_w = \pi|_{\pi^{-1}(\Delta^{\mathcal{I}_w})}.$$

In particular $p_{(M_0,\emptyset,\emptyset)} = p$ and $\pi_{(M_0,\emptyset,\emptyset)} = \pi$. We now claim the following:

Lemma 15. *For all node $u = w \cdot (M, r.B, e) \in W_p$, we have $(p, p_u, \pi') \in S_M$ where $\pi' = (\lambda_u)^{-1} \circ \pi_u|_\Delta$ with $\Delta = \pi_u^{-1}(\lambda_u(\Delta^{\mathcal{J}_M}))$.*

Proof. We proceed by induction on elements of W_p, starting from its finitely many maximal elements w.r.t. $\leq$.

Base case. Assume $u = w \cdot (M, r.B, e) \in W_p$ is maximal for $\leq$. that is for all $u' \in M_0^+$ s.t. $u \leq u'$, we have $u' \notin W_p$. In particular, for all $v \in \mathsf{var}(p)$, $\pi(v) \in \lambda_{u'}(\mathcal{J}_{M'})$, where M' is the mosaic of node u', implies $\pi(v) \in \Delta^{\mathcal{I}_{\mathsf{base}}}$. Therefore $\pi_u^{-1}(\lambda_u(\Delta^{\mathcal{J}_M})) = \mathsf{var}(p_u)$ and, since $\mathcal{I}_{\mathsf{base}}$ is preserved in every mosaic (Condition 2 in the definition), it follows that p_u fully embeds in $\lambda_u(\mathcal{J}_M)$ via π_u. From S_M being saturated, we thus have $(p, p_u, (\lambda_u)^{-1} \circ \pi_u)$ which is the desired triple.

Induction case. Consider $u = w \cdot (M, r.B, e) \in W_p$ and assume the property holds for all $u \in W_p$ with $u \leq u'$. We cover p_u by the set of all $p_{u \cdot (M', r'.B', e')}$ s.t. $u \cdot (M', r'.B', e') \in W_p$ and by the remaining atoms $p_{=u}$ from p_u. Notice that this is not a partition of p_u as those subqueries may overlap. In particular, two distinct subqueries $p_{u \cdot (M_1, r_1.B_1, e_1)}$ and $p_{u \cdot (M_2, r_2.B_2, e_2)}$ may only share those variables which are mapped to $\Delta^{\mathcal{I}_{\mathsf{base}}}$ and, if $e_1 = e_2$, on e_1 (that is also e_2 in this case). Respectively, $p_{u \cdot (M', r'.B', e')}$ and $p_{=u}$ may only share variables that are mapped either on $\Delta^{\mathcal{I}_{\mathsf{base}}}$ or on e'. We cover as well the homomorphism π_u by all $\pi_{u \cdot (M', r'.B', e')}$ s.t. $u \cdot (M', r'.B', e') \in W_p$ and by the homomorphism $\pi_{=u}$ which is defined as the restriction of π_u to variables from $p_{=u}$. Notice that from S_M being saturated, we obtain $(p, p_{=u}, (\lambda_u)^{-1} \circ \pi_{=u}) \in S_M$. We further apply the induction hypothesis on each $u \cdot (M', r'.B', e') \in W_p$, which provides corresponding triples $(p, p_{u \cdot (M', r'.B', e')}, \pi')$ in each $S_{M'}$. From Condition 3 in the definition of good mosaics, if follows that each $(p, p_{u \cdot (M', r'.B', e')}, \pi'')$ belongs to S_M, where π'' denotes the restriction of π' to range $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e'\}$. It remains to form the union (in the sense given in the definition of a saturated mosaic) of all those subsequent triples, along with $(p, p_{=u}, (\lambda_u)^{-1} \circ \pi_{=u})$, to obtain the desired triple. Let us first focus on two triples $(p, p_{u \cdot (M_1, r_1.B_1, e_1)}, \pi_1'')$ and $(p, p_{u \cdot (M_2, r_2.B_2, e_2)}, \pi_2'')$ and consider $v \in \mathsf{var}(p_{u \cdot (M_1, r_1.B_1, e_1)}) \cap \mathsf{var}(p_{u \cdot (M_2, r_2.B_2, e_2)})$. From our covering of p_u, we have seen that either $\pi_u(v) \in$

$\Delta^{\mathcal{I}_{\mathsf{base}}}$, or, if $e_1 = e_2$, that $\pi(v) = e_1$. In any case, restricting π'_1 and π'_2 to $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e_1\}$ preserved π''_1 and π''_2 being defined on v, and, since issuing from the same π_u, equal. Let us now move to a triple $(p, p_{u\cdot(M',r'.B',e')}, \pi'')$ and the triple $(p, p_{=u}, (\lambda_u)^{-1} \circ \pi_{=u})$. Let $v \in \mathsf{var}(p_{u\cdot(M',r'.B',e')}) \cap \mathsf{var}(p_{=u})$. From our covering of p_u, we know $\pi_u(v) \in \Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e'\}$. Again restricting to $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e'\}$ preserved π'' being defined on v and, since issuing from the same π_u, equal to $\pi_{=u}$ on v. Therefore, we can form the union of all those triples, since they are 2-by-2 compatible, which provides the desired triple in S_M. □

This conclude the proof of the "⇐" direction of Lemma 5, as existence of such a homomorphism p then yield a complete triple in S_{M_0}, contradicting M_0 being a mosaic.

We now turn to the "⇒" direction.

Assume $\mathcal{I}_{\mathsf{base}}$ can be extended to a model $\mathcal{I}$ of $\mathcal{T}$ that satisfies Conditions (a) to (c). Up to introducing $|\mathsf{TP}(\mathcal{T})|$ copies of each element from $\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}$, we can safely assume $\Delta^{\mathcal{I}}_{\mathsf{base}} = \Delta^{\mathcal{I}_{\mathsf{base}}}$. Indeed, the resulting interpretations is still a model of $\mathcal{K}$ and still satisfies Conditions (a) to (c).

Set $\mathcal{J}_{M_0} := \mathcal{I}_{\mathsf{base}}$ and $S_{M_0} := \{(p, \widehat{p}, h|_{\Delta^{\mathcal{I}_{\mathsf{base}}}}) \mid p \in q, \widehat{p} \subseteq p, h : \widehat{p} \to \mathcal{I}$ is a homomorphism$\}$. Condition (c) ensures in particular that S_{M_0} only contains incomplete triples. We claim $M_0 := (\mathcal{J}_{M_0}, S_{M_0})$ is accepting.

To prove this, we shall build a set of mosaics, whose every mosaic M is *not trivially bad*, i.e. M satisfy the base-case condition of a good mosaic, and which is *realized in* $\mathcal{I}$, meaning that $\mathcal{J}_M$ homomorphically embeds into $\mathcal{I}$. Observe that the initial mosaic M_0 satisfies both conditions. To pursue the construction, given any mosaic M satisfying the two conditions and some $e \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_M}$ with $A \sqsubseteq \exists r.B \in \mathcal{T}$, we show how to extract from $\mathcal{I}$ another non-trivially bad mosaic N to succeed to M. Since the number of mosaics is finite, every sequence of mosaics constructed in such a manner either leads to a trivially good mosaic (i.e. one with no such problematic element e) or loops back to an already explored mosaic satisfying the conditions. It follows that all mosaics in the set are accepting (in particular, M_0).

To formalize the construction, we shall introduce along with each mosaic M a function τ being a homomorphism $\mathcal{J}_M \to \mathcal{I}$. We also assume chosen, for every $r.A \in \Omega$, a function $\mathsf{succ}^{\mathcal{I}}_{r.A}$ that maps every element $e \in (\exists r.A)^{\mathcal{I}}$ to an element $e' \in \Delta^{\mathcal{I}}$ such that $(e, e') \in r^{\mathcal{I}}$ and $e' \in A^{\mathcal{I}}$.

Definition 4. *Base case: the construction begins with the pair* $(M_0, \mathsf{Id}_{\mathcal{J}_{M_0} \to \mathcal{I}})$*, where* $\mathsf{Id}_{\mathcal{J}_{M_0} \to \mathcal{I}}$ *denotes the identity function.*

Induction case: consider some already constructed pair (M_1, τ_1)*, and some* $e_1 \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_{M_1}}$ *with* $A \sqsubseteq \exists r.B \in \mathcal{T}$*. Set* $e'_1 := \tau_1(e_1)$*. Since* τ_1 *is a homomorphism and* $\mathcal{I}$ *is a model of* $\mathcal{T}$*, we obtain* $e'_1 \in (\exists r.B)^{\mathcal{I}}$ *and can set* $e'_2 := \mathsf{succ}^{\mathcal{I}}_{r.B}(e'_1)$*. If* $e'_2 \in \Delta^{\mathcal{I}_{\mathsf{base}}}$*, then we set* $e_2 := e'_2$*, otherwise we set* e_2 *to either* e^*_1 *or* e^*_2 *such that* $e_1 \neq e_2$*.*

We let τ_2 *be the function that maps elements of* $\Delta^{\mathcal{I}_{\mathsf{base}}}$ *to themselves,* e_1 *to* e'_1 *and* e_2 *to* e'_2*. We now define a new mosaic* M_2*. Its domain is* $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e_1, e_2\}$*. Its interpretation* $\mathcal{J}_{M_2}$ *is given by:*

$$\begin{array}{rcl} \Delta^{\mathcal{J}_{M_2}} & = & \Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e_1, e_2\} \\ C^{\mathcal{J}_{M_2}} & = & C^{\mathcal{J}_{M_0}} \cup \{e_k \mid e'_k \in C^{\mathcal{I}}, k = 1, 2\} \\ p^{\mathcal{J}_{M_2}} & = & p^{\mathcal{J}_{M_0}} \cup \{(e_1, e_2) \mid \mathcal{T} \models r \sqsubseteq p\} \\ & & \cup \{(e_2, e_1) \mid \mathcal{T} \models r^- \sqsubseteq p\} \end{array}$$

Its specification is $S_{M_2} = \{(p, \widehat{p}, \tau_2^{-1} \circ h|_{\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e'_1, e'_2\}}) \mid p \in q, \widehat{p} \subseteq p, h : \widehat{p} \to \mathcal{I}$ *is a homomorphism*$\}$.

We obtain a new pair (M_2, τ_2)*.*

Recalling that $\mathcal{I}$ is a model, of $\mathcal{K}$ it is then straightforward to verify that M_2 is a well-defined mosaic that is not trivially bad, satisfying all conditions to succeed to M_1 for e and $A \sqsubseteq \exists r.B$, and that τ_2 is indeed a homomorphism. These properties are verified by the next two lemmas.

Lemma 16. *Each pair* (M, τ) *built according to Definition 4 yields a well-defined and non-trivially bad mosaic* M *and a homomorphism* $\tau : \mathcal{J}_M \to \mathcal{I}$*.*

Proof. The base case consisting of verifying that M_0 is well-defined and non-trivially bad is indeed trivial, and $\mathsf{Id}_{\mathcal{J}_{M_0} \to \mathcal{I}}$ is indeed a homomorphism.

We move to the induction case: assume (M_1, τ_1) has already been obtained by the described procedure and M_1 is a well-defined and non-trivially bad mosaic and $\tau_1 : \mathcal{J}_{M_1} \to \mathcal{I}$ a homomorphism. Assume there exists some $e_1 \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_{M_1}}$ with $A \sqsubseteq \exists r.B \in \mathcal{T}$, and consider the pair (M_2, τ_2) obtained by applying Definition 4 on (M_1, τ_1) for this choice of e.

We first verify that τ_2 is a homomorphism:

- Let $u \in A^{\mathcal{J}_{M_2}}$. If $u \in A^{\mathcal{J}_{M_0}}$, then in particular $u \in \Delta^{\mathcal{I}_{\mathsf{base}}}$ hence $\tau_2(u) = u \in A^{\mathcal{J}_{M_0}} \subseteq A^{\mathcal{I}}$. Otherwise, $u = e_k$ for $k = 1$ or $k = 2$ with $e'_k \in A^{\mathcal{I}}$. In that case, notice $\tau_2(u) = e'_k$ which concludes.
- Let $(u, v) \in p^{\mathcal{J}_{M_2}}$. If $(u, v) \in p^{\mathcal{J}_{M_0}}$, then in particular $u, v \in \Delta^{\mathcal{I}_{\mathsf{base}}}$, hence $(\tau_2(u), \tau_2(v)) = (u, v) \in p^{\mathcal{J}_{M_0}} \subseteq p^{\mathcal{I}}$. Otherwise, if $(u, v) = (e_1, e_2)$ with $\mathcal{T} \models r \sqsubseteq p$, then notice that $(\tau_2(u), \tau_2(v)) = (e'_1, e'_2)$. Since e'_2 is the successor of e'_1 for $r.B$ in $\mathcal{I}$, and $\mathcal{I}$ models $\mathcal{T}$, we obtain $(e'_1, e'_2) \in p^{\mathcal{I}}$ as desired. Otherwise we have $(u, v) = (e_2, e_1)$ with $\mathcal{T} \models r^- \sqsubseteq p$, then notice that $(\tau_2(u), \tau_2(v)) = (e'_1, e'_1)$. Since e'_2 is the successor of e'_1 for $r.B$ in $\mathcal{I}$, and $\mathcal{I}$ models $\mathcal{T}$, we have $(e'_2, e'_1) \in p^{\mathcal{I}}$ as desired.

We now verify that M_2 is a well-defined mosaic.

- Regarding $\mathcal{J}_{M_2}$:
  1. By definition, we have $\Delta^{\mathcal{I}_{\mathsf{base}}} \subseteq \Delta^{\mathcal{J}_{M_2}} \subseteq \Delta^{\mathcal{I}_{\mathsf{base}}} \uplus \{e^*_1, e^*_2\}$ as desired;
  2. Also by definition, we have $\mathcal{J}_{M_2}|_{\Delta^{\mathcal{I}_{\mathsf{base}}}} = \mathcal{I}_{\mathsf{base}}$;
  3. From interpretations of concepts being inherited from $\mathcal{I}$ that satisfies Condition (b), it follows that indeed $\mathsf{tp}_{\mathcal{J}_{M_2}}(e^*_i) \in T_{\overline{\mathsf{core}}}$ if $e^*_i \in \Delta^{\mathcal{J}_{M_2}}$, for $i \in \{1, 2\}$;
  4. From $\mathcal{I}$, being a model of $\mathcal{K}$, it follows that $\mathcal{J}_{M_2}$ satisfies in particular all $\exists r.A \sqsubseteq B \in \mathcal{T}$ and all $r \sqsubseteq s \in \mathcal{T}$.

- Regarding S_{M_2}, it is the restriction of a saturated specification (up to τ_2, which essentially rename elements), and hence is also saturated. From $\mathcal{I}$ satisfying Condition (c), we obtain that S_{M_2} doesn't contain any complete triple. □

Lemma 17. *In the induction step of the definition of good mosaics, M_2 is a suitable mosaic for M_1 and $e_1 \in (A \sqcap \neg\exists r.B)^{\mathcal{J}_{M_1}}$ with $A \sqsubseteq \exists r.B \in \mathcal{T}$.*

Proof. We verify the three conditions:

1. Interpretations of concepts names being directly imported from $\mathcal{I}$, we immediately have $\mathsf{tp}_{\mathcal{J}_{M_1}}(e_1) = \mathsf{tp}_{\mathcal{J}_{M_2}}(e_1)$;
2. Since we set $e_2' := \mathsf{succ}^{\mathcal{I}}_{r.B}(e_1')$, it follows naturally that $e \in (\exists r.B)^{\mathcal{J}_{M_2}}$;
3. This is immediate as any partial homomorphism from either S_{M_1} (resp. S_{M_2}) is the restriction of a complete homomorphism to $\mathcal{I}$ to $\Delta^{\mathcal{J}_{M_1}}$ (resp. $\Delta^{\mathcal{J}_{M_2}}$), which can thus be restricted back to $\Delta^{\mathcal{J}_{M_2}}$ (resp. $\Delta^{\mathcal{J}_{M_1}}$). In fact, this proves: if $(p, \widehat{p}, h) \in S_{M_2}$, then $(p, \widehat{p}, h') \in S_{M_1}$ where h' is the restriction of h to range $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e_1\}$, but also "the converse", that is: if $(p, \widehat{p}, h) \in S_{M_1}$, then $(p, \widehat{p}, h') \in S_{M_2}$ where h' is the restriction of h to range $\Delta^{\mathcal{I}_{\mathsf{base}}} \cup \{e_1\}$. □

Theorem 2. *CQ evaluation on circumscribed $\mathcal{EL}$ KBs is* 2EXP*-hard w.r.t. combined complexity. This holds even with a single minimized concept name and no fixed concept names.*

Proof. We give a polynomial time reduction from (Boolean) CQ evaluation on $\mathcal{EL}$ KBs with closed concept names, which is 2EXP-hard (Ngo, Ortiz, and Simkus 2016). A *KB with closed concept names* takes the form $\mathcal{K}_\Sigma$ with $\mathcal{K}$ a KB and $\Sigma \subseteq \mathsf{N_C}$ a set of closed concept names. We say that an interpretation $\mathcal{I}$ *respects* Σ if for all $A \in \Sigma$, $d \in A^{\mathcal{I}}$ implies $A(d) \in \mathcal{A}$. Moreover, $\mathcal{I}$ is a *model* of $\mathcal{K}_\Sigma$ if $\mathcal{I}$ is a model of $\mathcal{K}$ and $\mathcal{I}$ respects Σ. Thus, the only instances of a closed concept name are those that are explicitly asserted in the ABox. Now the problem proved to be 2EXP-hard in (Ngo, Ortiz, and Simkus 2016) is: given a KB with closed concept names $\mathcal{K}_\Sigma$ and a Boolean CQ q, decide whether $\mathcal{K}_\Sigma \models q$. We remark that the formalism in (Ngo, Ortiz, and Simkus 2016) also admits closed role names, but these are not used in the hardness proof.

Let $\mathcal{K}_\Sigma$ be an $\mathcal{EL}$ KB with closed concept names, $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, and q a Boolean CQ. We construct a circumscribed $\mathcal{EL}$ KB $\mathsf{Circ_{CP}}(\mathcal{K}')$, with $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$, and a CQ q' such that $\mathcal{K}_\Sigma \models q$ iff $\mathsf{Circ_{CP}}(\mathcal{K}') \models q'$.

A natural first attempt is to use use $\mathcal{K}' = \mathcal{K}$ and $q' = q$, and to minimize the concept names in Σ while letting all other concept names vary. Apart from using more than a single minimized concept name, this does not work since the extension of closed concept names may be too large. As a simple example, consider the empty ABox and the TBox $\top \sqsubseteq \exists r.A$ with $A \in \Sigma$. This KB is unsatisfiable when A is closed, but satisfiable when A is minimized. We thus use a more refined approach.

Let CP be the circumscription pattern that minimizes the fresh concept name M and lets all other symbols vary. In what follows, we assemble $\mathcal{T}'$ and $\mathcal{A}'$ and define q'. For $\mathcal{A}'$, we start from $\mathcal{A}$ and extend with additional assertions. $\mathcal{T}'$ contains the CIs from $\mathcal{T}$ in a modified form, as well as additional CIs, see below for details.

We start with connecting the concept names in Σ to the minimized concept name M:

$$A \sqsubseteq M \quad \text{for all } A \in \Sigma \tag{1}$$

Much of the reduction is concerned with preventing non-asserted instances d of closed concept names $A \in \Sigma$, that is, elements d that satisfy A but are not an individual from $\mathsf{ind}(\mathcal{A})$ such that $A(d) \in \mathcal{A}$. There are three cases to be distinguished.

We first consider the case that $d = a \in \mathsf{ind}(\mathcal{A})$, but $A(d) \notin \mathcal{A}$. We mark such a with a fresh concept name L:

$$\overline{A}(a) \quad \text{for all } a \in \mathsf{ind}(\mathcal{A}) \text{ with } A(a) \notin \mathcal{A} \tag{2}$$

$$A \sqcap \overline{A} \sqsubseteq L \quad \text{for all } A \in \Sigma. \tag{3}$$

where $\overline{A}$ is a fresh concept name for each $A \in \Sigma$.

We next want to achieve that any model $\mathcal{I}$ that satisfies $a \in L^{\mathcal{I}}$ for some ABox individual a also satisfies $\mathcal{I} \models q$ and is thus ruled out as a countermodel against the query being entailed. To this end, we add a copy of q to $\mathcal{A}'$:

$$A(x) \quad \text{for all } A(x) \in \varphi \tag{4}$$

$$r(x, y) \quad \text{for all } r(x, y) \in \varphi \tag{5}$$

assuming w.l.o.g. that $\mathsf{var}(q) \cap \mathsf{ind}(\mathcal{A}) = \emptyset$.

We want the copy of q in $\mathcal{A}'$ to only become 'active' when the concept name L is made true at some individual from $\mathsf{ind}(\mathcal{A})$; otherwise, it should be 'dormant'. The individual names in $\mathsf{ind}(\mathcal{A})$, in contrast, should always be active. To achieve this, we include in $\mathcal{A}'$ the following assertions where X is a fresh concept name indicating activeness and u is a fresh role name:

$$X(a) \quad \text{for all } a \in \mathsf{ind}(\mathcal{A}), \tag{6}$$

$$u(a, x) \quad \text{for all } a \in \mathsf{Ind}(\mathcal{A}) \text{ and } x \in \mathsf{var}(q) \tag{7}$$

We then define the CQ q' to be

$$q' = q \wedge \bigwedge_{x \in \mathsf{var}(q)} X(x),$$

and add to $\mathcal{T}'$:

$$\exists u.L \sqsubseteq X. \tag{8}$$

Elements generated by existential quantifiers on the right-hand side of CIs in $\mathcal{T}$ should of course also be active. Moreover, dormant elements should not trigger CIs from $\mathcal{T}$ as this might generate 'unjustified' existential (thus active) elements. We deal with these issues as follows. For an $\mathcal{EL}$ axiom α, we define the axiom α_X as follows: For an $\mathcal{EL}$ concept C, we inductively define the concept C_X as follows:

$$A_X := X \sqcap A \qquad (\exists r.D)_X := X \sqcap \exists r.D_X$$

$$\top_X := X \qquad (D \sqcap D')_X := D_X \sqcap D'_X$$

We now include in $\mathcal{T}'$ the following relativized version of $\mathcal{T}$:

$$C_X \sqsubseteq D_X \quad \text{for all } C \sqsubseteq D \in \mathcal{T} \tag{9}$$

Note that, when activating the copy of q in $\mathcal{A}'$ by making X true at all its individual names, then the CIs (9) apply also there and may make concept names A with $A \in \Sigma$ true at elements $x \in \mathsf{var}(q)$. Because of (1), this may lead to models in which the copy of q is activated to be incomparable to models in which it is not activated regarding $<_{\mathsf{CP}}$. This can be remedied by adding to $\mathcal{A}'$:

$$M(x) \quad \text{for all } x \in \mathsf{var}(q). \tag{10}$$

The second case of non-asserted instances d of closed concept names is that d is not from $\mathsf{ind}(\mathcal{A}) \cup \mathsf{var}(q)$. The idea is that models with such instances d should be non-minimal. We achieve this by including in $\mathcal{A}'$ a fresh individual t that must satisfy M, but not necessarily anything else. Since any existential instance d of a closed concept name must also satisfy M, we can find a model that is preferred w.r.t. $<_{\mathsf{CP}}$ by using t as a surrogate for d, that is, making true at t exactly the concept names true at d, rerouting all incoming and outgoing role edges from d to t, and then removing d from the extension of all concept and role names. We thus include in $\mathcal{A}'$:

$$M(t). \tag{11}$$

The third and last case of non-asserted instances d of closed concept names is that d is t or an individual in the copy of q in $\mathcal{A}'$. Since these individuals satisfy M, we may make true at them a closed concept name without producing a non-minimal model. To fix this problem, it suffices to prevent the individuals in $\mathsf{var}(q) \cup \{t\}$ to be used as witnesses for existential restrictions in $\mathcal{T}$, as this is the only possible reason for creating non-asserted instances of closed concept names of the described form. It is, however, easy to identify such witnesses as they must be active and satisfy the concept name X. We thus add

$$\overline{X}(a) \quad \text{for all } a \in \mathsf{var}(q) \cup \{t\} \tag{12}$$

$$u(x,a) \quad \text{for all } a \in \mathsf{var}(q) \cup \{t\} \text{ and } x \in \mathsf{var}(q) \tag{13}$$

$$X \sqcap \overline{X} \sqsubseteq L. \tag{14}$$

where $\overline{X}$ is a fresh concept name. This finishes the construction of $\mathcal{T}'$ and $\mathcal{A}'$.

Claim. $\mathcal{K}_\Sigma \models q$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models q'$.

"$\Leftarrow$". Let $\mathcal{K}_\Sigma \not\models q$. Then, there is a model $\mathcal{I}$ of $\mathcal{K}_\Sigma$ such that $\mathcal{I} \not\models q$. W.l.o.g. assume that $\Delta^\mathcal{I} \cap (\mathsf{var}(q) \cup \{t\}) = \emptyset$. We construct an interpretation $\mathcal{J}$ as follows:

$$\begin{aligned}
\Delta^\mathcal{J} &:= \Delta^\mathcal{I} \cup \mathsf{var}(q) \cup \{t\}\\
A^\mathcal{J} &:= A^\mathcal{I} \cup \{x \mid A(x) \in \mathcal{A}'\}\\
r^\mathcal{J} &:= r^\mathcal{I} \cup \{(x,y) \mid r(x,y) \in \mathcal{A}'\}\\
X^\mathcal{J} &:= \Delta^\mathcal{I}\\
M^\mathcal{J} &:= \bigcup_{A \in \Sigma} A^\mathcal{J} \cup \mathsf{var}(q) \cup \{t\}
\end{aligned}$$

for all concept names $A \notin \{X, M\}$ and role names r. From $\mathcal{I} \models \mathcal{K}$ and the construction of $X^\mathcal{J}$, it easily follows that $\mathcal{J} \models \mathcal{K}'$. To show $\mathcal{J} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$, we are left to prove that $\mathcal{J}$ is minimal.

Assume to the contrary that there is a model $\mathcal{J}'$ of $\mathcal{K}'$ with $\mathcal{J}' <_{\mathsf{CP}} \mathcal{J}$. As M is the only minimized concept name, there must be some $a \in M^\mathcal{J}$ with $a \notin M^{\mathcal{J}'}$. We must have $a \notin \mathsf{var}(q) \cup \{t\}$ since $M(x) \in \mathcal{A}'$ for $x \in \mathsf{var}(q) \cup \{t\}$ and $\mathcal{J}'$ is a model of $\mathcal{K}'$. But then, the definition of $M^\mathcal{J}$ implies $a \in A^\mathcal{J}$ with $A \in \Sigma$. By definition of $A^\mathcal{J}$, in turn, $a \in A^\mathcal{I}$ or $A(a) \in \mathcal{A}'$. In the former case, it follows from the fact that $\mathcal{I}$ respects Σ that $A(a) \in \mathcal{A} \subseteq \mathcal{A}'$, and thus $A(a) \in \mathcal{A}'$ in both cases. Now, CI (1) and $a \notin M^{\mathcal{J}'}$ imply that $\mathcal{J}'$ is not a model of $\mathcal{K}'$, a contradiction.

We are left to show that $\mathcal{J} \not\models q'$. Assume to the contrary that there is a homomorphism h from $\mathcal{D}_{q'}$ to $\mathcal{J}$. We derive a contradiction to $\mathcal{I} \not\models q$ by showing that h is also a homomorphism from $\mathcal{D}_q$ to $\mathcal{I}$. First, note that h is a function from $\mathsf{var}(q)$ to $\Delta^\mathcal{I}$ since $\mathsf{var}(q') = \mathsf{var}(q)$ and for each $x \in \mathsf{var}(q')$, we have $X(x) \in q'$ and thus $h(x) \in X^\mathcal{J} = \Delta^\mathcal{I}$. Finally, observe that $\mathcal{I}$ is identical to the restriction of $\mathcal{J}$ to $\Delta^\mathcal{I}$ and the concept and role names that occur in $\mathcal{K}'$ and q'.

"$\Rightarrow$". Assume that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \not\models q'$. Then there is a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$ with $\mathcal{I} \not\models q'$. We show that w.l.o.g. we may assume that $\mathcal{I}$ satisfies several additional conditions that shall prove to be convenient in what follows.

Claim 2. *There is a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$ with $\mathcal{I} \not\models q'$ that satisfies the following properties.*

(a) $a \in A^\mathcal{I}$ *implies* $A(a) \in \mathcal{A}'$ *for all* $a \in \mathsf{var}(q) \cup \{t\}$*,*

(b) $(d,e) \in r^\mathcal{I}$ *implies* $r(d,e) \in \mathcal{A}'$ *for all* $d, e \in \Delta^\mathcal{I}$ *with* $\{d,e\} \cap (\{\mathsf{var}(q)\} \cup \{t\}) \neq \emptyset$*, and*

(c) $(d,e) \in u^\mathcal{I}$ *implies* $u(d,e) \in \mathcal{A}'$ *for all* $d, e \in \Delta^\mathcal{I}$*.*

We may construct the desired $\mathcal{I}$ from any model $\mathcal{I}'$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$ with $\mathcal{I}' \not\models q'$, as follows:

$$\begin{aligned}
\Delta^\mathcal{I} &:= \Delta^{\mathcal{I}'}\\
A^\mathcal{I} &:= A^{\mathcal{I}'} \setminus \{a \in \mathsf{var}(q) \cup \{t\} \mid A(a) \notin \mathcal{A}'\}\\
u^\mathcal{I} &:= \{(x,y) \mid u(x,y) \in \mathcal{A}'\}\\
r^\mathcal{I} &:= r^{\mathcal{I}'} \setminus \{(x,y) \mid r(x,y) \notin \mathcal{A}' \text{ and}\\
&\qquad\qquad \{x,y\} \cap (\mathsf{var}(q) \cup \{t\}) \neq \emptyset\}
\end{aligned}$$

for all concept names A and role names $r \neq u$. Obviously, $\mathcal{I} \not\models q'$, $\mathcal{I} \models \mathcal{A}'$ and Points (a) to (b) hold by construction. To show that $\mathcal{I} \models \mathcal{T}'$, note the following:

- $\mathcal{I}$ obviously satisfies the CIs (1) by construction and the fact that $\mathcal{I}'$ is a model of $\mathcal{K}'$.
- $\mathcal{I}$ satisfies CIs (3), (8) and (14) as $L^{\mathcal{I}'}$ must be empty—otherwise Assertions (7) and CIs (8) activate the copy of q in $\mathcal{A}'$ and thus there is a trivial homomorphism from q' to $\mathcal{I}'$, contradicting $\mathcal{I}' \not\models q'$.

- For the CIs (9), observe that $X^{\mathcal{I}'} \cap (\mathsf{var}(q) \cup \{t\}) = \emptyset$ following the same argument and using Assertions (12) and CI (14). Hence

$$X^{\mathcal{I}'} \cap A^{\mathcal{I}'} = X^{\mathcal{I}} \cap A^{\mathcal{I}}$$

and

$$(X^{\mathcal{I}'} \times X^{\mathcal{I}'}) \cap r^{\mathcal{I}'} = (X^{\mathcal{I}} \times X^{\mathcal{I}}) \cap r^{\mathcal{I}}$$

for all concept names A and role names $r \neq u$. Consequently, $C_X^{\mathcal{I}'} = C_X^{\mathcal{I}}$. Together with the fact that $\mathcal{I}' \models C_X \sqsubseteq D_X$, $\mathcal{I}$ must also satisfy $C_X \sqsubseteq D_X$.
- We are left to show that $\mathcal{I}$ is minimal; however, this is easy to see as $\mathcal{I}'$ is minimal and $\mathcal{I} \leq_{\mathsf{CP}} \mathcal{I}'$.

This finishes the proof of Claim 2.

We aim to show that the interpretation $\mathcal{J}$, obtained as the restriction of $\mathcal{I}$ to domain $X^{\mathcal{I}}$, is a model of $\mathcal{K}_\Sigma$ with $\mathcal{J} \not\models q$. We assume that the properties from Claim 2 are satisfied, and Property (a) yields $\Delta^{\mathcal{J}} \cap (\mathsf{var}(q) \cup \{t\}) = \emptyset$. Assertion (6) yields $\mathsf{ind}(\mathcal{A}) \subseteq \Delta^{\mathcal{J}}$ and since $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{I}$ is a model of $\mathcal{A}'$, $\mathcal{J}$ must be a model of $\mathcal{A}$. Since $\mathcal{J}$ is the restriction of $\mathcal{I}$ to $X^{\mathcal{I}}$ and $\mathcal{I}$ satisfies CIs (9), $\mathcal{J}$ must satisfy $\mathcal{T}$ as well.

To see that $\mathcal{J} \not\models q$, assume to the contrary that there is a homomorphism h from q to $\mathcal{J}$. As $\mathcal{J}$ is the restriction of $\mathcal{I}$ to $X^{\mathcal{I}}$ and q' is the extension of q that adds atoms $X(x)$ for all variables x, h must also be a homomorphism from q' to $\mathcal{I}$. This contradicts $\mathcal{I} \not\models q'$.

We are left to argue that $\mathcal{J}$ respects Σ. Assume to the contrary that there is some $A \in \Sigma$ and $d_0 \in A^{\mathcal{J}}$ with $A(d_0) \notin \mathcal{A}$. We have $d_0 \in X^{\mathcal{I}}$ by construction of $\mathcal{J}$. First assume that $d_0 \in \mathsf{ind}(\mathcal{A})$. Then (2), (3), (7), and (8) activate the copy of q in $\mathcal{A}'$, giving a homomorphism from q to $\mathcal{I}$, contradicting $\mathcal{I} \not\models q'$.

The more laborious case is $d_0 \notin \mathsf{ind}(\mathcal{A})$. As we show in the following, we may then construct a model $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}'$, contradicting the minimality of $\mathcal{I}$.

Intuitively, in the construction of $\mathcal{I}'$ we replace d_0 with t by making true at t all concept names that are true at d_0, making false all concept names at d_0, and "redirecting" all incoming and outgoing edges of d_0 to t. This forces us to make X true at t, and by (8) and (12) to (14) we must activate the copy of q in $\mathcal{A}'$. But then the CIs (9) apply to that copy. To satisfy them, we make all concept and role names true in the copy of q. In detail:

$$\begin{aligned}
\Delta^{\mathcal{I}'} &:= \Delta^{\mathcal{I}} \\
A^{\mathcal{I}'} &:= (A^{\mathcal{I}} \setminus \{d_0\}) \cup \{t \mid d_0 \in A^{\mathcal{I}}\} \cup \mathsf{var}(q) \\
L^{\mathcal{I}'} &:= \Delta^{\mathcal{I}} \\
r^{\mathcal{I}'} &:= (r^{\mathcal{I}} \setminus \{(d,e) \mid d_0 \in \{d,e\}\}) \\
&\quad \cup \{(e,t) \mid (e,d_0) \in r^{\mathcal{I}} \text{ and } e \neq d_0\} \\
&\quad \cup \{(t,e) \mid (d_0,e) \in r^{\mathcal{I}} \text{ and } e \neq d_0\} \\
&\quad \cup \{(t,t) \mid (d_0,d_0) \in r^{\mathcal{I}}\} \\
&\quad \cup (\mathsf{var}(q) \times \mathsf{var}(q))
\end{aligned}$$

for all concept names $A \neq L$ and role names r.

Because $\mathcal{I}$ satisfies Assertions (10) and (11) and $d_0 \in \Delta^{\mathcal{J}}$ cannot be from $\mathsf{var}(q) \cup \{t\}$, we have $M^{\mathcal{I}'} = M^{\mathcal{I}} \setminus \{d_0\}$ by construction of $\mathcal{I}'$. Thus $\mathcal{I}' <_{\mathsf{CP}} \mathcal{I}$, as desired. It remains to prove that $\mathcal{I}'$ is a model of $\mathcal{K}'$. We have $\mathcal{I}' \models \mathcal{A}'$ because $\mathcal{I} \models \mathcal{A}'$ and the restrictions of $\mathcal{I}$ and of $\mathcal{I}'$ to $\mathsf{ind}(\mathcal{A}')$ are identical. Before we argue that $\mathcal{I}'$ is a model of $\mathcal{T}'$, we show that the redirection does not "damage" concept extensions. We denote with $\mathsf{sub}(\mathcal{T})$ the set of all concepts that occur in $\mathcal{T}$, closed under subconcepts.

Claim 3. *The following hold for all $C \in \mathsf{sub}(\mathcal{T})$:*

1. $t \in C_X^{\mathcal{I}'}$ *iff* $d_0 \in C_X^{\mathcal{I}}$, *and*
2. $x \in C_X^{\mathcal{I}'}$ *iff* $x \in C_X^{\mathcal{I}}$ *for all* $x \in \Delta^{\mathcal{I}} \setminus (\mathsf{var}(q) \cup \{t\} \cup \{d_0\})$.

We prove Points 1 and 2 simultaneously by induction on the structure of C, making a case distinction according to the topmost operator in C:

- $C = \top$. Point 1 holds since $t \in X^{\mathcal{I}'}$ and $d_0 \in X^{\mathcal{I}}$ and Point 2 holds since by construction of $\mathcal{I}'$, $x \in X^{\mathcal{I}'}$ iff $x \in X^{\mathcal{I}}$ for all $x \in \Delta^{\mathcal{I}} \setminus (\mathsf{var}(q) \cup \{t\} \cup \{d_0\})$.
- $C = A$. Immediate by construction of $\mathcal{I}'$ and Point (a) of Claim (2).
- $C = C_1 \sqcap C_2$. Straightforward using the semantics and the induction hypothesis.
- $C = \exists r.D$. Note that Point (b) of Claim (2) and the construction of $r^{\mathcal{I}'}$ entail the following for all $x, y \in \Delta^{\mathcal{I}}$:

$$(t,y) \in r^{\mathcal{I}'} \quad \text{iff} \quad (d_0,y) \in r^{\mathcal{I}} \tag{†}$$

$$(x,t) \in r^{\mathcal{I}'} \quad \text{iff} \quad (x,d_0) \in r^{\mathcal{I}} \tag{‡}$$

For the "⇒"-direction of Point 1, let $t \in (\exists r.D_X)^{\mathcal{I}'}$, i.e., let there be some $y \in D_X^{\mathcal{I}'}$ with $(t,y) \in r^{\mathcal{I}'}$. Via (†), we conclude that $(d_0,y) \in r^{\mathcal{I}}$, and Point 2 of the induction hypothesis yields $y \in D_X^{\mathcal{I}}$—together, $d_0 \in C_X^{\mathcal{I}}$. The "⇐"-direction of Point 1 is analogous.

For the "⇒"-direction of Point 2, let $x \in (\exists r.D_X)^{\mathcal{I}'}$ such that $x \notin \mathsf{var}(q) \cup \{t\} \cup \{d_0\}$. Then, there must be some $y \in D_X^{\mathcal{I}'}$ with $(x,y) \in r^{\mathcal{I}'}$. The construction of $r^{\mathcal{I}'}$ implies that either $y \neq d_0$ with $(x,y) \in r^{\mathcal{I}}$ or $y = t$ hold. In case of $y \neq d_0$ with $(x,y) \in r^{\mathcal{I}}$, we furthermore have $y \notin \mathsf{var}(q) \cup \{t\}$ by Point (b). Therefore, Point 2 of the induction hypothesis implies $y \in D_X^{\mathcal{I}}$; ergo $x \in C_X^{\mathcal{I}}$. In case of $y = t$, (‡) implies $(x,d_0) \in r^{\mathcal{I}}$, and Point 1 of the induction hypothesis yields $d_0 \in D_X^{\mathcal{I}}$—together, we again have $x \in C_X^{\mathcal{I}}$.

For the "⇐"-direction of Point 2, let $x \in C_X^{\mathcal{I}}$. Then there is some $y \in D_X^{\mathcal{I}}$ with $(x,y) \in r^{\mathcal{I}}$. By Point (b) of Claim (2), we have $y \notin \mathsf{var}(q) \cup \{t\}$. Ergo, if $y \neq d_0$, $y \in D_X^{\mathcal{I}}$ follows from Point 2 of the induction hypothesis, and $(x,y) \in r^{\mathcal{I}'}$ stems from the construction of $r^{\mathcal{I}'}$—together, $x \in C_X^{\mathcal{I}'}$. If otherwise $y = d_0$, we have $(x,t) \in r^{\mathcal{I}'}$ by (‡) and $t \in D_X^{\mathcal{I}'}$ by Point 1 of the induction hypothesis, i.e., $x \in C_X^{\mathcal{I}'}$.

This finishes the proof of Claim 3.

We will also rely on the following, which follows directly from the construction of $\mathcal{I}'$.

Observation 1. *$\mathsf{var}(q) \subseteq C_X^{\mathcal{I}'}$ for all $C \in \mathsf{sub}(\mathcal{T})$.*

We are now in the position to prove $\mathcal{I}' \models \mathcal{T}'$. To see that $\mathcal{I}'$ satisfies the CIs (1), let $x \in A^{\mathcal{I}'}$ with $A \in \Sigma$. If $x \in \mathsf{var}(q) \cup \{t\}$, then $x \in M^{\mathcal{I}}$ due to Assertions (10) and (11), and thus $x \in M^{\mathcal{I}'}$ by construction of $\mathcal{I}'$. Otherwise, we have $x \in A^{\mathcal{I}}$ by construction of $A^{\mathcal{I}'}$, yielding $x \in M^{\mathcal{I}}$ via Assertions (1). We must have $x \neq d_0$ and thus $x \in M^{\mathcal{I}'}$ by construction of $\mathcal{I}'$. $\mathcal{I}'$ obviously satisfies CIs (3) and (14) by $L^{\mathcal{I}'} = \Delta^{\mathcal{I}}$.

For CI (9), let $x \in C_X^{\mathcal{I}'}$ for some $C \sqsubseteq D \in \mathcal{T}$. We know that $x \neq d_0$ as $d_0 \notin X^{\mathcal{I}'}$. If $x \in \mathsf{var}(q)$, then Claim 1 implies $x \in D_X^{\mathcal{I}'}$. If, otherwise, $x \in \Delta^{\mathcal{I}} \setminus (\mathsf{var}(q) \cup \{d_0\})$, then Claim 3 and the fact that $\mathcal{I}$ satisfies $C_X \sqsubseteq D_X$ clearly yields $x \in D_X^{\mathcal{I}}$.

For CIs (8), let $x \in \exists(u.L)^{\mathcal{I}'}$. Point (c) of Claim (2) and the construction of $\mathcal{A}'$ entail that $x \in \mathsf{var}(q)$. By construction of $\mathcal{I}'$, we get $x \in X^{\mathcal{I}'}$ as desired.

□

C Proofs for Section 3.3

Lemma 7. *Let $q(\bar{x})$ be a UCQ and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. If $\mathcal{I}$ is a countermodel against $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q(\bar{a})$, then so is $\mathcal{I}'/\!\sim_{|q|+1}$.*

We prove $\mathcal{J} = \mathcal{I}'/\!\sim_{|q|+1}$ is indeed a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ that doesn't admit a homomorphism for the query. The key to proving the latter is to exhibit suitable local homomorphisms from $\mathcal{J} = \mathcal{I}'/\!\sim_{|q|+1}$ back to $\mathcal{I}'$. The existence of a homomorphism for $p \in q$ in $\mathcal{J}$ would then contradicts $\mathcal{I}'$ being a countermodel of q in the first place. Indeed, a homomorphism of $p \in q$ in $\mathcal{J}$ would map each connected component C of p into a $|q|$-neighborhood $\mathcal{N}_{|q|}^{\mathcal{J},[\Delta^{\mathcal{I}}_{\mathsf{base}}]}([c])$, for some $c \in \Delta^{\mathcal{I}'}$ and where $\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ stands for the set $\{\overline{e} \mid e \in \Delta^{\mathcal{I}}_{\mathsf{base}}\}$. By exhibiting a homomorphism $\rho_c : \mathcal{N}_{|q|}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\overline{c}) \to \mathcal{N}_{|q|}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(c)$ such that $\rho_c^{-1}(\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}) \subseteq \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, we can find a homomorphism of C in $\mathcal{I}'$. Such homomorphisms for p's connected components together form a homomorphism of the full q in $\mathcal{I}'$.

Except for the use of Lemma 2, concerned with circumscription, we roughly follow (Manière 2022), with slight simplifications as we are not considering negative role inclusions. For all n, all $c \in \Delta^{\mathcal{I}'}$, and all $d \in \mathcal{N}_n^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$, we denote $w^d_{n,c}$ the (unique) word such that $d = p_{n,c} w^d_{n,c}$. Let us first formulate two remarks concerning the constructed interpretation $\mathcal{J}$.

Remark 2. *Combining Conditions 1 and 2.(b) from the definition of $\sim_n$, gives: if $d_1 \sim_n d_2$, then $\mathsf{tp}_{\mathcal{I}'}(d_1) = \mathsf{tp}_{\mathcal{I}'}(d_2)$.*

Remark 3. *If $d_1 \sim_n d_2$, then $d_1 \sim_m d_2$ for any $m \leq n$.*

We now define homomorphisms ρ_c, mentioned in the proof sketch, inductively on $\mathcal{N}_k^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\overline{c})$ with k increasing from 0 to $|q|$. Starting from the element $\overline{c} \in \mathcal{N}_0^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\overline{c})$, we can naturally carry it back as $\rho_c(\overline{c}) = c \in \mathcal{N}_0^{\mathcal{I}',\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(c)$. Assume now that we have defined $\rho_c(\overline{d})$ for some $\overline{d} \in \mathcal{N}_n^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\overline{c})$ and that we are moving further to an element $\overline{e} \in \mathcal{N}_{n+1}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\overline{c})$ along an edge $(\overline{d},\overline{e})$ in $\mathcal{J}$. In the case of $\overline{e} \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, the following lemma produces a candidate $\rho_c(\overline{e})$, namely e', which is to $\rho_c(\overline{d})$, namely d', what $\overline{e}$ is to $\overline{d}$.

Lemma 18. *Given two elements $\overline{d},\overline{e} \in \Delta^{\mathcal{J}} \setminus \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, if there exists a role p from $\mathsf{N_R}^{\pm}$ such that $(\overline{d},\overline{e}) \in p^{\mathcal{J}}$, then there exists a unique element $r.B \in \Omega$ such that one of the two following conditions is satisfied:*

r^+. *$|e| \equiv |d| + 1 \mod 2|q| + 3$, $w^e_{|q|+1,e} = w^d_{|q|,d} r.B$ and $\mathcal{T} \models r \sqsubseteq p$. Furthermore, for all $d' \sim_k d$, the element $e' = d'r.B$ belongs to $\Delta^{\mathcal{I}'}$ and satisfies $e' \sim_{k-1} e$.*

r^-. *$|d| \equiv |e| + 1 \mod 2|q| + 3$, $w^d_{|q|+1,d} = w^e_{|q|,e} r.B$ and $\mathcal{T} \models r^- \sqsubseteq p$. Furthermore, for all $d' \sim_k d$, we have e' such that $d' = e'r.B$ and the prefix e' satisfies $e' \sim_{k-1} e$.*

Proof. Notice the two conditions are mutually exclusive: $|e| \equiv |d| + 1 \mod 2|q| + 3$ and $|d| \equiv |e| + 1 \mod 2|q| + 3$ would imply $0 \equiv 2 \mod 2|q| + 3$, which is impossible as $2|q| + 3 > 2$. Furthermore, in each case $r.B$ is defined as the last letter of the word $w^e_{|q|+1,e}$ (resp $w^d_{|q|+1,d}$), which is unique and does not depends on the choice of e (resp d) nor on p. This proves the uniqueness.

We now focus on the existence and the additional property. From the definition of $p^{\mathcal{J}}$, there exist $(d_0, e_0) \in p^{\mathcal{I}'}$ such that $\overline{d_0} = \overline{d}$ and $\overline{e_0} = \overline{e}$. Recall $\overline{d},\overline{e} \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, hence $d_0, e_0 \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$. Therefore the definition of $p^{\mathcal{I}'}$, yields two cases:

- We have $e_0 = d_0 r.B$ with $\mathcal{T} \models r \sqsubseteq p$. It follows that $|e_0| = |d_0| + 1 \mod 2|q| + 3$ and $w^{e_0}_{|q|+1,e_0} = w^{d_0}_{|q|+1-1,d_0} r.B$, immediately yielding the same properties for d and e as $(\overline{d_0},\overline{e_0}) = (\overline{d},\overline{e})$.
  Consider now d' and $1 \leq k \leq |q| + 1$ s.t. now $d' \sim_k d$. Transitivity gives $d' \sim_k d_0$, and we have in particular $w^{d'}_{k,d'} = w^{d_0}_{k,d_0}$. Recall that $e_0 = d_0 r.B$, hence Condition 2.(a) from $d' \sim_k d_0$ ensures $d'r.B$ is well-defined. Notice it is now sufficient to prove $d'r.B \sim_{k-1} e_0$: that is because $\overline{e} = \overline{e_0}$, hence transitivity will conclude the proof. It should be clear that $w^{d'r.B}_{k-1,d'r.B} = w^{e_0}_{k-1,e_0}$ and $|d'r.B| \equiv |e_0| \mod 2|q| + 3$. To prove Condition 2, it suffices to remark that $e_0 = d_0 r.B$ ensures that the characterization via paths of the k-neighborhood of d_0 fully determines the characterization of the $(k-1)$ neighborhood of e_0. But since the former coincides with the characterization of the k-neighborhood of d' (using Condition 2 of $\overline{d_0} = \overline{d'}$), that fully decides the characterization of the $(k-1)$-neighborhood of e', we are done.
- We have $d_0 = e_0 r.B$ with $\mathcal{T} \models r^- \sqsubseteq p$. It follows that $|d_0| \equiv |e_0| + 1 \mod 2|q| + 3$ and $w^{d_0}_{|q|+1,d_0} =$

$w^{e_0}_{|q|+1-1,e_0} r.B$, immediately yielding the same properties for d and e as $(\overline{d_0}, \overline{e_0}) = (\overline{d}, \overline{e})$.

Let now $1 \leq k \leq |q| + 1 + 1$ be an integer and $d' \sim_k d$. Transitivity gives $d' \sim_k d_0$, and we have in particular $w^{d'}_{1,d'} = w^{d_0}_{1,d_0} = r.B$ (it is here important to have $k \geq 1$!). That is d' ends by $r.B$, and therefore we can indeed have prefix e' such that $d' = e'r.B$. □

Notice the "strength" of the equivalence relation $\sim_k$ between $\overline{e}$ and $\rho_c(\overline{e})$ decreases as we move further in the neighbourhood of $\overline{c}$. However, since we start from $\rho_c(\overline{c}) := c \sim_{|q|+1} c$ and explore a $|q|$-neighbourhood, the index remains at least 1. This is essential as $\sim_1$ notably encodes relations to elements of $\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ as required by Condition 2.(c).

It remains to abstract from the particular choice of $\overline{d}$, which is likely not to be the only element of $\mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c})$ connected to $\overline{e}$. Taking a closer look at Lemma 18, we observe that $\rho_c(\overline{e})$, that is e', is obtained either by adding a letter to $\rho_c(\overline{d})$, that is d', or by removing the last letter of $\rho_c(\overline{d})$, and that these letters coincide with those in the suffixes of elements d and e. Therefore, when moving from $\overline{c}$ to $\overline{e}$ and ignoring self-cancelling steps, each added letter must appear in the suffix of e and, similarly, each removed letter must appear in the suffix of c.

The challenge is therefore to quantify the number of additions and removals to build $\rho_c(\overline{e})$ directly from c and $\overline{e}$. The next definition captures the relative difference of letters between $\overline{c}$ and $\overline{e}$, encoded in $|c|$ and $|e| \mod 2|q| + 3$.

Definition 5. *Let $\overline{c} \in \Delta^{\mathcal{J}}$ and $n \leq |q|$. The* relative depth *of $\overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c})$ from $\overline{c}$ is the integer $\delta_{\overline{c}}(\overline{e}) \in [-n, n]$ such that $|e| \equiv |c| + \delta_{\overline{c}}(\overline{e}) \mod 2|q| + 3$.*

Remark 4. *By induction on $n \leq |q|$, it is straightforward to see that $\delta_{\overline{c}}(\overline{e})$ is well defined. Unicity is ensured by $\delta_{\overline{c}}(\overline{e}) \leq n \leq |q|$. A consequence of Lemma 18 is that for the smallest $n \leq |q|$ such that $\overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c})$ we have $\delta_{\overline{c}}(\overline{e}) = n \mod 2$.*

We can now identify how many additions and removals cancelled each other. Indeed, if it takes n steps to reach $\overline{e}$ from $\overline{c}$, with relative difference of $\delta := \delta_{\overline{c}}(\overline{e})$, then $n - |\delta|$ is the length of the self-cancelling path, hence: $\frac{n-|\delta|}{2}$ cancelled additions and $\frac{n-|\delta|}{2}$ cancelled removals. Therefore, the actual number of additions is $\frac{n-|\delta|}{2} + \delta$ if $\delta \geq 0$, or $\frac{n-|\delta|}{2}$ if $\delta \leq 0$, that is in both cases $\frac{n+\delta}{2}$. Similarly we obtain $\frac{n-\delta}{2}$ for the actual number of removals. The next theorem formalizes all these intuitions: $\rho_{n,c}(\overline{e})$ (in non-trivial cases) is obtained by removing the $\frac{n-\delta}{2}$ last letters of c and keeping the $\frac{n+\delta}{2}$ last letters from the suffix of e. It is then a technicality to verify these syntactical operations on words make sense in the domain of $\mathcal{I}'$.

Lemma 19. *For all $c \in \Delta^{\mathcal{I}'}$ and all $n \leq |q|$, the following mapping $\rho_{n,c}(\overline{e})$:*

$$\mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c}) \to \mathcal{N}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}_n(c)$$

$$\overline{e} \mapsto \begin{cases} \rho_{n-1,c}(\overline{e}) & \text{if } \overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{n-1}(\overline{c}) \\ e & \text{if } \overline{e} \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}} \\ p_{\frac{n-\delta_{\overline{c}}(\overline{e})}{2},c}\, w^{e}_{\frac{n+\delta_{\overline{c}}(\overline{e})}{2},e} & \text{otherwise} \end{cases}$$

is a homomorphism satisfying $\rho_{n,c}(\overline{e}) \sim_{|q|+1-n} e$ and $\rho^{-1}_{n,c}(\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}) \subseteq \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$.

Proof. Let $c \in \Delta^{\mathcal{I}'}$. We proceed by induction on $n \leq |q|$ and prove along a technical statement. Property $\rho_{n,c}(\overline{e}) \sim_{|q|+1-n} e$ already ensures $w^{\rho_{n,c}(\overline{e})}_{|q|+1-n,\rho_{n,c}(\overline{e})} = w^{e}_{|q|+1-n,e}$; we reinforce this latter fact as follows. If $e \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c}) \setminus \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{n-1}(\overline{c})$, then:

$$w^{\rho_{n,c}(\overline{e})}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},\rho_{n,c}(\overline{e})} = w^{e}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},e} \qquad (*)$$

It is indeed a stronger statement since $-n \leq \delta_{\overline{c}}(\overline{e}) \leq n$ leads to $0 \leq \frac{n-\delta_{\overline{c}}(\overline{e})}{2} \leq n$, hence $|q| + 1 - n \leq |q| + 1 - \frac{n-\delta_{\overline{c}}(\overline{e})}{2}$. Property $*$ therefore provides a more precise information about the suffix of $\rho_{n,c}e$.

Base case: $n = 0$. Let $\overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_0(\overline{c})$, hence $\overline{e} = \overline{c}$. If $\overline{c} \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, then $\rho_{0,c}e = e = c$. Otherwise we have $\delta_{\overline{c}}(\overline{e}) = 0$, hence $\rho_{0,c}e = p_{0,c}w^{c}_{0,c} = c$. In both cases $\rho_{0,c}e = c$, and it is straightforward that all the desired properties hold. In particular, agreeing that $\mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{-1}(\overline{c})$ can reasonably be set to $\emptyset$, our technical statement holds.

Induction case. Assume the statement holds for $0 \leq n - 1 < |q|$. Let $\overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_n(\overline{c})$. If $\overline{e} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{n-1}(\overline{c})$, then the induction hypothesis applies directly on $\overline{e}$ and provides (stronger versions of) the desired properties. Otherwise, we have by definition of neighbourhoods an element $\overline{d} \in \mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{n-1}(\overline{c})$, not belonging to $\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ nor to $\mathcal{N}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}_{n-2}(\overline{c})$, and a role $p \in \mathsf{N}_{\mathsf{R}}^{\pm}$ such that $(\overline{d}, \overline{e}) \in p^{\mathcal{J}}$. We apply the induction hypothesis on $\overline{d}$, which gives $\rho_{n-1,c}(\overline{d}) = p_{\frac{n-1-\delta_{\overline{c}}(\overline{d})}{2},d}\, w^{d}_{\frac{n-1+\delta_{\overline{c}}(\overline{d})}{2},d}$ since $\overline{d} \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$. We further distinguish between $\overline{e} \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\overline{e} \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$, the latter subcase yielding two subcases by applying Lemma 18 and distinguishing between Cases $\mathrm{r}^+$and r^-. We have therefore three cases to treat.

$\overline{e} \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$**.** We have $\rho_{n,c}(\overline{e}) = e$ and the only non-trivial property to prove is that $e \in \mathcal{N}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\mathsf{base}}}_n(c)$. Recall the induction hypothesis ensures in particular $\rho_{n-1,c}(\overline{d}) \sim_1 d$. Condition 2.(c) from the definition of $\sim_1$ ensures $(\rho_{n-1,c}(\overline{d}), e) \in p^{\mathcal{I}'}$, which provides the desired property.

$\mathbf{r^+}$**.** Case r^+ensures $|e| = |d| + 1 \mod 2|q| + 3$, hence $\delta_{\overline{c}}(\overline{e}) = \delta_{\overline{c}}(\overline{d}) + 1$, and $w^{e}_{|q|+1,e} = w^{d}_{|q|+1-1,d} r.B$. Therefore, our element $\rho_{n,c}(\overline{e})$ of interest simplifies as:

$$\begin{aligned}\rho_{n,c}(\overline{e}) &= p_{\frac{n-\delta_{\overline{c}}(\overline{e})}{2},c}\, w^{e}_{\frac{n+\delta_{\overline{c}}(\overline{e})}{2},e}\\ &= p_{\frac{n-(\delta_{\overline{c}}(\overline{d})+1)}{2},c}\, w^{e}_{\frac{n+(\delta_{\overline{c}}(\overline{d})+1)}{2},e}\\ &= p_{\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},c}\, w^{e}_{\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2}+1,e}\\ &= p_{\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},c}\, w^{d}_{\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2},d}\, r.B\\ &= \rho_{n-1,c}(\overline{d})r.B,\end{aligned}$$

which is well-defined and satisfies $\rho_{n,c}(\overline{e}) \sim_{|q|+1-n} e$ from Lemma 18. Recalling that the induction hypothesis gives $\rho_{n-1,c}(\overline{d}) \in \mathcal{N}_{n-1}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\text{base}}}(c)$, it follows that $\rho_{n,c}(\overline{e}) \in \mathcal{N}_{n}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\text{base}}}(c)$. Furthermore, notice that $\overline{e}$ and $\overline{d}$ satisfy all conditions of our additional statement. Since in Case r$^+$we have $\mathcal{T} \models r \sqsubseteq p$, reusing $\rho_{n,c}(\overline{e}) = \rho_{n-1,c}(\overline{d})r.B$ immediately yields $(\rho_{n-1,c}(\overline{d}), \rho_{n,c}(\overline{e})) \in p^{\mathcal{I}'}$.

Checking that Property $*$ holds is now a technicality, and recall that since $d \in \mathcal{N}_{n-1}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c}) \setminus \mathcal{N}_{n-2}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c})$, we can apply it to d by induction hypothesis. We hence have:

$$\begin{aligned}&w^{\rho_{n,c}(\overline{e})}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},\rho_{n,c}(\overline{e})}\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2}-1,\rho_{n-1,c}(\overline{d})} r.B\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{(n-1)+1-(\delta_{\overline{c}}(\overline{d})+1)}{2}-1,\rho_{n-1,c}(\overline{d})} r.B\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2}-1,\rho_{n-1,c}(\overline{d})} r.B\\ &\quad= w^{d}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2}-1,d} r.B\\ &\quad= w^{e}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},e}.\end{aligned}$$

r$^-$. Case r$^-$ensures $|e| = |d| - 1 \mod 2|q|+3$, hence $\delta_{\overline{c}}(\overline{e}) = \delta_{\overline{c}}(\overline{d}) - 1$, and $w^{d}_{|q|+1,d} = w^{e}_{|q|+1-1,e}r.B$. By induction hypothesis, element $\rho_{n-1,c}(\overline{d}) = p_{\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},d} w^{d}_{\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2},d}$ is well-defined. Notice Property $*$ on d (which, again can be applied as $d \in \mathcal{N}_{n-1}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c}) \setminus \mathcal{N}_{n-2}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c})$) gives more precise information on the suffix of $\rho_{n-1,c}(\overline{d})$ than the definition of $\rho_{n-1,c}(\overline{d})$, because $n \leq |q|+1$ leads to $\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2} + 1 \leq |q|+1 - \frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2}$. Therefore, $w^{d}_{\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2}+1,d}$ is itself a suffix of $w^{d}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},d}$, which equals $w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},\rho_{n-1,c}(\overline{d})}$. Hence we obtain:

$$\begin{aligned}\rho_{n-1,c}(\overline{d}) &= p_{\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2}+1,d}\, w^{d}_{\frac{(n-1)+\delta_{\overline{c}}(\overline{d})}{2}+1,d}\\ &= p_{\frac{n-\delta_{\overline{c}}(\overline{e})}{2},d}\, w^{d}_{\frac{n+\delta_{\overline{c}}(\overline{e})}{2}+1,d}\\ &= p_{\frac{n-\delta_{\overline{c}}(\overline{e})}{2},d}\, w^{e}_{\frac{n+\delta_{\overline{c}}(\overline{e})}{2},e}\, r.B\\ &= \rho_{n,c}(\overline{e})r.B\end{aligned}$$

Lemma 18 now ensures $\rho_{n,c}(\overline{e}) \sim_{|q|+1-n} e$. Recalling that the induction hypothesis gives $\rho_{n-1,c}(\overline{d}) \in \mathcal{N}_{n-1}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\text{base}}}(c)$, it follows that $\rho_{n,c}(\overline{e}) \in \mathcal{N}_{n}^{\mathcal{I}',\Delta^{\mathcal{I}}_{\text{base}}}(c)$. Furthermore, notice that $\overline{e}$ and $\overline{d}$ satisfy all conditions of our additional statement. Since in Case r$^-$we have $\mathcal{T} \models r^- \sqsubseteq p$, reusing $\rho_{n-1,c}(\overline{d}) = \rho_{n,c}(\overline{e})r.B$ immediately yields $(\rho_{n-1,c}(\overline{d}), \rho_{n,c}(\overline{e})) \in p^{\mathcal{I}'}$.

Again, we check Property $*$ holds:

$$\begin{aligned}&w^{\rho_{n,c}(\overline{e})}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},\rho_{n,c}(\overline{e})} r.B\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2}+1,\rho_{n-1,c}(\overline{d})}\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{(n-1)+1-(\delta_{\overline{c}}(\overline{d})-1)}{2}+1,\rho_{n-1,c}(\overline{d})}\\ &\quad= w^{\rho_{n-1,c}(\overline{d})}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},\rho_{n-1,c}(\overline{d})}\\ &\quad= w^{d}_{|q|+1-\frac{(n-1)-\delta_{\overline{c}}(\overline{d})}{2},d}\\ &\quad= w^{e}_{|q|+1-\frac{n-\delta_{\overline{c}}(\overline{e})}{2},e} r.B\end{aligned}$$

We now verify that $\rho_{n,c}$ is a homomorphism.

- Let $\overline{u} \in A^{\mathcal{J}} \cap \mathcal{N}_{n}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c})$. By definition of $A^{\mathcal{J}}$, we have $e \in A^{\mathcal{I}'}$. Since $n \leq |q|$ we have $\rho_{n,c}(\overline{u}) \sim_1 e$, hence applying Remark 2 we obtain $\rho_{n,c}(\overline{u}) \in A^{\mathcal{I}'}$.
- Let $(\overline{u},\overline{v}) \in r^{\mathcal{J}} \cap (\mathcal{N}_{n}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c}) \times \mathcal{N}_{n}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c}))$. If $\overline{u} \in \overline{\Delta^{\mathcal{I}}_{\text{base}}}$ or $\overline{v} \in \overline{\Delta^{\mathcal{I}}_{\text{base}}}$, then Condition 2.(c) from the definition of $\sim_1$ applies on $\rho_{n,c}(\overline{u})$ or on $\rho_{n,c}(\overline{v})$ (recall $\rho_{n,c}(\overline{u}) \sim_1 u$ and $\rho_{n,c}(\overline{v}) \sim_1 v$) and gives $(\rho_{n,c}(\overline{u}), \rho_{n,c}(\overline{v})) \in r^{\mathcal{J}}$. Otherwise $\overline{u} \notin \overline{\Delta^{\mathcal{I}}_{\text{base}}}$ and $\overline{v} \notin \overline{\Delta^{\mathcal{I}}_{\text{base}}}$. Let n_1, n_2 be the minimum integers such that $\overline{u} \in \mathcal{N}_{n_1}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c})$ and $\overline{v} \in \mathcal{N}_{n_2}^{\mathcal{J},\overline{\Delta^{\mathcal{I}}_{\text{base}}}}(\overline{c})$. Since $(\overline{u},\overline{v}) \in r^{\mathcal{J}}$, we have $n_1 - n_2 \in \{-1,0,1\}$. Definitions of $\delta_{\overline{c}}(\overline{u})$ and $\delta_{\overline{c}}(\overline{v})$ lead to $|u| - |v| \equiv \delta_{\overline{c}}(\overline{u}) - \delta_{\overline{c}}(\overline{v}) \mod 2|q|+3$. Lemma 18 gives $|u| \equiv |v| \pm 1 \mod 2|q|+3$. Recall $\delta_{\overline{c}}(\overline{u}), \delta_{\overline{d}}(\overline{v}) \in [-|q|,|q|]$, hence $-2|q|-1 \leq \delta_{\overline{c}}(\overline{u}) - \delta_{\overline{c}}(\overline{u}) \mp 1 \leq 2|q|+1$. Since $\delta_{\overline{c}}(\overline{u}) - \delta_{\overline{d}}(\overline{v}) \mp 1 \equiv 0 \mod 2|q|+3$ and $2|q|+1 < 2|q|+3$, we must have $\delta_{\overline{c}}(\overline{u}) - \delta_{\overline{c}}(\overline{v}) = \pm 1$. Joint to Remark 4, it excludes the case $n_1 - n_2 = 0$. We are hence left with $n_1 = n_2 \pm 1$. Applying our additional property with $k := \max(n_1,n_2)$ gives $(\rho_{n,c}(\overline{u}), \rho_{n,c}(\overline{v})) \in r^{\mathcal{I}'}$.

Finally, $\rho_{n,c}^{-1}(\Delta^{\mathcal{I}}_{\text{base}}) \subseteq \overline{\Delta^{\mathcal{I}}_{\text{base}}}$ is a straightforward consequence of $\rho_{n,c}(\overline{u}) \sim_1 u$ (and again, recall elements from $\Delta^{\mathcal{I}}_{\text{base}}$ are alone in their equivalent class!). □

Let us now complete the proof of Lemma 7 with Lemma 19 in hand.

Proof of Lemma 7. We first prove that $\mathcal{J}$ is a model of $\mathcal{K}$ by considering each possible shape of assertions and axioms:

- $A(a)$**.** Since $\mathcal{I}'$ is a model, we have $a \in A^{\mathcal{I}'}$. Therefore, the definition of $A^{\mathcal{J}}$ gives $\overline{a} = a \in A^{\mathcal{J}}$.
- $p(a,b)$**.** Since $\mathcal{I}'$ is a model, we have $(a,b) \in p^{\mathcal{I}'}$. Therefore, the definition of $p^{\mathcal{J}}$ gives $(\overline{a},\overline{b}) = (a,b) \in p^{\mathcal{J}}$.

- $\top \sqsubseteq A$. Let $u \in \top^{\mathcal{J}} = \Delta^{\mathcal{J}}$. By definition of $\Delta^{\mathcal{J}}$, there exists $u_0 \in \Delta^{\mathcal{I}'}$ such that $\overline{u_0} = u$. Since $u_0 \in \top^{\mathcal{I}'}$ and $\mathcal{I}'$ is a model, it ensures $u_0 \in A^{\mathcal{I}'}$. Therefore the definition of $A^{\mathcal{J}}$ gives $u = \overline{u_0} \in A^{\mathcal{J}}$.
- $A_1 \sqcap A_2 \sqsubseteq A$. Let $u \in (A_1 \sqcap A_2)^{\mathcal{J}}$. By definition of $A_1^{\mathcal{J}}$ and $A_2^{\mathcal{J}}$, there exists $u_1 \in A_1^{\mathcal{I}'}$ and $u_2 \in A_2^{\mathcal{I}'}$ with $\overline{u_1} = \overline{u_2} = u$. Remark 2 ensures u_1 and u_2 satisfy the same concepts, that is in particular $u_1 \in (A_1 \sqcap A_2)^{\mathcal{I}'}$. Since $\mathcal{I}'$ is a model, it ensures $u_1 \in A^{\mathcal{I}'}$, yielding by definition of $A^{\mathcal{J}}$ that $u = \overline{u_1} \in A^{\mathcal{J}}$.
- $A_1 \sqsubseteq \exists r.A_2$. Let $u \in A_1^{\mathcal{J}}$. By definition of ${A_1}^{\mathcal{J}}$ there exists $u_0 \in {A_1}^{\mathcal{I}'}$ with $\overline{u_0} = u$. Since $\mathcal{I}'$ is a model, it ensures there exists $v_0 \in {A_2}^{\mathcal{I}'}$ with $(u_0, v_0) \in r^{\mathcal{I}'}$. By definition of ${A_2}^{\mathcal{J}}$ and $r^{\mathcal{J}}$, the element $v := \overline{v_0}$ satisfies both $v \in {A_2}^{\mathcal{J}}$ and $(u, v) \in r^{\mathcal{J}}$, that is $u \in (\exists r.A_2)^{\mathcal{J}}$.
- $\exists r.A_1 \sqsubseteq A_2$. Let $u \in (\exists r.A_1)^{\mathcal{J}}$, that is, there exists $v \in A_1^{\mathcal{J}}$ with $(u, v) \in r^{\mathcal{J}}$. By definition of $A_1^{\mathcal{J}}$ and $r^{\mathcal{J}}$, there exist $(u_0, v_0) \in r^{\mathcal{I}'}$ and $v_1 \in A_1^{\mathcal{I}'}$ such that $\overline{u_0} = u$ and $\overline{v_0} = \overline{v_1} = v$. Remark 2 ensures v_0 and v_1 satisfy the same concepts, in particular $u_0 \in (\exists r.A_1)^{\mathcal{I}'}$. Since $\mathcal{I}'$ is a model, this ensures $u_0 \in A_2^{\mathcal{I}'}$, yielding by definition of $A_2^{\mathcal{J}}$ that $u = \overline{u_0} \in A_2^{\mathcal{J}}$.
- $A \sqsubseteq \neg B$. By contradiction, assume $u \in A^{\mathcal{J}} \cap B^{\mathcal{J}}$. By definition there exists $v \in A^{\mathcal{I}'}$ and $w \in B^{\mathcal{I}'}$ with $\overline{v} = \overline{w} = u$. Remark 2 ensures v and w satisfy the same concepts, contradicting $\mathcal{I}'$ being a model.
- $\neg B \sqsubseteq A$. Let $u \in \neg B^{\mathcal{J}}$. By definition of $\Delta^{\mathcal{J}}$, there exists $v \in \mathcal{I}'$ such that $\overline{v} = u$. Since $u \notin B^{\mathcal{J}}$, we have $v \notin B^{\mathcal{I}'}$. Hence $\mathcal{I}'$ being a model gives $v \in A^{\mathcal{I}'}$, yielding by definition $u = \overline{v} \in A^{\mathcal{J}}$.
- $p \sqsubseteq r$. Let $(u, v) \in p^{\mathcal{J}}$. By definition of $p^{\mathcal{J}}$, there exists $(u_0, v_0) \in p^{\mathcal{I}'}$ such that $\overline{u_0} = u$ and $\overline{v_0} = v$. Since $\mathcal{I}'$ is a model, it ensures $(u_0, v_0) \in r^{\mathcal{I}'}$, hence $(\overline{u_0}, \overline{v_0}) = (u, v) \in r^{\mathcal{J}}$ by definition of $r^{\mathcal{J}}$.

It remains to prove that $\mathcal{J}$ is minimal w.r.t. $<_{\mathsf{CP}}$. We use Lemma 2 with $\mathcal{I}$ as reference model, and it suffices to show that the preconditions of that lemma are satisfied. Using Remark 2, we reduce to the study of types in $\mathcal{I}'$ and thus concludes as in the proof of Lemma 3.

We now prove $\mathcal{J} \not\models q(\bar{a})$. By contradiction, assume we have $p \in q$ and a homomorphism $\pi : p \to \mathcal{J}$ s.t. $p(\bar{x}) = \bar{a}$. Consider the set of variables $\mathbf{v}_\pi := \{v \mid v \in \bar{y}, \pi(v) \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}\}$. Let $\mathcal{C}$ denote the set of connected components of $\mathbf{v}_\pi$ in $p_{|\mathbf{v}_\pi}$ (that is the query obtained by keeping only those atoms containing variables from $\mathbf{v}_\pi$). For each connected component $C \in \mathcal{C}$, choose a reference variable $v_C \in C$. Since π is a homomorphism and $|C| \leq |q|$, every variable $v \in C$ satisfies $\pi(v) \in \mathcal{N}_{|q|}^{\mathcal{J}, \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\pi(v_C))$. Let $d_C \in \Delta^{\mathcal{I}'}$ denote one's favorite representative for the class of $\pi(v_C)$ (that is $\overline{d_C} = \pi(v_C)$). From Lemma 19, we have a homomorphism $\rho_C : \mathcal{N}_{|q|}^{\mathcal{J}, \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}}(\pi(v_C)) \to \mathcal{N}_{|q|}^{\mathcal{I}', \Delta^{\mathcal{I}}_{\mathsf{base}}}(d_C)$. Using these ρ_C, one per $C \in \mathcal{C}$, we define:

$$
\begin{array}{rcl}
\pi' : \bar{x} \cup \bar{y} & \to & \Delta^{\mathcal{I}'} \\
v & \mapsto & \begin{cases} \rho_C(\pi(v)) & \text{if } v \in C, C \in \mathcal{C} \\ e & \text{if } \pi(v) = \overline{e} \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}} \end{cases}
\end{array}
$$

Since each ρ_C is a homomorphism (again Lemma 19), we can check the overall π' is also a homomorphism:

- Consider $A(v) \in q$. If $v \in C$ for some $C \in \mathcal{C}$, then ρ_C being a homomorphism gives $\pi'(v) \in A^{\mathcal{I}'}$. Otherwise $\pi(v) = \overline{e} \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, but since π is a homomorphism we have $\pi(v) \in A^{\mathcal{J}}$. Since $\overline{e} = \{e\}$ and by definition of $A^{\mathcal{J}}$, it ensures $e \in A^{\mathcal{I}'}$, that is $\pi'(v) \in A^{\mathcal{I}'}$.
- Consider $r(u, v) \in q$.
  - If both $\pi(u), \pi(v) \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, then we can find $C \in \mathcal{C}$ such that $u, v \in C$, and then we use ρ_C being a homomorphism.
  - If both $\pi(u), \pi(v) \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, then the definition of $r^{\mathcal{J}}$ provides $(u_0, v_0) \in r^{\mathcal{I}'}$ with $\overline{u_0} = \pi(u) \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ and $\overline{v_0} = \pi(v) \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$. Hence $\overline{u_0} = \{u_0\}$ and $\overline{v_0} = \{v_0\}$, which gives $(\pi'(u), \pi'(v)) \in r^{\mathcal{I}'}$.
  - If $\pi(u) \notin \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$ and $\pi(v) \in \overline{\Delta^{\mathcal{I}}_{\mathsf{base}}}$, then we have $\pi'(u) = \rho_C(\pi(u))$ for some $C \in \mathcal{C}$. Lemma 19 ensures $\pi'(u) \sim_1 \pi(u)$, and since π is a homomorphism, we also have $(\pi(u), \pi(v)) \in r^{\mathcal{J}}$. Therefore we can apply Condition 2.(c) and we obtain $(\pi'(u), \pi'(v)) \in r^{\mathcal{I}'}$.

In particular, π' is a homomorphism s.t. $\pi'(\bar{x}) = \bar{a}$, hence the desired contradiction with $\mathcal{I}'$ being a countermodel. □

Theorem 3. *UCQ evaluation on circumscribed $\mathcal{ALCHIO}$ KBs is in Π^p_2 w.r.t. data complexity.*

Proof. Assume that we are given a circumscribed $\mathcal{ALCHI}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and a UCQ q. We describe a Σ^p_2 procedure to decide whether $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$.

We first guess an interpretation $\mathcal{I}$ with $|\Delta^{\mathcal{I}}| \leq |\mathcal{A}| + (2^{|\mathcal{T}|+2} + 1)^{3|q|}$. We next check in polynomial time that $\mathcal{I}$ is a model of $\mathcal{K}$ and that $\mathcal{I}$ is minimal w.r.t. $<_{\mathsf{CP}}$ by co-guessing a model $\mathcal{J}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. If one of the checks fails, we reject. Otherwise we verify, for each CQ p in q, whether there is a homomorphism from p to $\mathcal{I}$. This is done brute-force, in time $O(|\Delta^{\mathcal{I}}|^{|q|})$. We accept if there is no such homomorphism, and reject otherwise. This procedure is correct due to Lemma 7. □

Theorem 4. *AQ evaluation on circumscribed $\mathcal{EL}$ KBs is Π^p_2-hard. This holds even with a single minimized concept name and no fixed concept names.*

Proof. We give a polynomial time reduction from $\forall\exists$3SAT (Stockmeyer 1976), that is, the problem to decide whether a given $\forall\exists$-3CNF sentence is true. Thus let $\forall\bar{x}\exists\bar{y}\,\varphi$ be such a sentence where $\bar{x} = x_1 \cdots x_m$, $\bar{y} = y_1 \cdots y_n$, and $\varphi = \bigwedge_{i=1}^{\ell} \bigvee_{j=1}^{3} L_{ij}$ with $L_{ij} = v$ or $L_{ij} = \neg v$ for some $v \in \{x_1, \ldots x_m, y_1, \ldots, y_n\}$. We construct a circumscribed $\mathcal{EL}$

KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and an atomic query q such that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q$ iff $\forall\bar{x}\exists\bar{y}\,\varphi$ is true.

The circumscription pattern minimizes the concept name Min and lets all other concept names vary, and the instance query is $q = \mathsf{Goal}(c_0)$. We now describe how to construct the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, not strictly separating $\mathcal{T}$ from $\mathcal{A}$. We first represent φ in $\mathcal{A}$ using the following assertions:

$$\mathsf{Var}(v) \quad \text{for all } v \in \bar{x} \cup \bar{y} \tag{15}$$
$$\mathsf{pos}_j(c_i, v) \quad \text{if } L_{ij} = v \tag{16}$$
$$\mathsf{neg}_j(c_i, v) \quad \text{if } L_{ij} = \neg v \tag{17}$$
$$\mathsf{First}(c_0), \mathsf{Last}(c_\ell) \tag{18}$$
$$\mathsf{next}(c_i, c_{i+1}) \quad 0 \leq i < \ell \tag{19}$$

We next require each variable to be connected via the role name eval to two instances of Min, a "positive" one and a "negative" one:

$$\mathsf{Var} \sqsubseteq \exists\mathsf{eval}.(\mathsf{Min} \sqcap \mathsf{Pos}) \tag{20}$$
$$\mathsf{Var} \sqsubseteq \exists\mathsf{eval}.(\mathsf{Min} \sqcap \mathsf{Neg}). \tag{21}$$

As potential witnesses for the above existential restrictions, we provide several instances of Min, a positive and a negative one for each truth value:

$$\mathsf{Min}(t_+) \quad \mathsf{Min}(t_-) \quad \mathsf{Min}(f_+) \quad \mathsf{Min}(f_-) \tag{22}$$
$$\mathsf{True}(t_+) \quad \mathsf{True}(t_-) \quad \mathsf{False}(f_+) \quad \mathsf{False}(f_-) \tag{23}$$
$$\mathsf{Pos}(t_+) \quad \mathsf{Neg}(t_-) \quad \mathsf{Pos}(f_+) \quad \mathsf{Neg}(f_-) \tag{24}$$

If such an instance is used as a witness, then the respective variable is assigned the corresponding truth value:

$$\exists\mathsf{eval}.\mathsf{True} \sqsubseteq \mathsf{True} \tag{25}$$
$$\exists\mathsf{eval}.\mathsf{False} \sqsubseteq \mathsf{False} \tag{26}$$

We next introduce a mechanism for "freezing" the valuation of the $\bar{x}$ variables in the sense that this valuation must be identical in all models that are smaller w.r.t. '$<_{\mathsf{CP}}$'. This reflects the fact that the $\bar{x}$ variables are chosen prior to the $\bar{y}$ variables. Freezing is achieved by the following:

$$\mathsf{freeze}_{\mathsf{True}}(x_{\mathsf{True}}, x) \quad \text{for all } x \in \bar{x} \tag{27}$$
$$\mathsf{freeze}_{\mathsf{False}}(x_{\mathsf{False}}, x) \quad \text{for all } x \in \bar{x} \tag{28}$$
$$\exists\mathsf{freeze}_{\mathsf{True}}.\mathsf{True} \sqsubseteq \mathsf{Min} \tag{29}$$
$$\exists\mathsf{freeze}_{\mathsf{False}}.\mathsf{False} \sqsubseteq \mathsf{Min} \tag{30}$$

Note that if a variable $x \in \bar{x}$ satisfies True, then x_{True} must satisfy Min, and likewise for False. Since Min is minimized, any model that is smaller w.r.t. '$<_{\mathsf{CP}}$' must thus have the same valuation for the variables in $\bar{x}$.

There is a problem with this approach, though. On the one hand, Min acts as an incentive to use t_+, t_-, f_+, and f_- as witnesses for (20) and (21). On the other hand, freezing implies that if a variable x does so, then it triggers an instance of Min on x_{True} or x_{False}. There can hence be models in which fresh instances e^+ of $\mathsf{Min} \sqcap \mathsf{Pos}$ and e^- of $\mathsf{Min} \sqcap \mathsf{Neg}$ are created instead, leading to x satisfying neither True nor False. Since Min is minimized, however, it must then hold that $e^+ = e^-$, *i.e.* x is connected to an instance of $\mathsf{Pos} \sqcap \mathsf{Neg}$. Such models are easily detected due to our use of positive and negative instances of Min. We make sure that problematic models satisfy the query and are thus ruled out as a countermodel:

$$\mathsf{checks}(c_0, v) \quad \text{for all } v \in \bar{x} \cup \bar{y} \tag{31}$$
$$\exists\mathsf{checks}.(\exists\mathsf{eval}.(\mathsf{Pos} \sqcap \mathsf{Neg})) \sqsubseteq \mathsf{Goal} \tag{32}$$

Now that we can assume that each variable is given a proper truth value, we encode the evaluation of φ. The truth value of each clause is represented by concept name True or False holding on the corresponding individual c_i. Let $X_i = \{\exists\mathsf{pos}_i.\mathsf{False}, \exists\mathsf{neg}_i.\mathsf{True}\}$ and $X = X_1 \times X_2 \times X_3$, and add:

$$\exists\mathsf{pos}_i.\mathsf{True} \sqsubseteq \mathsf{True} \quad \text{for } 1 \leq i \leq 3 \tag{33}$$
$$\exists\mathsf{neg}_i.\mathsf{False} \sqsubseteq \mathsf{True} \quad \text{for } 1 \leq i \leq 3 \tag{34}$$
$$C_1 \sqcap C_2 \sqcap C_3 \sqsubseteq \mathsf{False} \quad \text{for all } (C_1, C_2, C_3) \in X \tag{35}$$

We next determine the truth value of φ by walking along the next-chain of clauses in $\mathcal{A}$. If the formula evaluates to true, then we make Goal true at c_0 and thus satisfy the query. If it evaluates to false, we make Min true at c_0. This reflects the existential quantification over the variables $\bar{y}$: when they exist, we prefer valuations that make φ true since making φ false gives us an additional instance of the minimized concept name. Formally:

$$\exists\mathsf{next}.(\mathsf{Last} \sqcap \mathsf{True}) \sqsubseteq \mathsf{Last} \tag{36}$$
$$\mathsf{First} \sqcap \mathsf{Last} \sqsubseteq \mathsf{Goal} \tag{37}$$
$$\exists\mathsf{next}.\mathsf{False} \sqsubseteq \mathsf{False} \tag{38}$$
$$\mathsf{False} \sqcap \mathsf{First} \sqsubseteq \mathsf{Min} \tag{39}$$

It remains to solve another technical issue. As explained above, we make Min true not only at the individuals t_+, t_-, f_+, f_-, but also at some individuals $x_{\mathsf{True}}, x_{\mathsf{False}}$ and possibly at c_0. However, we do not want these individuals to be used as witnesses for (20) and (21) as then there could be variables that do not receive a truth value. We can again detect this and then make Min true at a fresh individual a:

$$\mathsf{checks}(a, x_{\mathsf{True}}), \mathsf{checks}(a, x_{\mathsf{False}}) \quad \text{for all } x \in \bar{x} \tag{40}$$
$$\mathsf{checks}(a, c_0) \tag{41}$$
$$\exists\mathsf{checks}.\mathsf{Pos} \sqsubseteq \mathsf{Min} \tag{42}$$
$$\exists\mathsf{checks}.\mathsf{Neg} \sqsubseteq \mathsf{Min}. \tag{43}$$

We now set $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ with $\mathcal{T}$ and $\mathcal{A}$ as defined above. It is easily verified that $\mathcal{T}$ does not depend on the instance of $\forall\exists$3SAT. It remains to prove the following claim:

Claim. $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models \mathsf{Goal}(c_0)$ iff $\forall\bar{x}\,\exists\bar{y}\,\varphi(\bar{x}, \bar{y})$.

To prepare for the proof of the claim, we first observe that every valuation V for $\bar{x} \cup \bar{y}$ gives rise to a corresponding model $\mathcal{I}_V$ of $\mathcal{K}$. We use domain $\Delta^{\mathcal{I}_V} = \mathsf{ind}(\mathcal{A})$ and set

$$\begin{aligned} A^{\mathcal{I}_V} &= \{a \mid A(a) \in \mathcal{A}\} \\ r^{\mathcal{I}_V} &= \{(a, b) \mid r(a, b) \in \mathcal{A}\} \end{aligned}$$

for all concept names $A \in \{\mathsf{Var}, \mathsf{First}, \mathsf{Pos}, \mathsf{Neg}\}$ and all role names

$$r \in \{\mathsf{pos}_i, \mathsf{neg}_i, \mathsf{next}, \mathsf{freeze}_{\mathsf{True}}, \mathsf{freeze}_{\mathsf{False}}, \mathsf{checks}\}.$$

We interpret eval according to V, that is

$$\begin{aligned}\mathsf{eval}^{\mathcal{I}_V} = \{&(v,t_+),(v,t_-) \mid v \in \bar{x}\cup\bar{y} \text{ and } V(v)=1\} \cup \\ &\{(v,f_+),(v,f_-) \mid v \in \bar{x}\cup\bar{y} \text{ and } V(v)=0\}.\end{aligned}$$

The valuation V also assigns a truth value to each clause, let $V(c_i)$ denote the value of the i^{th} clause. Put

$$\begin{aligned}\mathsf{True}^{\mathcal{I}_V} &= \{t_+,t_-\} \cup \\ &\quad \{v \mid v \in \bar{x}\cup\bar{y} \text{ and } V(v)=1\} \cup \\ &\quad \{c_i \mid 1 \le i \le \ell \text{ and } V(c_i)=1\} \\ \mathsf{False}^{\mathcal{I}_V} &= \{f_+,f_-\} \cup \\ &\quad \{v \mid v \in \bar{x}\cup\bar{y} \text{ and } V(v)=0\} \cup \\ &\quad \{c_i \mid 1 \le i \le \ell \text{ and } V(c_i)=1\}.\end{aligned}$$

Let q be largest such that valuation V makes the last q of the ℓ clauses in φ true. Put

$$\mathsf{Last}^{\mathcal{I}_V} = \{c_\ell\} \cup \{c_{\ell-1},\dots,c_{\ell-v+1}\}.$$

It remains to interpret Min and Goal. As an abbreviation, set

$$\begin{aligned}M &= \{t_+,t_-,f_+,f_-,\} \cup \\ &\quad \{x_{\mathsf{True}} \mid x \in \bar{x} \text{ and } V(x)=1\} \\ &\quad \{x_{\mathsf{False}} \mid x \in \bar{x} \text{ and } V(x)=0\}.\end{aligned}$$

Now if V satisfies all clauses in φ, then put

$$\begin{aligned}\mathsf{Min}^{\mathcal{I}_V} &= M \\ \mathsf{Goal}^{\mathcal{I}_V} &= \{c_0\}\end{aligned}$$

and otherwise put

$$\begin{aligned}\mathsf{Min}^{\mathcal{I}_V} &= M \cup \{c_0\} \\ \mathsf{Goal}^{\mathcal{I}_V} &= \emptyset.\end{aligned}$$

It is straightforward to verify that $\mathcal{I}_V$ is indeed a model of $\mathcal{K}$. Moreover, $\mathcal{I}_V \models q$ if and only if V satisfies all clauses in φ. Now for the actual proof of the claim

"⇒". Assume that $\forall\bar{x}\exists\bar{y}\,\varphi$ is false, and thus $\exists\bar{x}\forall\bar{y}\,\neg\varphi$ is true and there is a valuation $V_{\bar{x}} : \bar{x} \to \{0,1\}$ such that $\forall\bar{y}\,\neg\varphi'$ holds where φ' is obtained from φ by replacing every variable $x \in \bar{x}$ with the truth constant $V_{\bar{x}}(x)$. Consider any extension V of $V_{\bar{x}}$ to the variables in $\bar{y}$. Then the interpretation $\mathcal{I}_V$ is a model of $\mathcal{K}$ with $\mathcal{I}_V \not\models q$, to show that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q$ it remains to prove that there is no model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$.

Assume to the contrary that there is such a $\mathcal{J}$. Then

$$\begin{aligned}\mathsf{Min}^{\mathcal{J}} \subsetneq \mathsf{Min}^{\mathcal{I}_v} &= \{t_+,t_-,f_+,f_-,c_0\} \cup \\ &\quad \{x_{\mathsf{True}} \mid x \in \bar{x} \text{ and } V(x)=1\} \\ &\quad \{x_{\mathsf{False}} \mid x \in \bar{x} \text{ and } V(x)=0\}.\end{aligned}$$

$\mathcal{J}$ being a model of $\mathcal{A}$ implies $\{t_+,t_-,f_+,f_-\} \subseteq \mathsf{Min}^{\mathcal{J}}$.

Since $\mathcal{J}$ satisfies (15), (20) and (21), we can choose for each $v \in \bar{x}\cup\bar{y}$ an element $v^+ \in \mathsf{Pos}^{\mathcal{J}} \cap \mathsf{Min}^{\mathcal{J}}$ such that $(v,v^+) \in \mathsf{eval}^{\mathcal{J}}$. Since $\mathcal{J}$ satisfies (40) to (43) and $a \notin \mathsf{Min}^{\mathcal{J}}$, we must have $v^+ \in \{t_+,t_-,f_+,f_-\}$. From (25) and (26), it follows that $v \in \mathsf{True}^{\mathcal{J}} \cup \mathsf{False}^{\mathcal{J}}$. Let $V_{\mathcal{J}}$ be the valuation defined by setting, for all $v \in \bar{x}\cup\bar{y}$:

$$V_{\mathcal{J}}(v) = \begin{cases} 1 & \text{if } v \in \mathsf{True}^{\mathcal{J}} \\ 0 & \text{otherwise.}\end{cases}$$

From (27) to (30), we further obtain that

$$\begin{aligned}\mathsf{Min}^{\mathcal{J}} \supseteq &\{x_{\mathsf{True}} \mid x \in \bar{x} \text{ and } V_{\mathcal{J}}(x)=1\} \\ &\{x_{\mathsf{False}} \mid x \in \bar{x} \text{ and } V_{\mathcal{J}}(x)=0\}.\end{aligned}$$

which implies that $\mathsf{Min}^{\mathcal{I}_V} \setminus \mathsf{Min}^{\mathcal{J}} = \{c_0\}$ and, through $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, that V and $V_{\mathcal{J}}$ agree on $\bar{x}$. From $\forall\bar{y}\,\neg\varphi'$, it follows that $V_{\mathcal{J}} \not\models \varphi$ and thus the soundness of the evaluation of φ implemented by (33) to (39) yields $c_0 \in \mathsf{Min}^{\mathcal{J}}$, a contradiction.

"⇐". Assume that $\forall\bar{x}\,\exists\bar{y}\,\varphi(\bar{x},\bar{y})$ is true and let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. We have to show that $\mathcal{I} \models q$.

Since $\mathcal{I}$ satisfies (15), (20) and (21), we can choose for every $v \in \bar{x}\cup\bar{y}$ two elements $v^+ \in \mathsf{Pos}^{\mathcal{I}} \cap \mathsf{Min}^{\mathcal{I}}$ and $v^- \in \mathsf{Neg}^{\mathcal{I}} \cap \mathsf{Min}^{\mathcal{I}}$ such that $(v,v^+),(v,v^-) \in \mathsf{eval}^{\mathcal{I}}$. If $v^+ = v^-$ for some v, then by (31) and (32) we have $c_0 \in \mathsf{Goal}^{\mathcal{I}}$, thus $\mathcal{I} \models q$ and we are done. For the remaining proof, we may thus assume $v^+ \neq v^-$ for all $v \in \bar{x}\cup\bar{y}$.

We next argue that for all $v \in \bar{x}\cup\bar{y}$,

$$\{v^+,v^-\} \cap \{t_+,t_-,f_+,f_-\} \neq \emptyset \qquad (*)$$

Assume to the contrary that $(*)$ is violated by some $v_0 \in \bar{x}\cup\bar{y}$, that is, $v_0^+,v_0^- \notin \{t_+,t_-,f_+,f_-\}$. Then we can find a model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$, contradicting the minimality of $\mathcal{I}$. We distinguish two cases:

1. $a \in \mathsf{Min}^{\mathcal{I}}$.
   We construct $\mathcal{J}$ by choosing $\Delta^{\mathcal{J}} = \mathsf{ind}(\mathcal{A})$ and interpreting the concept names Var and First as well as the role names pos_i, neg_i, next, $\mathsf{freeze}_{\mathsf{True}}$, $\mathsf{freeze}_{\mathsf{False}}$, checks as in the initially defined interpretations $\mathcal{I}_V$—note that this is independent of V. We further set

$$\begin{aligned}\mathsf{eval}^{\mathcal{J}} &= \{(v,a) \mid v \in \bar{x}\cup\bar{y}\} \\ \mathsf{Pos}^{\mathcal{J}} &= \{t_+,f_+,a\} \\ \mathsf{Neg}^{\mathcal{J}} &= \{t_-,f_-,a\} \\ \mathsf{True}^{\mathcal{J}} &= \{t_+,t_-\} \\ \mathsf{False}^{\mathcal{J}} &= \{f_+,f_-\} \\ \mathsf{Last}^{\mathcal{J}} &= \{c_\ell\} \\ \mathsf{Min}^{\mathcal{J}} &= \{t_+,t_-,f_+,f_-,a\} \\ \mathsf{Goal}^{\mathcal{J}} &= \emptyset.\end{aligned}$$

   It can be verified that $\mathcal{J}$ is a model of $\mathcal{K}$. Moreover, at least one of the (distinct!) v_0^+,v_0^- is not in $\mathsf{Min}^{\mathcal{J}}$ and thus $\mathsf{Min}^{\mathcal{J}} \subsetneq \mathsf{Min}^{\mathcal{I}}$ and $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$.

2. $a \notin \mathsf{Min}^{\mathcal{I}}$.
   We construct $\mathcal{J}$ exactly as in Case 1, but with a replaced by v_0^+ in $\mathsf{Pos}^{\mathcal{J}}$, $\mathsf{Neg}^{\mathcal{J}}$, and $\mathsf{Min}^{\mathcal{J}}$.

We have thus established $(*)$.

It follows from $(*)$ together with (23), (25), and (26) that $v \in \mathsf{True}^{\mathcal{J}} \cup \mathsf{False}^{\mathcal{J}}$ for all $v \in \bar{x} \cup \bar{y}$ and thus we find a valuation V such that $V(v) = 1$ implies $v \in \mathsf{True}^{\mathcal{J}}$ and $V(v) = 0$ implies $v \in \mathsf{False}^{\mathcal{J}}$ for all v. We show below that $V \models \varphi$. Then the soundness of the evaluation of φ implemented by (33) to (39) yields $c_0 \in \mathsf{Goal}^{\mathcal{I}}$ and thus $\mathcal{I} \models q$ as desired.

Assume to the contrary of what remains to be shown that $V \not\models \varphi$. Soundness of evaluation ensures $c_0 \in \mathsf{Min}^{\mathcal{I}}$. Since $\forall \bar{x}\, \exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ is true, we find a valuation V' that agrees with V on the variables in $\bar{x}$ and that satisfies φ. Now consider the model $\mathcal{I}_{V'}$ of $\mathcal{K}$. It satisfies

$$\begin{aligned}\mathsf{Min}^{\mathcal{I}_{V'}} \quad = \quad & \{t_+, t_-, f_+, f_-, \} \cup \\ & \{x_{\mathsf{True}} \mid x \in \bar{x} \text{ and } V(x) = 1\} \\ & \{x_{\mathsf{False}} \mid x \in \bar{x} \text{ and } V(x) = 0\}.\end{aligned}$$

and it is easy to verify that $\mathsf{Min}^{\mathcal{I}_{V'}} \subseteq \mathsf{Min}^{\mathcal{I}}$. But then $\mathcal{I}_{V'} < \mathcal{I}_V$ since $c_0 \in \mathsf{Min}^{\mathcal{I}}$. □

D Proofs for Section 4.1

Theorem 5. *UCQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{core}}$ KBs is 2EXP-hard w.r.t. combined complexity. This holds even with a single minimized concept name, no fixed concept names and no disjointness constraints.*

Proof. We reduce from (Boolean) UCQ evaluation on DL-Lite$^{\mathcal{H}}$ KBs with closed concept names, which is defined in the expected way and known to be 2EXP-hard (Ngo, Ortiz, and Simkus 2016). Let $\mathcal{K}_\Sigma$ be a DL-Lite$^{\mathcal{H}}$ KB with closed concept names, $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, and let q be a Boolean UCQ. We construct a circumscribed DL-Lite$^{\mathcal{H}}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$, with $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$, and a CQ q' such that $\mathcal{K}_\Sigma \models q$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models q'$.

As in the proof of Theorem 2, we minimize only a single concept name M and include $\mathcal{A}$ in $\mathcal{A}'$. To construct $\mathcal{T}'$, we start from $\mathcal{T}$ and extend with additional concept inclusions. For q', we start from q and extend with additional disjuncts. We will actually use individuals $a \in \mathsf{Ind}(\mathcal{A})$ as constants in q'. These can be eliminated by introducing a fresh concept name A_a, extending $\mathcal{A}$ with $A_a(a)$, and replacing in each CQ in q' the constant a with a fresh variable x_a and adding the atom $A_a(x_a)$.

As in the proof of Theorem 2, we include in $\mathcal{T}'$ the CI

$$A \sqsubseteq M \quad \text{for all } A \in \Sigma \tag{44}$$

and then have to rule out non-asserted instances of closed concept names. To rule out non-asserted instances inside of $\mathsf{ind}(\mathcal{A})$, we add to q' the disjunct $A(a)$ for all $A \in \Sigma$ and $a \in \mathsf{ind}(\mathcal{A})$ with $A(a) \notin \mathcal{A}$.

To rule out instances of closed concept names outside of $\mathsf{ind}(\mathcal{A})$, add

$$M(t) \tag{45}$$

where t is a fresh individual. As in the proof of Theorem 2, this guarantees that models with instances of closed concept names outside of $\mathsf{ind}(\mathcal{A})$ are not minimal. To ensure that no closed concept names are made true at t, we extend q' with the disjunct $r(x, t)$ for every role name r used in $\mathcal{T}$.

Claim. $\mathcal{K}_\Sigma \models q$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models q'$.

The proof is rather straightforward, we omit the details. □

We now work towards a proof of Theorem 6, first establishing the following.

Lemma 8. *Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a DL-Lite$_{\mathsf{bool}}$ cKB with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, $q(\bar{x})$ a UCQ, and $\bar{a} \in \mathsf{ind}(\mathcal{A})^{|\bar{x}|}$. If $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q(\bar{a})$, then there exists a countermodel $\mathcal{I}$ with $|\Delta^{\mathcal{I}}| \leq (|\mathcal{A}| + 2^{|\mathcal{T}|})^{|q|^2(|\mathcal{T}|+1)}$.*

Refining the unraveling begins by noticing the set Ω now only contains elements with shape $r\top$ as DL-Lite$_{\mathsf{bool}}$ TBoxes are considered. We drop the $\top$ concept for simplicity. Recall we assume chosen a representative $e_t \in \Delta^{\mathcal{I}}$ for each non-core type $t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})$. We define the set $\mathcal{P}_{\mathsf{bool}}$ of bool-paths through $\mathcal{I}$ along with a mapping h that assigns to each $p \in \mathcal{P}_{\mathsf{bool}}$ an element of $\Delta^{\mathcal{I}}$:

- each element d of the set
  $$\Delta^{\mathcal{I}}_{\mathsf{base}} := \mathsf{ind}(\mathcal{A}) \cup \Delta^{\mathcal{I}}_{\mathsf{core}} \cup \{e_t \mid t \in \mathsf{TP}_{\overline{\mathsf{core}}}(\mathcal{I})\}.$$
  is a path in $\mathcal{P}_{\mathsf{bool}}$ and $h(d) = d$;
- if $p \in \mathcal{P}_{\mathsf{bool}}$ with $h(p) = d$ and $r \in \Omega$ such that:
  (a) $f(d, r)$ is defined and not from $\Delta^{\mathcal{I}}_{\mathsf{core}}$ and
  (b) p does not end by r^-, which we denote $\mathsf{tail}(p) \neq r^-$.
  then $p' = pr$ is a path in $\mathcal{P}_{\mathsf{bool}}$ and $h(p') = f(d, r)$.

For every role r, define

$$\begin{aligned}R_r = \{&(a, b) \mid a, b \in \mathsf{ind}(\mathcal{A}), \mathcal{K} \models r(a, b)\} \cup \\ \{&(d, e) \mid d, e \in \Delta^{\mathcal{I}}_{\mathsf{core}}, (d, e) \in r^{\mathcal{I}}\} \cup \\ \{&(p, p') \mid p' = pr\} \cup \\ \{&(p, e) \mid e = f(h(p), r) \in \Delta^{\mathcal{I}}_{\mathsf{core}}, \mathsf{tail}(p) \neq r^-\}.\end{aligned}$$

Now the bool-unraveling of $\mathcal{I}$ is defined by setting

$$\begin{aligned}\Delta^{\mathcal{I}'} &= \mathcal{P}_{\mathsf{bool}} \\ A^{\mathcal{I}'} &= \{p \in \mathcal{P}_{\mathsf{bool}} \mid h(p) \in A^{\mathcal{I}}\} \\ r^{\mathcal{I}'} &= R_r \cup \{(e, d) \mid (d, e) \in R_{r^-}\}\end{aligned}$$

for all concept names A and role names r. It is easy to verify that h is a homomorphism from $\mathcal{I}'$ to $\mathcal{I}$. It is a technicality to verify that $\mathcal{I}'_{\mathsf{bool}}$ satisfies the same properties as the previous unraveling $\mathcal{I}'$, that is: if $\mathcal{I}$ is a countermodel for q over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, then so is $\mathcal{I}'_{\mathsf{bool}}$. The key additional property satisfied by $\mathcal{I}'_{\mathsf{bool}}$ is the following:

Lemma 20. *For any $r \in \mathsf{N_R}$ and $d_1 \in \Delta^{\mathcal{I}'_{\mathsf{bool}}} \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$, there is at most one element $d_2 \in \Delta^{\mathcal{I}'_{\mathsf{bool}}}$ such that $(d_1, d_2) \in r^{\mathcal{I}'_{\mathsf{bool}}}$.*

Proof. Unfolding the definition of $r^{\mathcal{I}'_{\mathsf{bool}}}$ and recalling $d_1 \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$, we obtain that if $d_1 \in \Delta^{\mathcal{I}'_{\mathsf{bool}}} \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $(d_1, d_2) \in r^{\mathcal{I}'_{\mathsf{bool}}}$, then either:

- $(d_1, d_2) \in R_r$ and either $d_2 = d_1 r$ (Case 1) or $d_2 = f(h(d_1), r) \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\mathsf{tail}(d_1) \neq r^-$ (Case 2).

- $(d_2, d_1) \in R_{r^-}$ and we have $d_1 = d_2 r^-$ (Case 3).

To verify uniqueness, let us assume there exist d_2 and d_2' s.t. $(d_1, d_2'), (d_1, d_2') \in r^{\mathcal{I}'_{\mathsf{bool}}}$. We want to prove $d_2 = d_2'$. From the small analysis above, there are 3×3 cases to consider:

1. if $d_2 = d_1 r$, then:
   1. if $d_2' = d_1 r$, we obtain directly $d_2' = d_2$.
   2. if $d_2' = f(h(d_1), r) \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\mathsf{tail}(d_1) \neq r^-$, then it contradicts $d_1 r \in \mathcal{P}_{\mathsf{bool}}$ (Condition (a)).
   3. if $d_1 = d_2' r^-$, then $d_2 = d_2' r^- r$, which contradicts $d_2 \in \mathcal{P}_{\mathsf{bool}}$ (Condition (b)).
2. if $d_2 = f(h(d_1), r) \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\mathsf{tail}(d_1) \neq r^-$, then:
   1. if $d_2' = d_1 r$, then same argument as 1.2.
   2. if $d_2' = f(h(d_1), r) \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\mathsf{tail}(d_1) \neq r^-$, then $d_2' = d_2$ from h and f being functions.
   3. if $d_1 = d_2' r^-$, then it contradicts $\mathsf{tail}(d_1) \neq r^-$.
3. if $d_1 = d_2 r^-$, then:
   1. if $d_2' = d_1 r$, then same argument as 1.3.
   2. if $d_2' = f(h(d_1), r) \in \Delta^{\mathcal{I}}_{\mathsf{base}}$ and $\mathsf{tail}(d_1) \neq r^-$, then same argument as 2.3.
   3. if $d_1 = d_2' r^-$, we obtain directly $d_2' = d_2$.

□

Repeated applications of Lemma 20 ensure that each partial homomorphism of a CQ $p \in q$ in the non-$\Delta^{\mathcal{I}}_{\mathsf{base}}$ part of the bool-unraveling can be completed uniquely in a maximal such partial homomorphism (see further Lemma 21). This motivates a refined notion of the neighborhood of an element d, restricting the (usual) neighborhood to those elements e that can be reached by a homomorphism of some connected subquery involving both d and e. Formally, for $n \geq 0$ and $\Delta \subseteq \Delta^{\mathcal{I}}$, we use $\mathcal{N}^{\mathcal{I},\Delta}_{n,\mathsf{bool}}(d)$ to denote the *n-bool-neighborhood of d in $\mathcal{I}$ up to Δ*, that is, the set of all elements $e \in \mathcal{N}^{\mathcal{I},\Delta}_{n}(d)$ such that there exists a connected subquery $p' \subseteq p$ for some $p \in q$ and a homomorphism $\pi : p' \to \mathcal{N}^{\mathcal{I},\Delta}_{n}(d)$ s.t. $d, e \in \pi(\mathsf{var}(p'))$.

The central property allowing bool-neighbourhoods to improve our construction is the following polynomial bound on their size in the bool-unraveling.

Lemma 21. *Let $\mathcal{I}$ be a model of $\mathcal{K}$ and $\mathcal{I}'_{\mathsf{bool}}$ its bool-unraveling. Consider $d \in \Delta^{\mathcal{I}'_{\mathsf{bool}}} \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$, then* $\left|\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(d)\right| \leq |q|^2(|\mathcal{T}| + 1)$.

Proof. Let $c \in \Delta^{\mathcal{I}'_{\mathsf{bool}}} \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}$ in a first step, we prove the number of elements in $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}_{\mathsf{base}}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}$ is at most $|q|^2$. In a second step, we notice each element $e \in \mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}_{\mathsf{base}}}}_{n,\mathsf{bool}}(c) \cap \Delta^{\mathcal{I}_{\mathsf{base}}}$ must be connected to an element $d \in \mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}_{\mathsf{base}}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}_{\mathsf{base}}}$ by construction of (usual) neighborhoods. However, from Lemma 20, each such element d is connected to at most $|\mathcal{T}|$ elements. From the first step it follows there are at most $|q|^2 \cdot |\mathcal{T}|$ elements in $e \in \mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}_{\mathsf{base}}}}_{n,\mathsf{bool}}(c) \cap \Delta^{\mathcal{I}_{\mathsf{base}}}$, hence the claimed bound.

It remains to prove the first step. We start by proving that if the connected subquery $p' \subseteq p$ for some $p \in q$ and the variable v_0 that shall map on c are fixed, then all homomorphisms $p' \to (\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}})$ mapping v_0 on c are equal. Consider two such homomorphisms π_1 and π_2. We proceed by induction on the variables v of p' being connected. For $v = v_0$, we have $\pi_1(v_0) = \pi_2(v_0)$ by definition. For a further term v, we use the induction hypothesis, that is the existence of an atom $r(v', v) \in p'$ (or the other way around) such that $\pi_1(v') = \pi_2(v')$. Recall π_1 and π_2 are homomorphisms of p' in $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$, in particular $\pi_1(v'), \pi_2(v') \notin \Delta^{\mathcal{I}}_{\mathsf{base}}$, hence we can apply Lemma 20, yielding $\pi_1(v) = \pi_2(v)$.

This proves that, for a fixed $v_0 \in \mathsf{var}(p)$, each connected subquery $p' \subseteq p$ admitting a homomorphism in $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ defines at most $|p|$ new neighbors, but also that if $p' \subseteq p'' \subseteq p$ are two such subqueries, then the neighbors defined by p' are subsumed by those defined by p'' (the restriction to the variables of p' of the unique homomorphism of p'' mapping v_0 on c must coincide with the unique homomorphism of p' mapping v_0 on c). Still for a fixed v_0, consider now two connected subqueries $p_1, p_2 \subseteq p$, each admitting a (unique) homomorphism π_1 resp. π_2, to $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ mapping v_0 to c, and each maximal, w.r.t. the inclusion, for this property. By the previous property, we know π_1 and π_2 coincide on $\mathsf{var}(p_1) \cap \mathsf{var}(p_2)$. Therefore, $p_1 \cup p_2$ admits a homomorphism to $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ mapping v_0 to c, being $\pi_1 \cup \pi_2$. But since p_1 and p_2 are assumed maximal for this property, we must have $p_1 = p_2$.

Therefore, for a fixed $v_0 \in \mathsf{var}(p)$, there is a unique maximal connected subquery $p_{\mathsf{max}} \subseteq p$ admitting a homomorphism in $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$ and mapping v_0 to c. As previously seen, the neighbors defined by p_{max} subsume those defined by other subqueries of p, and since the homomorphism for p_{max} is unique, it defines at most $|p|$ neighbors. This holds for each possible choices of term v_0, hence a total number of possible neighbors issued by p bounded by $|p|^2$. Iterating over each $p \in q$, we hence obtain the claimed bound of at most $|q|^2$ elements in $\mathcal{N}^{\mathcal{I}'_{\mathsf{bool}},\Delta^{\mathcal{I}}_{\mathsf{base}}}_{n,\mathsf{bool}}(c) \setminus \Delta^{\mathcal{I}}_{\mathsf{base}}$. □

Using bool-neighborhoods in the quotient construction presented in Section 3.3, the number of equivalence classes drops, which yieldings Lemma 8.

Lemma 22. *For all $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$, $\mathcal{I}_{\mathcal{P}}$ is a model of $\mathcal{K}$.*

Proof. Let $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$, we have:

- $\mathcal{I}_{\mathcal{P}}$ models $\mathcal{A}$ as it inherits interpretations of concepts and roles on $\mathsf{ind}(\mathcal{A})$ from $\mathcal{I}$, which is a model of $\mathcal{A}$.
- Axioms in $\mathcal{T}$ with shape $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ or $\neg B \sqsubseteq A$ are satisfied since $\mathcal{I}_{\mathcal{P}}$ inherits interpretations of concept names from $\mathcal{I}$, which is a model of $\mathcal{T}$.
- Axioms in $\mathcal{T}$ with shape $A \sqsubseteq \exists r$ are witnessed with an r-edge pointing to w_r.

- Axioms in $\mathcal{T}$ with shape $\exists r \sqsubseteq A$ are satisfied since every element in $\mathcal{I}_\mathcal{P}$ having some r-edge already has one in $\mathcal{I}$ (and interpretations of concept names are preserved).
- Role inclusions $r \sqsubseteq s$ are satisfied on $\mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A})$ from $\mathcal{I}$ being a model of $\mathcal{T}$ in the first place, otherwise directly from the definition of $s^{\mathcal{I}_\mathcal{P}}$.

This proves $\mathcal{I}_\mathcal{P} \models \mathcal{K}$ as desired. □

Lemma 9. *The following are equivalent:*

1. *There exists a model $\mathcal{I}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$;*
2. *There exist $\mathcal{P} \subseteq \Delta^\mathcal{I}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$ and a family $(\mathcal{J}_e)_{e \in \Delta^\mathcal{I}}$ of models of $\mathcal{K}$ such that $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P} \cup \{e\}}$ and $\mathcal{J}_e|_{\Delta^{\mathcal{I}_\mathcal{P}}} = \mathcal{J}_{e'}|_{\Delta^{\mathcal{I}_\mathcal{P}}}$ for all $e, e' \in \Delta^\mathcal{I}$.*

To piece together the $\mathcal{J}_e$ and reconstruct an eventual $\mathcal{J}$, we first prove the following property:

Lemma 23. *Let $\mathcal{P}_1, \mathcal{P}_2 \subseteq \Delta^\mathcal{I}$ and $\mathcal{J}_1$, $\mathcal{J}_2$ two models of $\mathcal{K}$ s.t. $\mathcal{J}_1 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1}$ and $\mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_2}$. If $\mathcal{J}_1|_{\Delta^{\mathcal{J}_2}} = \mathcal{J}_2|_{\Delta^{\mathcal{J}_1}}$, then $\mathcal{J}_1 \cup \mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}$.*

Proof. Assume $\mathcal{J}_1|_{\Delta^{\mathcal{J}_2}} = \mathcal{J}_2|_{\Delta^{\mathcal{J}_1}}$. It is easily verified that $\mathcal{J}_1 \cup \mathcal{J}_2$ is also a model of $\mathcal{K}$ since $\mathcal{J}_1$ and $\mathcal{J}_2$ agree on their shared domain. We now check that all four conditions from the definition of $<_{\mathsf{CP}}$ are satisfied:

1. From $\mathcal{J}_1 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1}$ we have $\Delta^{\mathcal{J}_1} = \Delta^{\mathcal{I}_{\mathcal{P}_1}}$ and similarly for $\mathcal{J}_2$ we have $\Delta^{\mathcal{J}_2} = \Delta^{\mathcal{I}_{\mathcal{P}_2}}$. It follows from its definition that $\Delta^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}} = \Delta^{\mathcal{I}_{\mathcal{P}_1}} \cup \Delta^{\mathcal{I}_{\mathcal{P}_2}}$. Therefore we have as desired $\Delta^{\mathcal{J}_1 \cup \mathcal{J}_2} = \Delta^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$.
2. Let $A \in \mathsf{F}$. From $\mathcal{J}_1 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1}$ we have $A^{\mathcal{J}_1} = A^{\mathcal{I}_{\mathcal{P}_1}}$ and similarly for $\mathcal{J}_2$ we have $A^{\mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_2}}$. It follows from its definition that $A^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}} = A^{\mathcal{I}_{\mathcal{P}_1}} \cup A^{\mathcal{I}_{\mathcal{P}_2}}$. Therefore we have as desired $A^{\mathcal{J}_1 \cup \mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$.
3. Let $A \in \mathsf{M}$ such that $A^{\mathcal{J}_1 \cup \mathcal{J}_2} \not\subseteq A^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$. We may assume w.l.o.g. that A is minimal w.r.t. $\prec$ for this property. We thus have $e \in A^{\mathcal{J}_1 \cup \mathcal{J}_2} \setminus A^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$. We either have $e \in \Delta^{\mathcal{J}_1}$ or $e \in \Delta^{\mathcal{J}_2}$. We treat the case $e \in \Delta^{\mathcal{J}_1}$, the arguments for $e \in \Delta^{\mathcal{J}_2}$ are similar. From $\mathcal{J}_1 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1}$, there exists a concept $B_1 \prec A$ s.t. $B_1^{\mathcal{J}_1} \subsetneq B_1^{\mathcal{I}_{\mathcal{P}_1}}$. We chose such a B_1 that is minimal for this property, that is s.t. for all $B \prec B_1$, we have $B^{\mathcal{J}_1} = B^{\mathcal{I}_{\mathcal{P}_1}}$ (recall A is minimal for its property so that such a B must verify $B^{\mathcal{J}_1 \cup \mathcal{J}_2} \subseteq B^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$ hence $B^{\mathcal{J}_1} \subseteq B^{\mathcal{I}_{\mathcal{P}_1}}$). If $B_1^{\mathcal{J}_2} \subseteq B_1^{\mathcal{I}_{\mathcal{P}_2}}$, then we are done as we obtain $B_1^{\mathcal{J}_1 \cup \mathcal{J}_2} \subsetneq B_1^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$. Otherwise, we have $B_1^{\mathcal{J}_2} \not\subseteq B_1^{\mathcal{I}_{\mathcal{P}_2}}$ and from $\mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_2}$, there exists a concept $B_2 \prec B_1$ s.t. $B_2^{\mathcal{J}_2} \subsetneq B_2^{\mathcal{I}_{\mathcal{P}_2}}$. Since B_1 has been chosen minimal, we have in particular $B_2^{\mathcal{J}_1} = B_2^{\mathcal{I}_{\mathcal{P}_1}}$, which yields $B_2^{\mathcal{J}_1 \cup \mathcal{J}_2} \subsetneq B_2^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$.
4. From $\mathcal{J}_1 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1}$, there exists a concept $A_1 \in \mathsf{M}$ s.t. $A_1^{\mathcal{J}_1} \subsetneq A_1^{\mathcal{I}_{\mathcal{P}_1}}$ and for all $B \prec A_1$, we have $B^{\mathcal{J}_1} = B^{\mathcal{I}_{\mathcal{P}_1}}$. If $A_1^{\mathcal{J}_2} \subsetneq A_2^{\mathcal{P}_2}$ and for all $B \prec A_1$, $B^{\mathcal{J}_2} = B^{\mathcal{P}_2}$, then we obtain directly $A_1^{\mathcal{J}_1 \cup \mathcal{J}_2} \subsetneq A_1^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$ and for all $B \prec A_1$, $B^{\mathcal{J}_1 \cup \mathcal{J}_2} = B^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$ and we are done. Otherwise:
   - If $A_1^{\mathcal{J}_2} \not\subseteq A_1^{\mathcal{I}_{\mathcal{P}_2}}$, then from $\mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_2}$, there exists a concept $A_2 \prec A_1$ s.t. $A_2^{\mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_2}}$. Consider a minimal such A_2, that is s.t. for all $A \prec A_2$, we have $A^{\mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_2}}$ or $A^{\mathcal{J}_2} \not\subseteq A^{\mathcal{I}_{\mathcal{P}_2}}$. Notice the second option cannot happen otherwise from $\mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_2}$ we could obtain $A_3 \prec A_2$ s.t. $A_2^{\mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_2}}$, contradicting the minimality of A_2. Therefore A_2 being minimal yields that for all $A \prec A_2$, we have $A^{\mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_2}}$. Now, from the minimality of A_1, we extend this to $\mathcal{J}_1 \cup \mathcal{J}_2$ and obtain $A_2^{\mathcal{J}_1 \cup \mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$ and for all $B \prec A_2$, $B^{\mathcal{J}_1 \cup \mathcal{J}_2} = B^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$.
   - If there exists $A_2 \prec A_1$ s.t. $A_2^{\mathcal{J}_2} \neq A_2^{\mathcal{I}_{\mathcal{P}_2}}$. Consider a minimal such A_2, that is s.t. for all $A \prec A_2$, we have $A^{\mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_2}}$. Notice we must have $A_2^{\mathcal{J}_2} \subseteq A_2^{\mathcal{I}_{\mathcal{P}_2}}$, otherwise from $\mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_2}$ we could obtain $A_3 \prec A_2$ s.t. $A_2^{\mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_2}}$, contradicting that for all $A \prec A_2$, we have $A^{\mathcal{J}_2} = A^{\mathcal{I}_{\mathcal{P}_2}}$. Therefore, since $A_2^{\mathcal{J}_2} \neq A_2^{\mathcal{I}_{\mathcal{P}_2}}$, it must be that $A_2^{\mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_2}}$. Now, from the minimality of A_1, we extend this to $\mathcal{J}_1 \cup \mathcal{J}_2$ and obtain $A_2^{\mathcal{J}_1 \cup \mathcal{J}_2} \subsetneq A_2^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$ and for all $B \prec A_2$, $B^{\mathcal{J}_1 \cup \mathcal{J}_2} = B^{\mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}}$.

Overall, this proves $\mathcal{J}_1 \cup \mathcal{J}_2 <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}_1 \cup \mathcal{P}_2}$ as desired. □

We are now ready to properly prove Lemma 9.

Proof of Lemma 9. "$1 \Rightarrow 2$". Assume there exists $\mathcal{J} \models \mathcal{K}$ s.t. $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. Based on $\mathcal{J}$, we build a subset $\mathcal{P} \subseteq \Delta^\mathcal{J}$ containing:

- for each role r s.t. $r^\mathcal{J} \neq \emptyset$, an element $w'_r \in (\exists r^-)^\mathcal{J}$;
- for each $A \in \mathsf{M}$ s.t. $A^\mathcal{J} \not\subseteq A^\mathcal{I}$, an element $e_A \in B^\mathcal{J} \setminus B^\mathcal{I}$ for some $B \prec A$ (Condition 3 in the definition of $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ ensures existence of such B and e_A);
- an element $e_\mathsf{M} \in A^\mathcal{J} \setminus A^\mathcal{I}$ for some $A \in \mathsf{M}$ s.t. $A^\mathcal{J} \subsetneq A^\mathcal{I}$ and for all $B \prec A$, $B^\mathcal{J} = B^\mathcal{I}$ (Condition 4 in the definition of $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ ensures existence of such A and e_M).

Notice $\mathcal{P}$ has size at most $2|\mathcal{T}| + 1$. We now build $\mathcal{J}_e$ as:

$$\begin{aligned}
\Delta^{\mathcal{J}_e} &= \Delta^{\mathcal{I}_{\mathcal{P} \cup \{e\}}} \\
A^{\mathcal{J}_e} &= A^\mathcal{J} \cap \Delta^{\mathcal{J}_e} \\
r^{\mathcal{J}_e} &= r^\mathcal{J} \cap (\mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A})) \\
&\quad \cup \{(e, w'_s) \mid e \in (\exists s)^\mathcal{J}, \mathcal{T} \models s \sqsubseteq r\} \\
&\quad \cup \{(w'_s, e) \mid e \in (\exists s)^\mathcal{J}, \mathcal{T} \models s \sqsubseteq r^-\}
\end{aligned}$$

It is easily verified that $\mathcal{J}_e$ is a model of $\mathcal{K}$, and we now prove $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P} \cup \{e\}}$. We check that all four conditions from the definition of $<_{\mathsf{CP}}$ are satisfied:

1. By definition, we have $\Delta^{\mathcal{J}_e} = \Delta^{\mathcal{I}_{\mathcal{P} \cup \{e\}}}$.
2. Let $A \in \mathsf{F}$. Definitions of $\mathcal{I}_{\mathcal{P} \cup \{e\}}$ and $\mathcal{J}_e$ ensure $A^\mathcal{I} \cap \Delta^{\mathcal{I}_{\mathcal{P} \cup \{e\}}} = A^{\mathcal{I}_{\mathcal{P} \cup \{e\}}}$ and $A^\mathcal{J} \cap \Delta^{\mathcal{I}_{\mathcal{P} \cup \{e\}}} = A^{\mathcal{J}_e}$. From $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, we get $A^\mathcal{I} = A^\mathcal{J}$, which thus yields $A^{\mathcal{I}_{\mathcal{P} \cup \{e\}}} = A^{\mathcal{J}_e}$.

3. Let $A \in \mathsf{M}$ such that $A^{\mathcal{J}_e} \not\subsetneq A^{\mathcal{I}_{\mathcal{P}\cup\{e\}}}$. Therefore $A^{\mathcal{J}} \not\subsetneq A^{\mathcal{I}}$, and recall we kept in $\mathcal{P}$ an element $e_A \in B^{\mathcal{J}} \setminus B^{\mathcal{I}}$ for some $B \prec A$ to belong to $\mathcal{P}$. Joint with $B^{\mathcal{J}_e} \subseteq B^{\mathcal{I}_{\mathcal{P}\cup\{e\}}}$ being trivial, e_A additionally witnesses that $B^{\mathcal{J}_e} \subsetneq B^{\mathcal{I}_{\mathcal{P}\cup\{e\}}}$.
4. Recall we kept in $\mathcal{P}$ an element $e_{\mathsf{M}} \in A^{\mathcal{J}} \setminus A^{\mathcal{I}}$ for some $A \in \mathsf{M}$ s.t. $A^{\mathcal{J}} \subsetneq A^{\mathcal{I}}$ and for all $B \prec A$, $B^{\mathcal{J}} = B^{\mathcal{I}}$. It gives immediately that $A^{\mathcal{J}_e} \subsetneq A^{\mathcal{I}_{\mathcal{P}\cup\{e\}}}$ and for all $B \prec A$, $B^{\mathcal{J}_e} = B^{\mathcal{I}_{\mathcal{P}\cup\{e\}}}$.

This proves $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}\cup\{e\}}$. It is straightforward from the definition of each $\mathcal{J}_e$ that for all e, e' in $\Delta^{\mathcal{I}}$, we have $\mathcal{J}_e|_{\Delta^{\mathcal{I}_\mathcal{P}}} = \mathcal{J}_{e'}|_{\Delta^{\mathcal{I}_\mathcal{P}}}$, concluding the proof of "1 ⇒ 2".

"2 ⇒ 1". Assume there exist $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$ and a family $(\mathcal{J}_e)_{e\in\Delta^{\mathcal{I}}}$ of models of $\mathcal{K}$ such that $\mathcal{J}_e <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}\cup\{e\}}$ and $\mathcal{J}_e|_{\Delta^{\mathcal{I}_\mathcal{P}}} = \mathcal{J}_{e'}|_{\Delta^{\mathcal{I}_\mathcal{P}}}$ for all $e, e' \in \Delta^{\mathcal{I}}$. Consider such a $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$, and family of models $\mathcal{J}_e$ for each $e \in \Delta^{\mathcal{I}}$. We build:

$$\mathcal{J} = \bigcup_{e\in\Delta^{\mathcal{I}}} \mathcal{J}_e$$

It is clear $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and since all $\mathcal{J}_e$ agree on the shared domain $\Delta^{\mathcal{I}_\mathcal{P}}$, we apply Lemma 23 to obtain $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. □

Theorem 6. *UCQ evaluation on circumscribed DL-Lite*$_{\mathsf{bool}}$ *KBs is in* coNExp *w.r.t. combined complexity.*

Proof. We exhibit a NExp procedure to decide the complement of our problem, that is existence of a countermodel for UCQ q over DL-Lite$_{\mathsf{bool}}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Our procedure starts by guessing a interpretation $\mathcal{I}$ whose domain has size at most $\mathcal{A} + 2^{|\mathcal{T}||q|}$. This can be done in exponentially many non-deterministic steps. It further checks whether $\mathcal{I}$ is a model of $\mathcal{K}$ that does not entail q, and rejects otherwise. This can essentially be done naively in $\left|\Delta^{\mathcal{I}}\right|^{|\mathcal{T}||q|}$ deterministic steps, that is still simply exponential w.r.t. our input. The procedure finally checks whether $\mathcal{I}$ complies with CP by iterating over each $\mathcal{P}$ with $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$ and over each model $\mathcal{J}_\mathcal{P}$ of $\mathcal{K}$ s.t. $\mathcal{J}_\mathcal{P} < \mathcal{I}_\mathcal{P}$. If, for a choice of $\mathcal{P}$ and $\mathcal{J}_\mathcal{P}$, we can find a $\mathcal{J}_{\mathcal{P},e}$ for each $e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_\mathcal{P}}$ s.t. $\mathcal{J}_{\mathcal{P},e} \models \mathcal{K}$ with $\mathcal{J}_{\mathcal{P},e} <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}\cup\{e\}}$ and $\mathcal{J}_{\mathcal{P},e}|_{\Delta^{\mathcal{J}_\mathcal{P}}} = \mathcal{J}_\mathcal{P}$, then our procedure rejects. Otherwise, that is all choices of $\mathcal{P}$ and $\mathcal{J}_\mathcal{P}$ led to the existence of a $e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_\mathcal{P}}$ without a fitting $\mathcal{J}_{\mathcal{P},e}$, then it accepts. Notice that: iterating over such $\mathcal{P}$ can be done in $\left|\Delta^{\mathcal{I}}\right|^{2|\mathcal{T}|+1}$ iterations since we require $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$. Computing $\mathcal{I}_\mathcal{P}$ follows directly from the choice of $\mathcal{P}$. Since each $\mathcal{I}_\mathcal{P}$ has polynomial size, iterating over each $\mathcal{J}_\mathcal{P}$ can be done naively with exponentially many steps. Further iterating over each $e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_\mathcal{P}}$ can be done in $\left|\Delta^{\mathcal{I}}\right|$ steps, which is at most exponential by construction. Since each $\mathcal{I}_{\mathcal{P}\cup\{e\}}$ has polynomial size, deciding the existence of $\mathcal{J}_{\mathcal{P},e}$ can be done naively in exponentially many steps. Overall, the procedure uses an exponential number of non-deterministic steps at the beginning and further performs several checks using exponentially many more deterministic steps.

It remains to prove there exists an accepting run iff there is a countermodel for q over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. If there exists an accepting run, then the corresponding guessed interpretation $\mathcal{I}$ is a model of $\mathcal{K}$ that does not entail q, and it must also comply with CP otherwise the "1 ⇒ 2" direction from Lemma 9 ensures the procedure would have rejected it. Conversely, if there exists a countermodel, then Lemma 8 ensures existence of a countermodel $\mathcal{I}$ whose domain has exponential size. This $\mathcal{I}$ can be guessed by the procedure which checks it is indeed a model of $\mathcal{K}$ not entailing q, hence do not reject it at first. The "2 ⇒ 1" direction from Lemma 9 further ensures $\mathcal{I}$ also passes the remaining check performed by the procedure as otherwise it would contradict $\mathcal{I}$ being a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. □

Theorem 7. *UCQ answering on circumscribed DL-Lite*$_{\mathsf{core}}$ *KBs is* coNExp*-hard w.r.t. combined complexity. This holds even with a single minimized concept name, no fixed concept names and no disjointness constraints.*

Proof. The proof proceeds by reduction from the complement of the NExp-complete Succinct-3COL problem. An instance of Succinct-3COL consists of a Boolean circuit C with $2n$ input gates. The graph G_C encoded by C has 2^n vertices, identified by binary encodings on n bits. Two vertices u and v, with respective binary encodings $u_1 \dots u_n$ and $v_1 \dots v_n$, are adjacent in G_C iff C returns True when given as input $u_1 \dots u_n$ on its first n gates and $v_1 \dots v_n$ on the second half. The problem of deciding if G_C is 3-colorable has been proven to be NExp-complete in (Papadimitriou and Yannakakis 1986).

Let C be a Boolean circuit with $2n$ input gates. Let $G = (V, E)$ be the corresponding graph with $V = \{v_1, v_2, \cdots, v_{2^n}\}$. We denote $\bar{i}$ the binary encoding on n bits of vertex v_i. We also identify t (True) and f (False) with their usual binary valuation 1 and 0, respectively. We construct a circumscribed DL-Lite$_{\mathsf{core}}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ and a UCQ q that encode a given problem instance. With the aim to restrict colors and truth values to those defined in the ABox (see axioms (50)), we minimize exactly the concept name Min and let others vary freely. Our reduction starts by representing the colors (46) and truth values (48) in the ABox. The role name neq makes the inequality relations within them explicit; see axioms (47) and (49).

$$\mathsf{Col}(c) \text{ for all } c \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\} \tag{46}$$

$$\mathsf{neq}(c, c') \text{ for all } c, c' \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\} \text{ with } c \neq c' \tag{47}$$

$$\mathsf{Val}_b(\mathsf{val}_b) \text{ for all } b \in \{\mathsf{f}, \mathsf{t}\} \tag{48}$$

$$\mathsf{neq}(\mathsf{val}_b, \mathsf{val}_{b'}) \text{ for all } b, b' \in \{\mathsf{f}, \mathsf{t}\} \text{ with } b \neq b' \tag{49}$$

$$\mathsf{Min}(x) \text{ for all } x \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}, \mathsf{val}_\mathsf{t}, \mathsf{val}_\mathsf{f}\} \tag{50}$$

We aim countermodels to contain 2^{2n} elements that each encode a pair $(v_k, v_l) \in V^2$. Towards this goal, we lay the foundation to branch a binary tree of depth n from a root a_{tree}, which will be later enforced via the constructed UCQ further down. The concept names $(\mathsf{Index}_i)_{i=1,\cdots,2n}$ encode the tree levels, which fork via the role names $\mathsf{next}_{i,\mathsf{t}}$ and $\mathsf{next}_{i,\mathsf{f}}$, encoding whether the i-th bit in $\bar{k} \cdot \bar{l}$ should be set to t or f, respectively. The desired elements will therefore be (a subset of) the extension Index_{2n}. A role name $(\mathsf{hBit}_j)_{j=1,\cdots,2n}$ will later set the actual bit values. Note that axioms (54) require any i-th tree level to set the bits 1 to i, so that the UCQ can later ensure the truth values to coincide with the choice via next. Formally:

$$\mathsf{Index}_0(a_{\mathsf{tree}}) \tag{51}$$

$$\mathsf{Index}_{i-1} \sqsubseteq \exists \mathsf{next}_{i,b} \tag{52}$$

$$\exists \mathsf{next}^-_{i,b} \sqsubseteq \mathsf{Index}_i \tag{53}$$

$$\mathsf{Index}_i \sqsubseteq \exists \mathsf{hBit}_j \qquad \text{for all } j \in \{1, \cdots, i\} \tag{54}$$

$$\exists \mathsf{hBit}^-_j \sqsubseteq \mathsf{Min} \qquad \text{for all } j \in \{1, \cdots, 2n\} \tag{55}$$

for all $i \in \{1, \cdots, 2n\}$ and $b \in \{\mathsf{t}, \mathsf{f}\}$.

We encode the color assignments of (v_k, v_l) only at the $2n$-th level via the role name hCol for v_k and hCol$'$ for v_l:

$$\mathsf{Index}_{2n} \sqsubseteq \exists \mathsf{hCol} \qquad \mathsf{Index}_{2n} \sqsubseteq \exists \mathsf{hCol}' \tag{56}$$

$$\exists \mathsf{hCol}^- \sqsubseteq \mathsf{Min} \qquad \exists \mathsf{hCol}'^- \sqsubseteq \mathsf{Min} \tag{57}$$

Finally, the computation of a gate g in C is enforced for each pair by an outgoing edge gate_g at the $2n$-th level. Add:

$$\mathsf{Index}_{2n} \sqsubseteq \exists \mathsf{gate}_g \tag{58}$$

$$\exists \mathsf{gate}_g \sqsubseteq \mathsf{Min} \tag{59}$$

for all gates g in C.

Let the constructed UCQ q be the disjunction over all subsequent queries. We first make sure that no color is used as a truth value (CQs (60) and (61)) or vice-versa (CQs (62) and (63)):

$$\exists x, y\, \mathsf{hBit}_j(x, y) \land \mathsf{Col}(y) \text{ for all } j \in \{1, \cdots, 2n\} \tag{60}$$

$$\exists x, y\, \mathsf{gate}_g(x, y) \land \mathsf{Col}(y) \text{ for all gates } g \text{ in } G' \tag{61}$$

$$\exists x, y\, \mathsf{hCol}(x, y) \land \mathsf{Val}_b(y) \text{ for all } b \in \{\mathsf{t}, \mathsf{f}\} \tag{62}$$

$$\exists x, y\, \mathsf{hCol}'(x, y) \land \mathsf{Val}_b(y) \text{ for all } b \in \{\mathsf{t}, \mathsf{f}\} \tag{63}$$

We next simultaneously enforce that a) the structure constructed by axioms (51)–(53) really is a tree, and b) that the valuation required by axioms (54) and (55) is consistent with the branching of the tree. We achieve this via CQs (64), which makes countermodels assign the bits by hBit as dictated by next, and via CQs (65), which makes countermodels propagate the bit assignment of a node downwards the tree to all successors:

$$\exists x, y, z_1, z_2\, \mathsf{next}_{i-1,b}(x, y) \land \mathsf{hBit}_j(y, z_1) \land \mathsf{neq}(z_1, z_2) \land \mathsf{Val}_b(z_2) \tag{64}$$

$$\exists x, y, z_1, z_2\, \mathsf{next}_{i-1,b}(x, y) \land \mathsf{hBit}_j(x, z_1) \land \mathsf{hBit}_j(x, z_2) \land \mathsf{neq}(z_1, z_2) \tag{65}$$

for all $i \in \{1, \cdots, 2n\}$, $j \in \{1, \cdots, i\}$ and $b \in \{\mathsf{t}, \mathsf{f}\}$. Observe that the above indeed enforces a tree.

We go on to construct CQs (66) and (67), which prohibit any two encoded pairs of vertices (v, u) and (v, u') or (v, u) and (u, u') to assign different colors to v or u, respectively:

$$\exists x, y, c, d, z_1, \cdots, z_n\, \mathsf{hCol}(x, c) \land \mathsf{hCol}(y, d) \land \mathsf{neq}(c, d) \land \bigwedge_{1 \le j \le n} \big(\mathsf{hBit}_j(y, z_j) \land \mathsf{hBit}_j(x, z_j)\big) \tag{66}$$

$$\exists x, y, c, d, z_1, \cdots, z_n\, \mathsf{hCol}(x, c) \land \mathsf{hCol}'(y, d) \land \mathsf{neq}(c, d) \land \bigwedge_{1 \le j \le n} \big(\mathsf{hBit}_j(y, z_j) \land \mathsf{hBit}'_{j+n}(x, z_j)\big) \tag{67}$$

To make sure that the computation of the boolean circuit is consistent, we construct CQs restricting gate_g to follow the logical operator of g. For this, we assume w.l.o.g. that C only contains unary NOT-gates, binary AND- and OR-gates, and $2n$ nullary INPUT-gates (each encoding a bit in $\bar{k} \cdot \bar{l}$). Given an INPUT-gate g from C, we denote the index of the encoded bit from $\bar{k} \cdot \bar{l}$ with i_g. Construct, for each INPUT-gate g from C, the CQ as given next:

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{hBit}_{i_g}(x, z) \land \mathsf{neq}(y, z) \tag{68}$$

For all NOT-gates g with their parent g', construct

$$\exists x, y\, \mathsf{gate}_g(x, y) \land \mathsf{gate}_{g'}(x, y) \tag{69}$$

For all AND-gates g with parents g_1, g_2, construct

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{t}(y) \land \mathsf{gate}_{g_1}(x, z) \land \mathsf{Val}_\mathsf{f}(z) \tag{70}$$

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{t}(y) \land \mathsf{gate}_{g_2}(x, z) \land \mathsf{Val}_\mathsf{f}(z) \tag{71}$$

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{f}(y) \land \mathsf{gate}_{g_1}(x, z) \land \mathsf{Val}_\mathsf{t}(z) \land \mathsf{gate}_{g_2}(x, z) \tag{72}$$

For all OR-gates g with parents g_1, g_2, construct

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{f}(y) \land \mathsf{gate}_{g_1}(x, z) \land \mathsf{Val}_\mathsf{t}(z) \tag{73}$$

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{f}(y) \land \mathsf{gate}_{g_2}(x, z) \land \mathsf{Val}_\mathsf{t}(z) \tag{74}$$

$$\exists x, y, z\, \mathsf{gate}_g(x, y) \land \mathsf{Val}_\mathsf{t}(y) \land \mathsf{gate}_{g_1}(x, z) \land \mathsf{Val}_\mathsf{f}(z) \land \mathsf{gate}_{g_2}(x, z) \tag{75}$$

Finally, we rule out the existence of monochromatic edges in countermodels via CQ (76):

$$\exists x, y, z\, \mathsf{hCol}(x, y) \land \mathsf{hCol}'(x, y) \land \mathsf{gate}_{\dot{g}}(x, z) \land \mathsf{Val}_\mathsf{t}(z) \tag{76}$$

where we here and in what follows denote by $\dot{g}$ the output gate of C.

Lemma 24. *G is 3-colorable iff* $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q$.

"$\Rightarrow$". Let G be 3-colorable. Then there exists a 3-coloring $\pi : V \to \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ such that $\pi(v) \neq \pi(u)$ for all $(v, u) \in V$. To show that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models q$, we construct a countermodel $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ as follows. Let $X := \bigcup_{j=1,\cdots,2n}\{a_{j,1}, \cdots, a_{j,2^j}\}$; then set

$$\Delta^{\mathcal{I}} := \mathsf{ind}(\mathcal{A}) \cup X$$
$$A^{\mathcal{I}} := \{a \mid A(a) \in \mathcal{A}\}$$
$$\mathsf{neq}^{\mathcal{I}} := \{(a,b) \mid \mathsf{neq}(a,b) \in \mathcal{A}\}$$
$$\mathsf{Index}_i^{\mathcal{I}} := \{a_{i,1}, \cdots, a_{i,2^i}\}$$
$$\mathsf{next}_{i,\mathsf{t}}^{\mathcal{I}} := \bigcup_{j=1,\cdots,2^{i-1}} \{(a_{i-1,j}, a_{i,2j})\}$$
$$\mathsf{next}_{i,\mathsf{f}}^{\mathcal{I}} := \bigcup_{j=1,\cdots,2^{i-1}} \{(a_{i-1,j}, a_{i,2j-1})\}$$
$$\mathsf{hBit}_i^{\mathcal{I}} := \bigcup_{j=1,\cdots,2^{i-1}} \big(\{(a, \mathsf{val_t}) \mid a \in R(a_{i,2j})\} \cup \{(a, \mathsf{val_f}) \mid a \in R(a_{i,2j-1})\}\big)$$

for all concept names $A \in \{\mathsf{Col}, \mathsf{Val_t}, \mathsf{Val_f}, \mathsf{Min}, \mathsf{Index}_0\}$ and $i \in \{1, \cdots, 2n\}$, where we use

- $a_{0,1}$ to denote a_{tree}, and
- $R(a_{k,l})$ to denote the set of instances that are reachable in $\mathcal{I}$ from $a_{k,l} \in X$ via the family of role names $\mathsf{next}_{i,\mathsf{t}}$ and $\mathsf{next}_{i,\mathsf{f}}$ (including $a_{k,l}$ itself).

We furthermore set $\mathsf{hCol}^{\mathcal{I}}, \mathsf{hCol}'^{\mathcal{I}}$ as follows. Given some $a_{2n,i} \in X$ with $i \in \{1, \cdots 2^{2n}\}$, we denote by $\vartheta(a_{2n,i})$ the encoded pair $(v_k, u_l) \in V^2$, i.e., such that for all $j \in \{1, \cdots, 2n\}$, we have $(a_{2n,i}, \mathsf{val}_b) \in \mathsf{hBit}_j^{\mathcal{I}}$ iff the j-th bit in $\bar{k} \cdot \bar{l}$ is b with $b \in \{\mathsf{t}, \mathsf{f}\}$.

$$\mathsf{hCol}^{\mathcal{I}} := \{(a_{2n,i}, \pi(v)) \mid \vartheta(a_{2n,i}) = (v,u)\}$$
$$\mathsf{hCol}'^{\mathcal{I}} := \{(a_{2n,i}, \pi(u)) \mid \vartheta(a_{2n,i}) = (v,u)\}$$

Finally, we set gate_g for all gates g in G_C. We denote the sub-circuit of G_C that contains exactly g and all its ancestors with G_C^g. Note that the output gate of G_C^g is g, i.e., $G_C^g(\bar{k}, \bar{l})$ computes the value of g in G_C given the input $(\bar{k}, \bar{l})$. Then, for all gates g, set:

$$\mathsf{gate}_g^{\mathcal{I}} := \{(a_{2n,i}, \mathsf{val}_b) \mid \vartheta(a_{2n,i}) = (v_k, u_l) \text{ and } G_C^g(\bar{k}, \bar{l}) = b\}$$

It is obvious that $\mathcal{I}$ is a model of $\mathcal{K}$. As $a \in \mathsf{Min}^{\mathcal{I}}$ iff $\mathsf{Min}(a)$ for all $a \in \Delta^{\mathcal{I}}$, and Min is the only minimized concept, $\mathcal{I}$ must furthermore be minimal.

To see that $\mathcal{I} \not\models q$, it is readily checked that

- $\mathcal{I}$ by definition has no answer to CQs (60)–(67),
- additionally, $\mathcal{I}$ has no answer to CQs (68)–(75), as otherwise, the computation of G_C would be inconsistent, and
- finally, $\mathcal{I}$ has no answer to CQ (76), as otherwise, π would not be a 3-coloring of G.

"$\Leftarrow$". Let $\mathsf{Circ_{CP}}(\mathcal{K}) \not\models q$. Then there is a model $\mathcal{I}$ of $\mathsf{Circ_{CP}}(\mathcal{K})$ such that $\mathcal{I} \not\models q$. We aim to extract a function $\pi : V \to \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ and to show that π is indeed a 3-coloring of G. This is rather laborious, as we have to prepare the final argument in five steps. In Claim (4)–(8), respectively, we

1. briefly argue that hBit and gate have the range $\{\mathsf{val_t}, \mathsf{val_f}\}$ and that hCol and hCol$'$ have the range $\{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$, which will be convenient later on,
2. identify within $\mathcal{I}$ the intended tree-structure,
3. show that each leaf indeed encodes a pair $(v_k, v_l) \in V^2$, setting precisely the bits from $\bar{k} \cdot \bar{l}$,
4. extract π from $\mathcal{I}$ and show that it is sound in the sense that any two leaves that encode the same vertex also assign it the same color, and
5. proof that the leaves correctly compute all gates.

We start with the ranges of hBit, gate and hCol:

Claim 4. *The following properties hold.*

- *both $(x,y) \in \mathsf{hBit}_i^{\mathcal{I}}$ and $(x,y) \in \mathsf{gate}_g^{\mathcal{I}}$ entail $y \in \{\mathsf{val_t}, \mathsf{val_f}\}$ for all $i \in \{1, \cdots, 2n\}$ and gates g in C, and*
- *both $(x,y) \in \mathsf{hCol}^{\mathcal{I}}$ and $(x,y) \in \mathsf{hCol}'^{\mathcal{I}}$ entail $y \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$.*

The proof is simple: As $\mathcal{I}$ is a model of $\mathcal{A}$, axioms (50) imply $\{\mathsf{r}, \mathsf{g}, \mathsf{b}, \mathsf{val_t}, \mathsf{val_f}\} \subseteq \mathsf{Min}^{\mathcal{I}}$. Furthermore, it is obvious that there cannot be any $a \in \mathsf{Min}^{\mathcal{I}}$ with $\mathsf{Min}(a) \notin \mathcal{A}$, since $\mathcal{I}$ would otherwise not be $<_{\mathsf{CP}}$ minimal (it would then be straightforward to construct a smaller model). Ergo, $\mathsf{Min}^{\mathcal{I}} = \{\mathsf{r}, \mathsf{g}, \mathsf{b}, \mathsf{val_t}, \mathsf{val_f}\}$.

The rest of the claim follows from axioms (46), (48), (55), (57), (59), CQs (60)–(63), and $\mathcal{I} \not\models q$: For example, in the case of hCol, consider some $(x,y) \in \mathsf{hCol}^{\mathcal{I}}$. By axiom (57), we have that $y \in \mathsf{Min}^{\mathcal{I}}$. CQ (62) and the fact that $\mathcal{I} \not\models q$ together yield $y \in \mathsf{Min}^{\mathcal{I}} \setminus (\mathsf{Val_t} \cup \mathsf{Val_f})$. As $\{\mathsf{val_t}, \mathsf{val_f}\} \subseteq \mathsf{Val_t} \cup \mathsf{Val_f}$ by axioms (48), we thus have that $y \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$. The other cases are analogous.

This finishes the proof of Claim (4).

We go on by identifying within $\mathcal{I}$ the intended tree-structure:

Claim 5. *For all $i \in \{1, \cdots, 2n\}$ there is a set $Y_i \subseteq \mathsf{Index}_i^{\mathcal{I}}$ that satisfies the following properties:*

1. *for all $y \in Y_i$, there is exactly one* parent *$p \in Y_{i-1}$ of y with $(p,y) \in \mathsf{next}_{i-1,b}^{\mathcal{I}}$ for some $b \in \{\mathsf{t}, \mathsf{f}\}$ (we set $Y_0 = \{a_{\mathsf{tree}}\}$),*
2. *for all $y \in Y_i$ and $j \in \{1, \cdots, i\}$, there is exactly one $b \in \{\mathsf{t}, \mathsf{f}\}$ such that $(y, \mathsf{val}_n) \in \mathsf{hBit}_j$,*
3. *for all $x, y \in Y_i$ with $x \neq y$, there is some $j \in \{1, \cdots, i\}$ such that $(x, \mathsf{val_t}) \in \mathsf{hBit}_j$ iff $(y, \mathsf{val_f}) \in \mathsf{hBit}_j$, and*
4. *$|Y_i| = 2^i$.*

The proof of Claim (5) goes via induction over i:

Base case $(i = 1)$**.** Axioms (51)–(53) require the existence of some $x, y \in \mathsf{Index}_1^{\mathcal{I}}$ such that $(a_{\mathsf{tree}}, x) \in \mathsf{next}_{1,\mathsf{t}}^{\mathcal{I}}$ and $(a_{\mathsf{tree}}, y) \in \mathsf{next}_{1,\mathsf{f}}^{\mathcal{I}}$. Set $Y_1 = \{x, y\}$. Property 1. of Claim (5) obviously holds as a_{tree} is, on one hand, the only element in Y_0, and, on the other hand, the parent of both x and y. Together with Claim (4), axioms (49), and (54), and CQs (64) (and the fact that $\mathcal{I} \not\models q$), we have that

$(x, \mathsf{val_t}) \in \mathsf{hBit}_1^{\mathcal{I}}$ and $(y, \mathsf{val_f}) \in \mathsf{hBit}_1^{\mathcal{I}}$, while $(x, \mathsf{val_f}) \notin \mathsf{hBit}_1^{\mathcal{I}}$ and $(y, \mathsf{val_t}) \notin \mathsf{hBit}_1^{\mathcal{I}}$, i.e., also properties 2. and 3. hold. The above furthermore entails $x \neq y$, which is why also property 4. must hold.

Induction step ($i > 1$). By the induction hypothesis, there is a set Y_{i-1} with the claimed properties. For all $y \in Y_{i-1}$, axioms (51)–(53) entail the existence of elements $a_y, a'_y \in \mathsf{Index}_i^{\mathcal{I}}$ such that $(y, a_y) \in \mathsf{next}_{i,\mathsf{t}}^{\mathcal{I}}$ and $(y, a'_y) \in \mathsf{next}_{i,\mathsf{f}}^{\mathcal{I}}$. Similar as in the base case, we deduct that $(a_y, \mathsf{val_t}) \in \mathsf{hBit}_i^{\mathcal{I}}$ and $(a'_y, \mathsf{val_f}) \in \mathsf{hBit}_i^{\mathcal{I}}$, while $(a_y, \mathsf{val_f}) \notin \mathsf{hBit}_i^{\mathcal{I}}$ and $(a'_y, \mathsf{val_t}) \notin \mathsf{hBit}_i^{\mathcal{I}}$ (†) via Claim (4), axioms (49) and (54), and CQs (64) (and the fact that $\mathcal{I} \not\models q$). Set $Y_i = \bigcup_{y \in Y_{i-1}} \{a_y, a'_y\}$.

To see that Y_i satisfies property 1., assume to the contrary that there is some $a \in Y_i$ with two parents from Y_{i-1}, i.e., that there are two elements $y, y' \in Y_{i-1}$ such that $(y, a) \in \mathsf{next}_{i-1,b}^{\mathcal{I}}$ and $(y', a) \in \mathsf{next}_{i-1,b'}^{\mathcal{I}}$ for some $b, b' \in \{\mathsf{t}, \mathsf{f}\}$. By property 3. of the induction hypothesis, there must be some $j \in \{1, \cdots, i-1\}$ such that $(x, \mathsf{val_t}) \in \mathsf{hBit}_j^{\mathcal{I}}$ iff $(x', \mathsf{val_f}) \in \mathsf{hBit}_j^{\mathcal{I}}$. Furthermore observe that Claim (4) and axioms (54) require a to assign the j-th bit, i.e., we have $(a, \mathsf{val_t}) \in \mathsf{hBit}_j^{\mathcal{I}}$ or $(a, \mathsf{val_f}) \in \mathsf{hBit}_j^{\mathcal{I}}$. In either case, due to axioms (49), we can find a homomorphism from CQ (65) to $\mathcal{I}$, which contradicts the fact that $\mathcal{I}$ is a countermodel to q. Ergo, Y_i must satisfy property 1.

Property 2. follows, in case of $j = i$, from (†), and, if otherwise $j \in \{1, \cdots, i-1\}$, from Y_{i-1} satisfying property 2. and the fact that Claim (4), axioms (49) and (54), and CQs (65) require all $a \in Y_i$ to assign the bits $1, \cdots, i-1$ exactly as their parents.

For property 3, consider some $a, a' \in Y_i$ with $a \neq a'$. If a and a' have the same parent x, property 3. follows from (†) and property 1. Otherwise, if a has a parent x and a' has a parent y such that $x \neq y$, observe that property 3. of the induction hypothesis implies the existence of some $j \in \{1, \cdots, i-1\}$ such that $(x, \mathsf{val_t}) \in \mathsf{hBit}_j$ iff $(y, \mathsf{val_f}) \in \mathsf{hBit}_j$. Property 3. then follows from the fact that Claim (4), axioms (49) and (54), and CQs (65) require a and a' to assign the bits $1, \cdots, i-1$ exactly as their parents.

Regarding property 4., observe that by (†), Y_i contains two elements a_y, a'_y with $a_y \neq a'_y$ for each $y \in Y_{i-1}$. Given any two elements $x, y \in Y_{i-1}$, it is furthermore easy to see that $\{a_x, a'_x\}$ and $\{a_y, a'_y\}$ are disjoint, as by property 2. and 3. of the induction hypothesis, there must be some bit $j \in \{1, \cdots, i-1\}$ that x and y assign differently, which subsequently must then also be assigned differently by $\{a_x, a'_x\}$ and $\{a_y, a'_y\}$ due to Claim (4), axioms (49) and (54), and CQs (65). As, thus, $|Y_i| = 2|Y_{i-1}|$, property 4. of the induction hypothesis implies $|Y_i| = 2^i$.

This finishes the proof of Claim (5).

We go on to identify the instances in $\mathcal{I}$ that encode pairs of vertices:

Claim 6. *We can identify a set $Y \subseteq \mathsf{Index}_{2n}^{\mathcal{I}}$ and a bijective function $\vartheta : Y \to V^2$ such that $\vartheta(x) = (v_k, u_l)$ implies $(x, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff the i-th bit in $\bar{k} \cdot \bar{l}$ is b for all $b \in \{\mathsf{t}, \mathsf{f}\}$ and $i \in \{1, \cdots, 2n\}$.*

Claim (6) builds upon Claim (5); set $Y = Y_{2n}$. By property 2. of Claim (5), each $y \in Y$ assigns a unique $b \in \{\mathsf{t}, \mathsf{f}\}$ to every i-th bit with $i \in \{1, \cdots, 2n\}$, encoded via $(y, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$. Ergo, we can find a function $\vartheta : Y \to V^2$ such that $\vartheta(y) = (v_k, u_l)$ implies $(y, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff the i-th bit in $\bar{k} \cdot \bar{l}$ is b for all $b \in \{\mathsf{t}, \mathsf{f}\}$ and $i \in \{1, \cdots, 2n\}$.

Note that ϑ must be surjective, as by $|Y| = 2^{2n}$ (property 4 of Claim (5)) and the fact that all $x, y \in Y$ pairwise assign at least one bit differently (property 3. of Claim (5)).

To see that ϑ is injective, consider some $x, y \in Y$ with $\vartheta(x) = (v_k, u_l) = \vartheta(y)$. By definition of ϑ, we have that $(x, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff $(y, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ for all with $b \in \{\mathsf{t}, \mathsf{f}\}$ and $i \in \{1, \cdots, 2n\}$, and, together with property 2 of Claim (5), even that $(x, \mathsf{val_t}) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff $(y, \mathsf{val_f}) \notin \mathsf{hBit}_i^{\mathcal{I}}$. Property 3. of Claim (5) then entails $x = y$.

This finishes the proof of Claim (6).

We go on to extract π from $\mathcal{I}$ and show that $\mathcal{I}$ is sound in the sense that any two elements $x, y \in Y$ that encode the same vertex also assign it the same color as defined by π.

Claim 7. *The relation $\pi : V \to \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ with*

$$\pi(v) := \{c \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\} \mid \exists y \in Y \text{ such that } \vartheta(y) = (v, u) \text{ and } (y, c) \in \mathsf{hCol}^{\mathcal{I}}\}.$$

is a function such that $(y, c_1) \in \mathsf{hCol}^{\mathcal{I}}$ and $(y, c_2) \in \mathsf{hCol}'^{\mathcal{I}}$ hold for all $y \in Y$ with $\vartheta(y) = (v, u)$, $\pi(v) = c_1$ and $\pi(u) = c_2$.

Indeed, π is a function: On one hand, Claim (4) and axioms (56) ensure the existence of some $c, d \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ such that $(y, c) \in \mathsf{hCol}^{\mathcal{I}}$ and $(y, d) \in \mathsf{hCol}'^{\mathcal{I}}$ for all $y \in Y$. On the other hand, $\pi(y)$ is unique for all $y \in Y$: Assume to the contrary that there are two elements $y, y' \in Y$ such that $\vartheta(y) = (v, u)$ with $(y, c) \in \mathsf{hCol}^{\mathcal{I}}$ and $\vartheta(y') = (v, u')$ with $(y', c') \in \mathsf{hCol}^{\mathcal{I}}$ and $c' \neq c$. By definition of ϑ, y and y' assign the bits with $1, \cdots, n$ the same, i.e., $(y, z) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff $(y', z) \in \mathsf{hBit}_i^{\mathcal{I}}$. Furthermore, $c' \neq c$ yields $(c', c) \in \mathsf{neq}^{\mathcal{I}}$ via axioms (47). Then, however, there must be a homomorphism from CQ (66) to $\mathcal{I}$, which contradicts the fact that $\mathcal{I} \not\models q$.

Now, consider some $y \in Y$ with $\vartheta(y) = (v, u)$, $\pi(v) = c_1$ and $\pi(u) = c_2$. We first show that $(y, c_1) \in \mathsf{hCol}^{\mathcal{I}}$. The definition of π requires the existence of some $y' \in Y$ such that $\vartheta(y') = (v, u')$ and $(y', c_1) \in \mathsf{hCol}^{\mathcal{I}}$, while there must also be some $c \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ such that $(y, c) \in \mathsf{hCol}^{\mathcal{I}}$ via Claim (4) and axioms (56). By $\vartheta(y) = (v, u)$ and $\vartheta(y') = (v, u')$, Claim (6) implies that $(y, \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ iff $(y', \mathsf{val}_b) \in \mathsf{hBit}_i^{\mathcal{I}}$ for all $i \in \{1, \cdots, n\}$ and $b \in \{\mathsf{t}, \mathsf{f}\}$. Then, we have that $(c_1, c) \notin \mathsf{neq}^{\mathcal{I}}$ via CQs (66) and the fact that $\mathcal{I} \not\models q$. Ergo, by Claim (4), and axioms (47) and (56), we have that $c = c_1$, i.e., $(y, c_1) \in \mathsf{hCol}^{\mathcal{I}}$.

We go on to show that $(y, c_2) \in \mathsf{hCol}'^{\mathcal{I}}$. Again, the definition of π requires the existence of some $y' \in Y$ such that $\vartheta(y') = (u, u')$ and $(y', c_2) \in \mathsf{hCol}^{\mathcal{I}}$, while there must also be some $c \in \{\mathsf{r}, \mathsf{g}, \mathsf{b}\}$ such that $(y, c) \in \mathsf{hCol}'^{\mathcal{I}}$ via Claim (4) and axioms (56). By $\vartheta(y) = (v, u)$ and $\vartheta(y') = (v, u')$, Claim (6) implies that $(y, \mathsf{val}_b) \in \mathsf{hBit}^{\mathcal{I}}_{i+n}$ iff $(y', \mathsf{val}_b) \in \mathsf{hBit}^{\mathcal{I}}_i$ for all $i \in \{1, \cdots, n\}$ and $b \in \{\mathsf{t}, \mathsf{f}\}$. Then, we have that $(c_2, c) \notin \mathsf{neq}^{\mathcal{I}}$ via CQs (67) and the fact that $\mathcal{I} \not\models q$. Ergo, by Claim (4), and axioms (47) and (56), we have that $c = c_1$, i.e., $(y, c_1) \in \mathsf{hCol}'^{\mathcal{I}}$.

This finishes the proof of Claim (7).

As a last step before proving that π indeed is a 3-coloring, we need to argue that the computation of the logical gates within $\mathcal{I}$ is sound.

Claim 8. *For all $y \in Y$, we have that $\vartheta(y) \in E$ implies $(y, \mathsf{var}_\mathsf{t}) \in \mathsf{gate}^{\mathcal{I}}_{\dot{g}}$, where $\dot{g}$ is the output gate of C.*

As $\vartheta(y) \in E$ implies $C^{\dot{g}}(\bar{k}, \bar{l}) = \mathsf{t}$, it suffices to show that $C^g(\bar{k}, \bar{l}) = b$ implies $(y, \mathsf{var}_b) \in \mathsf{gate}^{\mathcal{I}}_g$ for all $y \in Y$ with $\vartheta(y) = (u_k, v_l)$, gates g in C and $b \in \{\mathsf{t}, \mathsf{f}\}$. We do this per induction over the structure of C^g:

g **is an INPUT gate.** Let $C^g(\bar{k}, \bar{l}) = b$. Then, as g is an INPUT gate, the i_g-th bit in $\bar{k} \cdot \bar{l}$ must be b. Thus, $(y, \mathsf{var}_b) \in \mathsf{hBit}^{\mathcal{I}}_{i_g}$ by definition of ϑ. Claim (4) and axioms (58) entail $(y, \mathsf{var}_{b'}) \in \mathsf{gate}^{\mathcal{I}}_g$ for some $b' \in \{\mathsf{t}, \mathsf{f}\}$. Finally, $(y, \mathsf{var}_b) \in \mathsf{hBit}^{\mathcal{I}}_{i_g}$, axioms (49), CQs (68), and the fact that $\mathcal{I} \not\models q$ entail $b' = b$, i.e., $(y, \mathsf{var}_b) \in \mathsf{gate}^{\mathcal{I}}_g$.

g **is a NOT gate with parent** g'**.** Let $C^g(\bar{k}, \bar{l}) = b$. As g is a NOT gate with parent g', we have that $C^{g'}(\bar{k}, \bar{l}) = b'$ for some $b' \in \{\mathsf{t}, \mathsf{f}\}$ such that $b' \neq b$. By the induction hypothesis, we then have that $(y, \mathsf{var}_{b'}) \in \mathsf{gate}^{\mathcal{I}}_{g'}$. Claim (4) and axioms (58) entail $(y, \mathsf{var}_{b''}) \in \mathsf{gate}^{\mathcal{I}}_g$ for some $b'' \in \{\mathsf{t}, \mathsf{f}\}$. Due to $b' \neq b$, CQs (69) and the fact that $\mathcal{I} \not\models q$, it must be so that $b'' = b$, i.e., $(y, \mathsf{var}_b) \in \mathsf{gate}^{\mathcal{I}}_g$.

g **is a AND or OR gate.** We omit the details, as these cases are very similar to the NOT case, with the exception that the respective semantics of the logical operators need to be applied.

We are now in the position to show that π indeed is a 3-coloring. Assume to the contrary that there is an edge $(v, u) \in E$ with $\pi(v) = \pi(u) := c$. Since θ is a bijection as of Claim (6), there exists a unique $y \in Y$ such that $\vartheta(y) = (v, u)$. We have $(y, \pi(v)) \in \mathsf{hCol}^{\mathcal{I}}$ and $(y, \pi(u)) \in \mathsf{hCol}'^{\mathcal{I}}$ by Claim (7), i.e., $(y, c) \in \mathsf{hCol}^{\mathcal{I}}$ and $(y, c) \in \mathsf{hCol}'^{\mathcal{I}}$. Furthermore, by Claim (8), $(v, u) \in E$ entails $(y, \mathsf{var}_\mathsf{t}) \in \mathsf{gate}^{\mathcal{I}}_{\dot{g}}$. Together with axioms (48), there must be a homomorphism from CQ (76) to $\Delta^{\mathcal{I}}$. However, this contradicts the fact that $\mathcal{I}$ is a countermodel to q. Ergo, π must indeed be a 3-coloring, i.e., G is 3-colorable. □

E Proofs for Section 4.2

Lemma 10. *The following are equivalent:*

1. $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$*;*
2. $\mathcal{I}_{\mathcal{P}} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}_{\mathcal{P}})$ *for all* $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ *with* $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$.

Proof. Before proving the lemma, we recall the set E gathers elements e_{t_1,t_2} which will serve as references for others with same types. To lighten notation, we introduce the mapping $\mathsf{ref} : e \mapsto e_{\mathsf{tp}_{\mathcal{A}}(e), \mathsf{tp}_{\mathcal{I}}(e)}$ defined on $\Delta^{\mathcal{I}}$.

"1 ⇒ 2". Assume $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Consider $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ s.t. $|\mathcal{P}| \leq 2|\mathcal{T}| + 1$. We first verify that $\mathcal{I}_{\mathcal{P}}$ models $\mathcal{K}_{\mathcal{P}}$:

- $\mathcal{I}_{\mathcal{P}}$ models $\mathcal{A}_{\mathcal{P}}$ as it inherits interpretations of concepts on $\mathsf{ind}(\mathcal{A}')$ from $\mathcal{I}$, which is a model of $\mathcal{A}$.
- Axioms in $\mathcal{T}$ with shape $\top \sqsubseteq A$, $A_1 \sqcap A_2 \sqsubseteq A$, $A \sqsubseteq \neg B$ or $\neg B \sqsubseteq A$ are satisfied since $\mathcal{I}_{\mathcal{P}}$ inherits interpretations of concept names from $\mathcal{I}$, which is a model of $\mathcal{T}$.
- Axioms in $\mathcal{T}$ with shape $A \sqsubseteq \exists r$ are witnessed with an r-edge pointing to w_r.
- Axioms in $\mathcal{T}$ with shape $\exists r \sqsubseteq A$ are satisfied since every element in $\mathcal{I}_{\mathcal{P}}$ receiving some r-edge already received some in $\mathcal{I}$ (and interpretations of concept names are preserved).
- Role inclusions $r \sqsubseteq s$ are satisfied directly from the definition of $s^{\mathcal{I}_{\mathcal{P}}}$.

It remains to prove that $\mathcal{I}_{\mathcal{P}}$ also complies with CP. Assume to the contrary that there exists a model $\mathcal{J}'$ of $\mathcal{K}_{\mathcal{P}}$ with $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}}$. Notice that, for each role r s.t. $r^{\mathcal{J}'} \neq \emptyset$, there must be an element $w'_r \in (\exists r^-)^{\mathcal{J}'}$. We can chose such an element w'_r for each such role r, and build the following interpretation $\mathcal{J}$:

$$\begin{aligned}
\Delta^{\mathcal{J}} &= \Delta^{\mathcal{I}} \\
A^{\mathcal{J}} &= A^{\mathcal{J}'} \cup \{e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{J}'} \mid \mathsf{ref}(e) \in A^{\mathcal{J}'}\} \\
r^{\mathcal{J}} &= \{(a, b) \mid \mathcal{K} \models r(a, b)\} \\
&\quad \cup \{(e, w'_s) \mid \mathsf{ref}(e) \in (\exists s)^{\mathcal{J}'}, \mathcal{T} \models s \sqsubseteq r\} \\
&\quad \cup \{(w'_s, e) \mid \mathsf{ref}(e) \in (\exists s)^{\mathcal{J}'}, \mathcal{T} \models s \sqsubseteq r^-\}
\end{aligned}$$

It is easily verified that $\mathcal{J}$ is a model of $\mathcal{K}$, the most interesting cases being axioms in $\mathcal{T}$ with shape either $A \sqsubseteq \exists r$ or $\exists r \sqsubseteq A$. The former are satisfied using the witness w_r. The latter, when triggered on $a \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{J}'}$ due to a pair (a, b) s.t. $\mathcal{K} \models r(a, b)$, hold thanks to $\mathsf{ref}(a)$ having same ABox-type as a. We now prove $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, which will contradict $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. It suffices to prove that for each concept name A and $\odot \in \{\subseteq, \supseteq\}$, we have $A^{\mathcal{J}} \odot A^{\mathcal{I}}$ iff $A^{\mathcal{J}'} \odot A^{\mathcal{I}_{\mathcal{P}}}$. Let A be a concept name. We prove the claim for $\odot = \subseteq$ only; for $\odot = \supseteq$, the arguments are similar.

"⇒". Let $A^{\mathcal{J}} \subseteq A^{\mathcal{I}}$ and $d \in A^{\mathcal{J}'}$. In particular, $d \in A^{\mathcal{J}}$ by definition of $A^{\mathcal{J}}$. Thus, by hypothesis, $d \in A^{\mathcal{I}}$, and since $d \in \Delta^{\mathcal{J}'} = \Delta^{\mathcal{I}_{\mathcal{P}}}$ we obtain $d \in A^{\mathcal{I}_{\mathcal{P}}}$ by definition of $A^{\mathcal{I}_{\mathcal{P}}}$.

"⇐". Let $A^{\mathcal{J}'} \subseteq A^{\mathcal{I}_{\mathcal{P}}}$ and $d \in A^{\mathcal{J}}$; we need to show $d \in A^{\mathcal{I}}$. If $d \in \Delta^{\mathcal{J}'}$, then it is straightforward. Otherwise, we know that $\mathsf{ref}(d) \in A^{\mathcal{J}'}$ by definition of $A^{\mathcal{J}}$. Therefore $\mathsf{ref}(d) \in A^{\mathcal{I}_{\mathcal{P}}}$ by hypothesis, that is $A \in \mathsf{tp}_{\mathcal{I}}(\mathsf{ref}(d))$. Since

d and $\mathsf{ref}(d)$ have same type, it yields $A \in \mathsf{tp}_{\mathcal{I}}(d)$, that is $d \in A^{\mathcal{I}}$.

"2 ⇒ 1". Assume $\mathcal{I}_{\mathcal{P}} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}_{\mathcal{P}})$ for all $\mathcal{P} \subseteq \Delta^{\mathcal{I}}$ s.t. $|\mathcal{P}| \leq 2|\mathcal{T}|+1$. By hypothesis, we already have $\mathcal{I} \models \mathcal{K}$. By contradiction, assume now there exists $\mathcal{J} < \mathcal{I}$. Based on $\mathcal{J}$, we build a subset $\mathcal{P} \subseteq \Delta^{\mathcal{J}}$ containing:

- for each role r s.t. $r^{\mathcal{J}} \neq \emptyset$, an element $w'_r \in (\exists r^-)^{\mathcal{J}}$;
- for each $A \in \mathsf{M}$ s.t. $A^{\mathcal{J}} \not\subseteq A^{\mathcal{I}}$, an element $e_A \in B^{\mathcal{J}} \setminus B^{\mathcal{I}}$ for some $B \prec A$ (Condition 3 in the definition of $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ ensures existence of such B and e_A);
- an element $e_{\mathsf{M}} \in A^{\mathcal{J}} \setminus A^{\mathcal{I}}$ for some $A \in \mathsf{M}$ s.t. $A^{\mathcal{J}} \subsetneq A^{\mathcal{I}}$ and for all $B \prec A$, $B^{\mathcal{J}} = B^{\mathcal{I}}$ (Condition 4 in the definition of $<_{\mathsf{CP}}$ ensures the existence of such A and e_{M}).

Notice $\mathcal{P}$ has size at most $2|\mathcal{T}|+1$. We now build $\mathcal{J}'$ as:

$$\begin{aligned}
\Delta^{\mathcal{J}'} &= \Delta^{\mathcal{I}_{\mathcal{P}}}\\
A^{\mathcal{J}'} &= A^{\mathcal{J}} \cap \Delta^{\mathcal{J}'}\\
r^{\mathcal{J}'} &= \{(e, w'_s) \mid e \in (\exists s)^{\mathcal{J}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}}, \mathcal{T} \models s \sqsubseteq r\}\\
&\quad \cup \{(w'_s, e) \mid e \in (\exists s)^{\mathcal{J}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}}, \mathcal{T} \models s \sqsubseteq r^-\}
\end{aligned}$$

It is easily verified that $\mathcal{J}'$ is a model of $\mathcal{K}_{\mathcal{P}}$, and we now prove $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}}$, which will contradict $\mathcal{I}_{\mathcal{P}} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}_{\mathcal{P}})$. We now check that all four conditions from the definition of $<_{\mathsf{CP}}$ are satisfied:

1. By definition, we have $\Delta^{\mathcal{J}'} = \Delta^{\mathcal{I}_{\mathcal{P}}}$.
2. Let $A \in \mathsf{F}$. Definitions of $\mathcal{I}_{\mathcal{P}}$ and $\mathcal{J}'$ ensure $A^{\mathcal{I}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}} = A^{\mathcal{I}_{\mathcal{P}}}$ and $A^{\mathcal{J}} \cap \Delta^{\mathcal{I}_{\mathcal{P}}} = A^{\mathcal{J}'}$. From $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, we get $A^{\mathcal{I}} = A^{\mathcal{J}}$, which thus yields $A^{\mathcal{I}_{\mathcal{P}}} = A^{\mathcal{J}'}$.
3. Let $A \in \mathsf{M}$ such that $A^{\mathcal{J}'} \not\subseteq A^{\mathcal{I}_{\mathcal{P}}}$. Therefore $A^{\mathcal{J}} \not\subseteq A^{\mathcal{I}}$, and recall we kept in $\mathcal{P}$ an element $e_A \in B^{\mathcal{J}} \setminus B^{\mathcal{I}}$ for some $B \prec A$ to belong to $\mathcal{P}$. Joint with $B^{\mathcal{J}'} \subseteq B^{\mathcal{I}_{\mathcal{P}}}$ being trivial, e_A additionally witnesses that $B^{\mathcal{J}'} \subsetneq B^{\mathcal{I}_{\mathcal{P}}}$.
4. Recall we kept in $\mathcal{P}$ an element $e_{\mathsf{M}} \in A^{\mathcal{J}} \setminus A^{\mathcal{I}}$ for some $A \in \mathsf{M}$ s.t. $A^{\mathcal{J}} \subsetneq A^{\mathcal{I}}$ and for all $B \prec A$, $B^{\mathcal{J}} = B^{\mathcal{I}}$. It gives immediately that $A^{\mathcal{J}'} \subsetneq A^{\mathcal{I}_{\mathcal{P}}}$ and for all $B \prec A$, $B^{\mathcal{J}'} = B^{\mathcal{I}_{\mathcal{P}}}$.

This proves $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}_{\mathcal{P}}$, contradicting $\mathcal{I}_{\mathcal{P}} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}_{\mathcal{P}})$ as desired. □

Theorem 8. *UCQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ KBs is in* coNP *w.r.t. data complexity.*

Proof. We exhibit an NP procedure to decide the complement of our problem, that is existence of a countermodel for UCQ q over DL-Lite$^{\mathcal{H}}_{\mathsf{bool}}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Our procedure starts by guessing an interpretation $\mathcal{I}$ whose domain has size at most $|\mathcal{A}| + (2^{|\mathcal{T}|+2}+1)^{3|q|}$. This can be done in linearly many non-deterministic steps (w.r.t. data complexity) It further checks whether $\mathcal{I}$ is a model of $\mathcal{K}$ that does not entail q, and rejects otherwise. This can essentially be done naively in $\left|\Delta^{\mathcal{I}}\right|^{|\mathcal{T}||q|}$ deterministic steps, that is still polynomial w.r.t. data complexity. The procedure finally checks whether $\mathcal{I}$ complies with CP by iterating over each $\mathcal{P}$ with $|\mathcal{P}| \leq 2|\mathcal{T}|+1$ and checking whether there exists a model $\mathcal{J}_{\mathcal{P}}$ of $\mathcal{K}$ s.t. $\mathcal{J}_{\mathcal{P}} < \mathcal{I}_{\mathcal{P}}$. If, for a choice of $\mathcal{P}$ and we can find such a $\mathcal{J}_{\mathcal{P}}$ then our procedure rejects. Notice that iterating over such $\mathcal{P}$ can be done in $\left|\Delta^{\mathcal{I}}\right|^{2|\mathcal{T}|+1}$ iterations since we require $|\mathcal{P}| \leq 2|\mathcal{T}|+1$. Computing $\mathcal{I}_{\mathcal{P}}$ follows directly from the choice of $\mathcal{P}$. Since each $\mathcal{I}_{\mathcal{P}}$ has constant size w.r.t. data complexity, iterating over each possible $\mathcal{J}_{\mathcal{P}}$ can be done naively with constantly many steps. Overall, the procedure uses a linear number of non-deterministic steps at the beginning and further performs several checks using a polynomial number of additional deterministic steps.

It remains to prove that there exists an accepting run iff there is a countermodel for q over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. If there exists an accepting run, then the corresponding guessed interpretation $\mathcal{I}$ is a model of $\mathcal{K}$ that does not entail q, and the "2 ⇒ 1" direction from Lemma 10 ensures it also comply with CP. Conversely, if there exists a countermodel, then Lemma 7 ensures existence of a countermodel $\mathcal{I}$ whose domain has size at most $|\mathcal{A}| + (2^{|\mathcal{T}|+2}+1)^{3|q|}$. This $\mathcal{I}$ can be guessed by the procedure, and as $\mathcal{I}$ is indeed a model of $\mathcal{K}$ not entailing q, will not be rejected. The "1 ⇒ 2" direction from Lemma 9 further ensures $\mathcal{I}$ also passes the remaining checks performed by the procedure. □

Theorem 9. *CQ evaluation on circumscribed DL-Lite$_{\mathsf{core}}$ KBs is* coNP*-hard w.r.t. data complexity. This holds even with a single minimized concept name, no fixed concept names, and no disjointness constraints.*

Proof. We reduce the complement of the graph 3-colorability problem (3Col) to evaluating the Boolean CQ:

$$q = \exists y_1\, \exists y_2\, \exists y\; \mathsf{edge}(y_1, y_2) \wedge \mathsf{hasCol}(y_1, y) \wedge \mathsf{hasCol}(y_2, y)$$

over the DL-Lite$_{\mathsf{pos}}$ TBox:

$$\mathcal{T} = \{\mathsf{Vertex} \sqsubseteq \exists\mathsf{hasCol}, \exists\mathsf{hasCol}^- \sqsubseteq \mathsf{Color}\},$$

where Color is minimized while all other predicates varying. Let CP be the resulting circumscription pattern. Given an instance $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of 3Col, we build an ABox $\mathcal{A}$ containing the following assertions:

$$\mathsf{Vertex}(v) \text{ for all } v \in \mathcal{V} \tag{77}$$
$$\mathsf{edge}(v_1, v_2) \text{ for all } \{v_1, v_2\} \in \mathcal{E} \tag{78}$$
$$\mathsf{Color}(c) \text{ for all } c \in \{r, g, b\} \tag{79}$$

We now prove the following claim:

$$\mathcal{G} \notin \text{3Col} \iff \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q$$

First notice that every model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ verifies $\mathsf{Color}^{\mathcal{I}} = \{r, g, b\}$. Indeed, by contradiction, if $\mathcal{I}$ contains $e \in \mathsf{Color}^{\mathcal{I}} \setminus \{r, g, b\}$, then we can build $\mathcal{J}$ a model of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by modifying $\mathcal{I}$ as follows:

- remove e from $\mathsf{Color}^{\mathcal{I}}$, that is $\mathsf{Color}^{\mathcal{J}} = \mathsf{Color}^{\mathcal{I}} \setminus \{e\}$;
- reroute hasCol as: $\mathsf{hasCol}^{\mathcal{J}} = \{(v, r) \mid v \in (\exists\mathsf{hasCol})^{\mathcal{I}}\}$.

It is then straightforward that $\mathcal{J}$ satisfies the desired properties, contradicting $\mathcal{I}$ being a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$.

"⇒". Assume $\mathcal{G} \notin$ 3Col and consider a model $\mathcal{I}$ of $\mathcal{J}$. By the above remark, we have $\mathsf{Color}^{\mathcal{I}} = \{r, g, b\}$ and from

$\mathcal{I}$ being a model of $\mathcal{K}$, we can find a mapping $\tau : \mathcal{V} \to \{r, g, b\}$ of $\mathcal{G}$ such that for all $v \in \mathcal{V}$, if $\tau(v) = c$ then $(v, c) \in \mathsf{hasCol}^{\mathcal{I}}$. Since, by assumption, $\mathcal{G} \notin$ 3Col, there exists an edge $\{v_1, v_2\} \in \mathcal{E}$ such that $\tau(v_1) = \tau(v_2) = c$ for some $c \in \{r, g, b\}$, which ensures $y_1 \mapsto v_1, y_2 \mapsto v_2, y \mapsto c$ is an homomorphism of q in $\mathcal{I}$.

"$\Leftarrow$". Assume $\mathcal{G} \in$ 3Col and consider a 3-coloring $\tau : \mathcal{V} \to \{r, g, b\}$. We build a model $\mathcal{I}_\tau$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ by interpreting all concepts and roles as in $\mathcal{A}$ except for hasCol, which is interpreted as:

$$\mathsf{hasCol}^{\mathcal{I}_\tau} = \{(v, c) \mid v \in \mathcal{V}, \tau(v) = c\}.$$

It is straightforward that $\mathcal{I}$ models $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and does not embed q (since τ is a 3-coloring). □

F Proofs for Section 5.1

We briefly discuss the variation of Proposition 1 mentioned in the main part of the paper. Assume given a circumscribed $\mathcal{ALCHIO}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, an AQ $A_0(x)$ and an individual $a_0 \in \mathsf{ind}(\mathcal{A})$. We again replace all occurrences of a nominal a with fresh concept A_a, add $A_a(a), B_a(a)$ to $\mathcal{A}$, minimize B_a and set $B_a \prec A$ for all $A \in \mathsf{M}$. However, in contrast with Proposition 1, we also add axioms $A_a \sqcap \neg B_a \sqsubseteq \exists r.(X \sqcap A_0)$ and $X \sqsubseteq Y$ to $\mathcal{T}$ for some fresh concept names X and Y and fresh role name r, and $Y(a_0)$ to $\mathcal{A}$. Finally, we also minimize Y with higher preference than concept from M, that is we set $Y \prec A$ for all $A \in \mathsf{M}$. Denoting $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}')$ the resulting circumscribed $\mathcal{ALCHI}$ KB, it can then be verified that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ iff $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}') \models A_0(a_0)$.

We now move to a proof of Theorem 11.

Theorem 11. *AQ evaluation on circumscribed $\mathcal{EL}$ KBs is* coNExp$^{\text{NP}}$*-hard w.r.t. combined complexity. This holds even without fixed concept names and with an empty preference order.*

Proof. Let $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ be a circumscribed $\mathcal{ALC}$ KB, with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, A_0 an AQ, and $a_0 \in \mathsf{ind}(\mathcal{A})$. We construct a circumscribed $\mathcal{EL}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$, with $\mathcal{K}' = (\mathcal{T}', \mathcal{A}')$, such that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0(a_0)$.

With $\mathcal{ELU}_\bot$, we mean the extension of $\mathcal{EL}$ with disjunction and $\bot$. It is well-known that every $\mathcal{ALC}$ TBox $\mathcal{T}$ can be rewritten in polynomial time into an $\mathcal{ELU}_\bot$ TBox $\mathcal{T}^*$ that is a conservative extension of $\mathcal{T}$ in the sense that every model of $\mathcal{T}^*$ is a model of $\mathcal{T}$ and every model of $\mathcal{T}$ can be extended to a model of $\mathcal{T}^*$ by interpreting the fresh concept and role names in $\mathcal{T}^*$ (Baader, Brandt, and Lutz 2005). It follows that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{T}^*, \mathcal{A}) \models A_0(a_0)$, assuming that all the fresh concept names in $\mathcal{T}^*$ vary in CP. In the following, we recall the rewriting, which proceeds in three steps:

1. *Remove value restrictions* by replacing every subconcept $\forall r.C$ with $\neg\exists r.\neg C$.
2. *Remove negation of compound concepts* by replacing every subconcept $\neg C$, with C compound, by $\neg X$ where X is a fresh concept name, and adding the CIs
$$C \sqsubseteq X \qquad X \sqsubseteq C.$$
3. *Remove negation entirely* by replacing every subconcept $\neg A$ with the fresh concept name $\overline{A}$, and adding the CIs
$$\top \sqsubseteq A \sqcup \overline{A} \qquad A \sqcap \overline{A} \sqsubseteq \bot.$$

We may thus assume w.l.o.g. that the TBox $\mathcal{T}$ in the given circumscribed KB $\mathcal{K}$ is an $\mathcal{ELU}_\bot$ TBox. We may further assume that disjunction occurs in $\mathcal{T}$ only in CIs of the form

$$B \sqsubseteq B_1 \sqcup B_2$$

where B, B_1, B_2 are concept names, by replacing every disjunction $C_1 \sqcup C_2$ with a fresh concept name X and adding the CIs

$$\begin{array}{lll} X \sqsubseteq Y_1 \sqcup Y_2 & Y_1 \sqsubseteq X & Y_2 \sqsubseteq X \\ C_i \sqsubseteq Y_i & Y_i \sqsubseteq C_i & \text{for } i \in \{1, 2\} \end{array}$$

where also Y_1, Y_2 are fresh concept names. Finally, we may assume that $\mathcal{T}$ contains only a single CI $B \sqsubseteq B_1 \sqcup B_2$ by replacing every such CI with

$$B \sqsubseteq \exists r_{B_1,B_2}.D \quad \exists r_{B_1,B_2}.D_1 \sqsubseteq B_1 \quad \exists r_{B_1,B_2}.D_2 \sqsubseteq B_2$$

where r_{B_1,B_2} is a fresh role name and D, D_1, D_2 are fresh concept names, and adding the CI

$$D \sqsubseteq D_1 \sqcup D_2.$$

We may clearly also assume that $\bot$ occurs in $\mathcal{T}$ only in the form $C \sqsubseteq \bot$ with C an $\mathcal{EL}$-concept. Note that the TBoxes resulting from these transformations are conservative extensions of the original ones. In CP, all of the freshly introduced concept names are varying.

We are now ready for the actual reduction, which combines ideas from the proofs of Theorems 2 and 4. To construct the TBox $\mathcal{T}'$, We start with relativized versions of the $\mathcal{EL}$-CIs in $\mathcal{T}$. For an $\mathcal{EL}$ concept C, inductively define the concept C_X as follows:

$$\begin{array}{ll} A_X := X \sqcap A & (\exists r.D)_X := X \sqcap \exists r.D_X \\ \top_X := X & (D \sqcap D')_X := D_X \sqcap D'_X \end{array}$$

We then include in $\mathcal{T}'$ all CIs

$$C_X \sqsubseteq D_X \tag{80}$$

such that $C \sqsubseteq D \in \mathcal{T}$ is neither the unique disjunctive CI $B \sqsubseteq B_1 \sqcup B_2$ nor of the form $C \sqsubseteq \bot$. To construct the ABox $\mathcal{A}'$, we start with the extension of $\mathcal{A}$ by

$$X(a) \quad \text{for all } a \in \mathsf{ind}(\mathcal{A}). \tag{81}$$

Of course, we need to compensate the removal of the mentioned CIs. To simulate the disjunctive CI $B \sqsubseteq B_1 \sqcup B_2$, we extend $\mathcal{T}'$ with

$$B \sqsubseteq \exists r_D.(D \sqcap \mathsf{Pos}) \qquad \exists r_D.D_1 \sqsubseteq B_1 \tag{82}$$
$$B \sqsubseteq \exists r_D.(D \sqcap \mathsf{Neg}) \qquad \exists r_D.D_2 \sqsubseteq B_2 \tag{83}$$

where D, D_1, D_2, Pos, and Neg are fresh concept names and r_D is a fresh role name. We update CP so that D is minimized. We further extend $\mathcal{A}$ with the assertions

$$D(d_1^+) \qquad D(d_1^-) \qquad D(d_2^+) \qquad D(d_2^-) \tag{84}$$
$$D_1(d_1^+) \qquad D_1(d_1^-) \qquad D_2(d_2^+) \qquad D_2(d_2^-) \tag{85}$$
$$\mathsf{Pos}(d_1^+) \qquad \mathsf{Neg}(d_1^-) \qquad \mathsf{Pos}(d_2^+) \qquad \mathsf{Neg}(d_2^-). \tag{86}$$

where $d_1^+, d_1^-, d_2^+, d_2^-$ are fresh individuals. Note that these individuals are *not* marked with X, exempting them from $\mathcal{T}$ is why the CIs from $\mathcal{T}$ are relativized in $\mathcal{T}'$. Note that the way we simulate disjunction is similar to the way in which we choose truth values in the proof of Theorem 4, and also the reason for using a positive and a negative witness is the same. We need to make sure that the positive and negative witness are distinct, which is achieved by otherwise making the query true. To this end, we introduce a fresh role name u and fresh concept names G, A_0'. We update CP so that G is minimized and extend $\mathcal{T}'$ with:

$$\mathsf{Pos} \sqcap \mathsf{Neg} \sqsubseteq \exists u.(G \sqcap A_0') \tag{87}$$

$$A_0 \sqsubseteq A_0' \tag{88}$$

We add $G(a_0)$ to $\mathcal{A}'$ and the new query is A_0'.

For the compensation of the removal of the CIs $C \sqsubseteq \bot$, we consider another fresh concept name M and update CP so that M is minimized. We further extend $\mathcal{T}'$ with

$$C \sqsubseteq M \sqcap \exists u.(G \sqcap A_0') \quad \text{for all } C \sqsubseteq \bot \in \mathcal{T}. \tag{89}$$

So if a concept C with $C \sqsubseteq \bot \in \mathcal{T}$ is satisfied, then we force the minimized concept name M to be non-empty and make the (new) query true.

Let CP' be the extended circumscription pattern. It remains to establish correctness of the reduction.

Claim. $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ iff $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0'(a_0)$.

To prepare for the proof of the claim, we first observe that every model $\mathcal{I}$ of $\mathcal{K}$ gives rise to a model $\mathcal{I}'$ of $\mathcal{K}'$ in a natural way. More precisely, we set $\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \uplus \{d_1^+, d_1^-, d_2^+, d_2^-\}$, define the extension of all concept and role names that occur in $\mathcal{K}$ exactly as in $\mathcal{I}$, and interpret the fresh concept and role names as follows:

$$\begin{aligned}
X^{\mathcal{I}'} &= \Delta^{\mathcal{I}} \\
D^{\mathcal{I}'} &= \{d_1^+, d_1^-, d_2^+, d_2^-\} \\
D_i^{\mathcal{I}'} &= \{d_i^+, d_i^-\} \text{ for } i \in \{1, 2\} \\
\mathsf{Pos}^{\mathcal{I}'} &= \{d_1^+, d_2^+\} \\
\mathsf{Neg}^{\mathcal{I}'} &= \{d_1^-, d_2^-\} \\
r_D^{\mathcal{I}'} &= \{(d, d_1^+), (d, d_1^-) \mid d \in (B \sqcap B_1)^{\mathcal{I}}\} \cup \\
&\quad\ \{(d, d_2^+), (d, d_2^-) \mid d \in (B \sqcap B_2)^{\mathcal{I}}\} \\
\mathsf{check}^{\mathcal{I}'} &= \{(a_0, a) \mid a \in \mathsf{ind}(\mathcal{A})\} \\
{A_0'}^{\mathcal{I}'} &= A_0^{\mathcal{I}} \\
u^{\mathcal{I}'} &= \emptyset \\
M^{\mathcal{I}'} &= \emptyset \\
G^{\mathcal{I}'} &= \{a_0\}
\end{aligned}$$

It is easily checked that $\mathcal{I}'$ is indeed a model of $\mathcal{K}'$. Now for the actual proof of the claim

"$\Leftarrow$". Assume that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \not\models A_0(a_0)$. Then there is a model $\mathcal{I}$ of $\mathcal{K}$ that is minimal w.r.t. CP and satisfies $\mathcal{I} \not\models A_0(a_0)$. Let $\mathcal{I}'$ be the corresponding model of $\mathcal{K}'$ defined above. We clearly have $\mathcal{I}' \not\models A_0'(a_0)$. To show that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \not\models A_0'(a_0)$, it thus remains to show that $\mathcal{I}'$ is minimal w.r.t. CP'.

Assume to the contrary that there is a model $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ of $\mathcal{K}'$. As $\mathcal{I}'$ interprets D, M, and G minimally among all models of $\mathcal{K}'$, we must have $D^{\mathcal{J}'} = D^{\mathcal{I}'} = \{d_1^+, d_1^-, d_2^+, d_2^-\}$, $M^{\mathcal{J}'} = \emptyset$, and $G^{\mathcal{J}'} = G^{\mathcal{I}'} = \{a_0\}$. We show that the restriction $\mathcal{J}$ of $\mathcal{J}'$ to domain $X^{\mathcal{J}'}$ is a model of $\mathcal{K}$. Moreover, $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$, $D^{\mathcal{J}'} = D^{\mathcal{I}'}$, $M^{\mathcal{J}'} = M^{\mathcal{I}'}$, and $G^{\mathcal{J}'} = G^{\mathcal{I}'}$ imply $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, which then yields a contradiction against the minimality of $\mathcal{I}$.

Based on the facts that $\mathcal{J}'$ is a model of $\mathcal{A}'$, $\mathsf{ind}(\mathcal{A}) \subseteq X^{\mathcal{J}'}$ due to (81), and $\mathcal{A} \subseteq \mathcal{A}'$, it is clear that $\mathcal{J}$ is a model of $\mathcal{A}$. Moreover, since $\mathcal{J}'$ is a model of $\mathcal{T}'$, by construction of $\mathcal{J}$ it is easy to see that $\mathcal{J}$ satisfies all CIs in $\mathcal{T}$ with the possible exceptions of the disjunctive CI $B \sqsubseteq B_1 \sqcup B_2$ and the CIs of the form $C \sqsubseteq \bot$. The former, however, is satisfied due to CIs (82) and (83) and since $D^{\mathcal{J}'} = \{d_1^+, d_1^-, d_2^+, d_2^-\}$. And the latter is satisfied due to the CIs (89) and since $M^{\mathcal{J}'} = \emptyset$.

"$\Rightarrow$". Assume that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \not\models A_0'(a_0)$. Then there is a model $\mathcal{I}'$ of $\mathcal{K}'$ that is minimal w.r.t. CP' and satisfies $\mathcal{I}' \not\models A_0'(a_0)$. We first argue that $G^{\mathcal{I}'} = \{a_0\}$ as, otherwise, we can find a model $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ of $\mathcal{K}'$, contradicting the minimality of $\mathcal{I}'$. We have $G^{\mathcal{I}'} \subseteq \{a_0\}$ since $\mathcal{I}'$ is a model of $\mathcal{A}'$. Assume to the contrary of what we want to show that the converse fails. Then we construct $\mathcal{J}'$ by modifying $\mathcal{I}'$ as follows:

- set $G^{\mathcal{J}'} = \{a_0\}$;
- add a_0 to ${A_0'}^{\mathcal{J}'}$;
- reroute u as $u^{\mathcal{J}'} = \{(e, a_0) \mid e \in (\exists u)^{\mathcal{I}'}\}$.

It is easy to see that, indeed, $\mathcal{J}'$ is still a model of $\mathcal{K}'$ and $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$.

Since $\mathcal{I}'$ satisfies CIs (82) and (83), for every $e \in B^{\mathcal{I}'}$ we can choose two elements $e^+ \in (D \sqcap \mathsf{Pos})^{\mathcal{I}'}$ and $e^- \in (D \sqcap \mathsf{Neg})^{\mathcal{I}'}$ such that $(e, e^+), (e, e^-) \in r_D^{\mathcal{I}'}$. If $e^+ = e^-$ for some e, then by (87) and $G^{\mathcal{I}'} = \{a_0\}$, we have $a_0 \in A_0^{\mathcal{I}'}$, which is impossible since $\mathcal{I}' \not\models A_0'(a_0)$. Thus $e^+ \neq e^-$ for all e.

We next argue that the following conditions are satisfied:

1. for all $e \in B^{\mathcal{I}'}$: $\{e^+, e^-\} \cap \{d_1^+, d_1^-, d_2^+, d_2^-\} \neq \emptyset$;
2. $M^{\mathcal{I}'} = \emptyset$.

If, in fact, any of the above conditions is violated, then we can again find a model $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$ of $\mathcal{K}'$, contradicting the minimality of $\mathcal{I}'$. We start with Condition 1.

Assume to the contrary that for some $e \in B^{\mathcal{I}'}$, $e^+, e^- \notin \{d_1^+, d_1^-, d_2^+, d_2^-\}$. Then we construct $\mathcal{J}'$ by modifying $\mathcal{J}$ as follows:

- remove e^- from $D^{\mathcal{J}'}$;
- add e^+ to $\mathsf{Neg}^{\mathcal{J}'}$;
- add a_0 to ${A_0'}^{\mathcal{J}'}$;
- add (e^+, a_0) to $u^{\mathcal{J}'}$.

It is easy to see that, indeed, $\mathcal{J}'$ is still a model of $\mathcal{K}'$ and $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$.

In the case of Condition 2, we can construct the model $\mathcal{J}'$ simply as $\mathcal{J}$ modified by setting $M^{\mathcal{J}'} = \emptyset$. The resulting $\mathcal{J}'$ is a model of $\mathcal{K}'$. The only potentially problematic item are the CIs (89). But since $\mathcal{I}' \not\models A_0'(a_0)$ and $G^{\mathcal{I}'} = \{a_0\}$, we can infer from (89) that $C^{\mathcal{I}'} = \emptyset$ for all $C \sqsubseteq \bot \in \mathcal{T}$, and thus the same holds for $\mathcal{J}'$. Consequently, the CIs (89) are never triggered in $\mathcal{J}'$ and it is safe to set $M^{\mathcal{J}'} = \emptyset$.

Let $\mathcal{I}$ be the restriction of $\mathcal{I}'$ to $X^{\mathcal{I}'}$. It can be verified that $\mathcal{I}'$ is a model of $\mathcal{A}$ and that it satisfies all CIs in $\mathcal{T}$ with the possible exceptions of the disjunctive CI $B \sqsubseteq B_1 \sqcup B_2$ and the CIs of the form $C \sqsubseteq \bot$. The former, however, is satisfied by Condition 1 and due to CIs (82) and (83). And the latter is satisfied by Condition 2 and due to CI (89).

Since $\mathcal{I}' \not\models A_0'(a_0)$ and by (88), we have $\mathcal{I} \not\models A_0(a_0)$. To show that $\mathcal{K} \not\models A_0(a_0)$, it thus remains to show that $\mathcal{I}$ is minimal w.r.t. $<_{\mathsf{CP}}$. Assume to the contrary that there is a model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$ and let $\mathcal{J}'$ be the corresponding model of $\mathcal{K}'$ constructed at the beginning of the correctness proof. Since $\mathcal{J}'$ interprets the minimized concept names D, M, and G in a minimal way among the models of $\mathcal{K}'$, it follows from $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ that $\mathcal{J}' <_{\mathsf{CP}'} \mathcal{I}'$, in contradiction to $\mathcal{I}'$ being minimal. □

G Proofs for Section 5.2

We now move to a proof of Theorem 13, which strongly combines DL-Lite$_{\mathsf{bool}}$ expressiveness with fixed concept names. Interestingly, fixed concept names are actually not needed. Indeed, given a circumscribed DL-Lite$_{\mathsf{bool}}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, each $A \in \mathsf{F}$ can be replaced by the two minimized predicates A and $\overline{A}$, where $\overline{A}$ is a fresh concept name set to be the complement of A via the DL-Lite$_{\mathsf{bool}}$ axioms $\overline{A} \sqsubseteq \neg A$ and $\neg A \sqsubseteq \overline{A}$ that can be added in the TBox. Concept names A and $\overline{A}$ are then given higher preference than any other concept name from the original set of minimized concept names M. It is then straightforward to verify that the resulting circumscribed DL-Lite$_{\mathsf{bool}}$ KB $\mathsf{Circ}_{\mathsf{CP}'}(\mathcal{K}')$ has the same models as $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ (up to the interpretation of $\overline{A}$, which always equals the interpretation of $\neg A$).

Theorem 13. *AQ evaluation on circumscribed DL-Lite$_{\mathsf{bool}}$ KBs is* CONEXP*-hard w.r.t. combined complexity.*

Proof. The proof proceeds by reduction from the complement of the NEXP-complete Succinct-3COL problem. An instance of Succinct-3COL consists of a Boolean circuit $\mathcal{C}$ with $2n$ input gates. The graph $\mathcal{G}_{\mathcal{C}}$ encoded by $\mathcal{C}$ has 2^n vertices, identified by binary encodings on n bits. Two vertices u and v, with respective binary encodings $u_1 \ldots u_n$ and $v_1 \ldots v_n$, are adjacent in $\mathcal{G}_{\mathcal{C}}$ iff $\mathcal{C}$ returns True when given as input $u_1 \ldots u_n$ on its first n gates and $v_1 \ldots v_n$ on the second half. The problem of deciding if $\mathcal{G}_{\mathcal{C}}$ is 3-colorable has been proven to be NEXP-complete in (Papadimitriou and Yannakakis 1986).

Let $\mathcal{C}$ be a Boolean circuit with $2n$ input gates. We build a DL-Lite$_{\mathsf{bool}}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ s.t. the AQ $\mathsf{Goal}(x)$ admits the individual a as a certain answer iff $\mathcal{C}$ does not belong to Succinct-3COL. Let us already clarify that CP uses three minimized concept names, namely Center, AllPairs and GoodCol, with the following preference relation $\prec$:

$$\mathsf{Center} \prec \mathsf{AllPairs} \prec \mathsf{GoodCol} \tag{90}$$

It also uses fixed concept names: each Bit_i for $1 \leq i \leq 2n$ and the six concept names $\mathsf{R}_1, \mathsf{G}_1, \mathsf{B}_1, \mathsf{R}_2, \mathsf{G}_2$ and B_2, which will be useful to respectively represent binary encodings and color assignments. All other predicates are varying.

Models of $\mathcal{K}$ contain a central element a, that we enforce via the ABox $\mathcal{A} = \{\mathsf{Center}(a)\}$ and which will be unique by virtue of the preference relation. This central element will further be used to detect undesired behaviors of the models via the AQ. In a model, each element represents a pair (v_1, v_2) of vertices by its combination of fixed concepts Bit_i with $1 \leq i \leq 2n$ that corresponds to the binary encodings of v_1 and v_2. To detect models that do not represent all possible pairs, we require each element representing a pair (v_1, v_2) to "send" at the center a binary sequence of $2n$ bits that must be different from the encoding of (v_1, v_2). This is achieved by the following axioms in $\mathcal{T}$:

$$\top \sqsubseteq \exists r_i \sqcup \exists \overline{r_i} \quad \text{for each } 1 \leq i \leq 2n \tag{91}$$

$$\exists r_i^- \sqsubseteq \mathsf{Center} \quad \text{for each } 1 \leq i \leq 2n \tag{92}$$

$$\exists \overline{r_i}^- \sqsubseteq \mathsf{Center} \quad \text{for each } 1 \leq i \leq 2n \tag{93}$$

$$\top \sqsubseteq \bigsqcup_{0 \leq i \leq 2n} (\exists r_i \sqcap \neg \mathsf{Bit}_i) \sqcup (\exists \overline{r_i} \sqcap \mathsf{Bit}_i) \tag{94}$$

We now require the number of sent sequences to be unique, if possible, via the minimized concept name AllPairs, triggered by the following axioms in $\mathcal{T}$:

$$\exists r_i^- \sqcap \exists \overline{r_i}^- \sqsubseteq \mathsf{AllPairs} \quad \text{for each } 1 \leq i \leq 2n \tag{95}$$

Therefore, if (at least) a pair is not represented in a model, then *all* elements can send the binary encoding of a same missing pair to avoid instantiating AllPairs. If, on the other hand, each pairs is represented (at least once), then the number of sent sequences cannot be unique and AllPairs must be satisfied on the central element a. To detect this with the AQ, we add the following axiom to $\mathcal{T}$:

$$\neg \mathsf{AllPairs} \sqsubseteq \mathsf{Goal} \tag{96}$$

Now that we can detect whether all pairs are represented, we move to the encoding of colors. The respective color of v_1 and v_2 is chosen locally via the following axioms in $\mathcal{T}$:

$$\top \sqsubseteq \mathsf{R}_1 \sqcup \mathsf{G}_1 \sqcup \mathsf{B}_1 \tag{97}$$

$$\top \sqsubseteq \mathsf{R}_2 \sqcup \mathsf{G}_2 \sqcup \mathsf{B}_2 \tag{98}$$

Recall concept names $\mathsf{R}_1, \mathsf{G}_1, \mathsf{B}_1$ and $\mathsf{R}_2, \mathsf{G}_2, \mathsf{B}_2$ are fixed. We additionally enforce that if an element represents a pair (v_1, v_2) corresponding to an edge $\{v_1, v_2\}$ in $\mathcal{G}_{\mathcal{C}}$, then it must assign different colors to v_1 and v_2. This is achieved by computing the output of $\mathcal{C}$ on input v_1 and v_2 via a fresh concept G for each gate G of $\mathcal{C}$ and adding the following axioms in $\mathcal{T}$:

$$G \equiv \mathsf{Bit}_i \quad \text{if } G \text{ is the } i^{\text{th}} \text{ input gate of } \mathcal{C} \tag{99}$$

$$G \equiv G_1 \sqcap G_2 \quad \text{if } G \text{ is a } \wedge\text{-gate with inputs } G_1, G_2 \tag{100}$$

$$G \equiv G_1 \sqcup G_2 \quad \text{if } G \text{ is a } \vee\text{-gate with inputs } G_1, G_2 \tag{101}$$

$$G \equiv \neg G' \quad \text{if } G \text{ is a } \neg\text{-gate with input } G' \tag{102}$$

Assuming the output gate of $\mathcal{C}$ is G_o, it now suffices to add the following axioms in $\mathcal{T}$:

$$G_o \sqcap \mathsf{R}_1 \sqcap \mathsf{R}_2 \sqsubseteq \bot \tag{103}$$

$$G_o \sqcap \mathsf{G}_1 \sqcap \mathsf{G}_2 \sqsubseteq \bot \tag{104}$$

$$G_o \sqcap \mathsf{B}_1 \sqcap \mathsf{B}_2 \sqsubseteq \bot \tag{105}$$

While the above allows to ensures each element e representing an edge $\{v_1, v_2\}$ assign different colors to v_1 and v_2, there might still be inconsistent colorings in the sense that v_1 might have been assigned another color by some other element e' representing an edge $\{v_1, v_3\}$. To detect such inconsistent colorings, the central element "scans" all colors assigned to a tested vertex, whose encoding correspond to a binary sequence of n bits represented by dedicated concepts T_i. To do so, the interpretation of each T_i on the center element is first copied by all elements via the following axioms in $\mathcal{T}$, defined for each $1 \leq i \leq n$:

$$\top \sqsubseteq \exists t_i \sqcup \exists \overline{t_i} \tag{106}$$

$$\exists t_i^- \sqsubseteq \mathsf{Center} \sqcap \mathsf{T}_i \qquad \exists \overline{t_i}^- \sqsubseteq \mathsf{Center} \sqcap \neg \mathsf{T}_i \tag{107}$$

$$\exists t_i \sqsubseteq \mathsf{T}_i \qquad \exists \overline{t_i} \sqsubseteq \neg \mathsf{T}_i \tag{108}$$

Each element representing a pair (v_1, v_2) then compares whether the encoding of either v_1 or v_2 matches the sequence T_i, via the following axioms:

$$\bigsqcap_{1 \leq i \leq n} ((\mathsf{Bit}_i \sqcap \mathsf{T}_i) \sqcup (\neg \mathsf{Bit}_i \sqcap \neg \mathsf{T}_i)) \equiv \mathsf{Test}_1 \tag{109}$$

$$\bigsqcap_{1 \leq i \leq n} ((\mathsf{Bit}_{n+i} \sqcap \mathsf{T}_i) \sqcup (\neg \mathsf{Bit}_{n+i} \sqcap \neg \mathsf{T}_i)) \equiv \mathsf{Test}_2 \tag{110}$$

It then sends the corresponding assigned color back to the central element via the following axioms in $\mathcal{T}$, defined for each $C \in \{\mathsf{R}, \mathsf{G}, \mathsf{B}\}$:

$$\mathsf{Test}_1 \sqcap C_1 \equiv \exists s_C \tag{111}$$

$$\mathsf{Test}_2 \sqcap C_2 \equiv \exists s_C \tag{112}$$

$$\exists s_C^- \sqsubseteq \mathsf{Center} \tag{113}$$

The center now detects whether it receives two different colors for the tested vertex:

$$\bigsqcup_{\substack{C, C' \in \{\mathsf{R}, \mathsf{G}, \mathsf{B}\} \\ C \neq C'}} (\exists s_C^- \sqcap \exists s_{C'}^-) \equiv \neg \mathsf{GoodCol} \tag{114}$$

where GoodCol is a minimized concept name, which forces the center element to detect an inconsistent coloring if it exists. This is further translated in term of the AQ with:

$$\neg \mathsf{GoodCol} \sqsubseteq \mathsf{Goal} \tag{115}$$

We are now done with the construction of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. It remains to prove that:

$$\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models \mathsf{Goal}(a) \iff \mathcal{C} \notin \text{Succinct-3COL}.$$

"$\Rightarrow$". Assume $\mathcal{C} \in$ Succinct-3COL and let $\rho : \{0, \dots, 2^n - 1\} \to \{\mathsf{R}, \mathsf{G}, \mathsf{B}\}$ be a (legal) 3-coloring of the encoded graph $\mathcal{G}_{\mathcal{C}}$. We build a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ based on ρ, whose domain $\Delta^{\mathcal{I}}$ contains a and one fresh element e_{v_1,v_2} per $(v_1, v_2) \in \{0, \dots, 2^n - 1\} \times \{0, \dots, 2^n - 1\}$. Model $\mathcal{I}$ further interprets concept and role names as follows:

$$\begin{aligned}
\mathsf{Center}^{\mathcal{I}} &= \mathsf{GoodCol}^{\mathcal{I}} = \mathsf{AllPairs}^{\mathcal{I}} = \{a\} \\
\mathsf{Bit}_i^{\mathcal{I}} &= \{e_{v_1,v_2} \mid i^{\text{th}} \text{ bit in } (v_1, v_2) \text{ encoding is } 1\} \\
C_i^{\mathcal{I}} &= \{e_{v_1,v_2} \mid \rho(v_i) = C\} \\
G &= \{e_{v_1,v_2} \mid G \in \mathcal{C} \text{ evaluates to 1 on input } (v_1, v_2)\} \\
\mathsf{Test}_i &= \{e_{v_1,v_2} \mid v_i = 0\} \\
\mathsf{Goal}^{\mathcal{I}} &= \mathsf{T}_i^{\mathcal{I}} = \emptyset \\
r_i^{\mathcal{I}} &= \{(e_{v_1,v_2}, a) \mid i^{\text{th}} \text{ bit in } (v_1, v_2) \text{ encoding is } 0\} \\
\overline{r_i}^{\mathcal{I}} &= \{(e_{v_1,v_2}, a) \mid i^{\text{th}} \text{ bit in } (v_1, v_2) \text{ encoding is } 1\} \\
t_i^{\mathcal{I}} &= \emptyset \\
\overline{t_i}^{\mathcal{I}} &= \{(e_{v_1,v_2}, a) \mid 0 \leq v_1, v_2 \leq 2^n - 1\} \\
s_C &= \{(e_{v_1,v_2}, a) \mid \rho(0) = C \\
&\qquad \text{and either } v_1 = 0 \text{ or } v_2 = 0\}
\end{aligned}$$

Notice we've arbitrarily chosen each element representing a pair to send back to the center its own encoding in which all bits have been flipped (see interpretations of roles r_i and $\overline{r_i}$), and the tested vertex to be 0 (interpretations of each T_i being empty and t_i, $\overline{t_i}$ and T_i being set accordingly). With the two above remarks, it is easily verified $\mathcal{I}$ models $\mathcal{K}$, and, by definition $a \notin \mathsf{Goal}^{\mathcal{I}}$. It remains to verify $\mathcal{I}$ complies with CP. By contradiction, assume there exists $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. Notice that from the fixed predicates, $\mathcal{J}$ still encodes one instance of each possible pair and the very same coloring ρ. From $\mathcal{J}$ being a model of $\mathcal{A}$, we have $a \in \mathsf{Center}^{\mathcal{J}}$. From $\mathsf{Center}^{\mathcal{I}} = \{a\}$ and Center being the most preferred minimized predicate, it follows that $\mathsf{Center}^{\mathcal{J}} = \{a\}$ as otherwise we would have $\mathcal{I} <_{\mathsf{CP}} \mathcal{J}$. Therefore, since each possible pair is represented at least once, the r_i and $\overline{r_i}$ mechanism ensures $a \in \mathsf{AllPairs}^{\mathcal{J}}$. Together with $\mathsf{AllPairs}^{\mathcal{I}} = \{a\}$ and AllPairs being second most preferred minimized predicates, we obtain $\mathsf{AllPairs}^{\mathcal{J}} = \{a\}$ (as otherwise, again, we would have $\mathcal{I} <_{\mathsf{CP}} \mathcal{J}$). Finally, since the coloring encoded in $\mathcal{J}$ must be exactly ρ, whatever the interpretations of concepts T_i, the concept $\exists s_C^- \sqcap \exists s_{C'}^-$ will always be empty for all $C, C' \in \{\mathsf{R}, \mathsf{G}, \mathsf{B}\}$ s.t. $C \neq C'$. Therefore, $a \in \mathsf{GoodCol}^{\mathcal{J}}$. Now, from GoodCol being also minimized and $\mathsf{GoodCol}^{\mathcal{I}} = \{a\}$, we obtain $\mathsf{GoodCol}^{\mathcal{J}} = \{a\}$. Overall $\mathcal{J}$ interprets the minimized predicates as $\mathcal{I}$ does, hence the desired contradiction of $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$.

"$\Leftarrow$". Assume $\mathcal{C} \notin$ Succinct-3COL. Consider a model $\mathcal{I}$ of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. First notice that $\mathsf{Center}^{\mathcal{I}} = \{a\}$. Indeed, $\mathcal{I}$ models $\mathcal{A}$ hence $a \in \mathsf{Center}^{\mathcal{I}}$, and, if there were $b \in$

$\mathsf{Center}^\mathcal{I}$ with $b \neq a$, then we could find a model $\mathcal{J}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by:

1. take $\Delta^\mathcal{J} = \Delta^\mathcal{I}$ and preserve all interpretations of fixed concept names;
2. setting $\mathsf{Center}^\mathcal{J} = \{a\}$;
3. replace every $(e, e') \in r_i^\mathcal{I}$ by $(e, a) \in r_i^\mathcal{J}$ and same for $\overline{r_i}$;
4. define accordingly $\mathsf{AllPairs}^\mathcal{J}$;
5. set for all $e \in \Delta^\mathcal{J}$: $e \in \mathsf{T}_i^\mathcal{J}$ iff $a \in \mathsf{T}_i^\mathcal{I}$;
6. set $t_i^\mathcal{J} = \{(e, a) \mid e \in \mathsf{T}_i^\mathcal{J}\}$ and $\overline{t_i}^\mathcal{J} = \{(e, a) \mid e \notin \mathsf{T}_i^\mathcal{J}\}$;
7. define accordingly interpretations of concepts evaluating the circuit and $\mathsf{Test}_1^\mathcal{J}$, $\mathsf{Test}_2^\mathcal{J}$;
8. set $s_C^\mathcal{J} = \{(e, a) \mid e \in ((\mathsf{Test}_1 \sqcap C_1) \sqcup (\mathsf{Test}_2 \sqcap C_2))^\mathcal{J}\}$;
9. define accordingly $\mathsf{GoodCol}^\mathcal{J}$ and $\mathsf{Goal}^\mathcal{J}$.

In particular, notice Step 1 removes b from the interpretation of Center.

We now want to prove that if a pair (v_1, v_2) is not represented in $\mathcal{I}$, that is:

$$\bigcap_{i \in \mathrm{Ones}(v_1,v_2)} \mathsf{Bit}_i^\mathcal{J} \cap \bigcap_{i \in \mathrm{Zeros}(v_1,v_2)} (\neg \mathsf{Bit}_i)^\mathcal{J} = \emptyset,$$

where $\mathrm{Ones}(v_1, v_2)$ is the set of $0 \leq i \leq 2n-1$ s.t. the i^{th} bit in the binary encoding of (v_1, v_2) is 1 and $\mathrm{Zeros}(v_1, v_2)$ is its complement, then the query must be satisfied. Indeed, if such a (v_1, v_2) exists, then we must have $\mathsf{AllPairs}^\mathcal{J} = \emptyset$, thus $a \in \mathsf{Goal}^\mathcal{J}$, as otherwise we can obtain a model $\mathcal{J}$ of $\mathcal{K}$ s.t. $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by:

1. take $\Delta^\mathcal{J} = \Delta^\mathcal{I}$ and preserve all interpretations of fixed concept names and Center;
2. set $r_i^\mathcal{J} = \Delta^\mathcal{I} \times \{a\}$ if $i \in \mathrm{Ones}(v_1, v_2)$ and $r_i^\mathcal{J} = \emptyset$ otherwise; set $\overline{r_i}^\mathcal{J} = \Delta^\mathcal{I} \times \{a\}$ if $i \in \mathrm{Zeros}(v_1, v_2)$ and $\overline{r_i}^\mathcal{J} = \emptyset$ otherwise;
3. follow steps 3 to 8 from the previous construction of $\mathcal{J}$.

In particular, Step 2 of the above complies with Equation 94 as (v_1, v_2) is not represented in $\mathcal{I}$. It further implies that $\mathsf{AllPairs}^\mathcal{J} = \emptyset$, which ensures the desired contradiction $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$.

We are thus left with the case in which each possible pair (v_1, v_2) is represented at least once in $\mathcal{I}$. From Equations 97 and 98, we know each element representing a pair assigns colors to its v_1 and v_2. From Equations 99 to 105, these colors cannot be the same if there is an edge $\{v_1, v_2\}$ in the graph $\mathcal{G}_\mathcal{C}$. Thus, since we assume $\mathcal{C} \notin$ Succinct-3COL, the overall choices of colors cannot be consistent: there must exists a vertex v_0 which is assigned a color C_1 by a element e_1 representing a pair p_1 and a different color C_2 by an element e_2 representing a pair p_2. Let us assume p_1 has shape (v_0, v_1) and p_2 has shape (v_0, v_2) (other cases works similarly). We can now prove that $\mathsf{GoodCol}^\mathcal{I} = \emptyset$, thus $a \in \mathsf{Goal}$, as otherwise we could find a model $\mathcal{J}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ by:

1. take $\Delta^\mathcal{J} = \Delta^\mathcal{I}$ and preserve all interpretations of fixed concept names, of Center and AllPairs, and of roles r_i and $\overline{r_i}$;
2. set for all $e \in \Delta^\mathcal{J}$: $e \in \mathsf{T}_i^\mathcal{J}$ iff $i \in \mathrm{Ones}(v_0)$;
3. set $t_i^\mathcal{J} = \{(e, a) \mid e \in \mathsf{T}_i^\mathcal{J}\}$ and $\overline{t_i}^\mathcal{J} = \{(e, a) \mid e \notin \mathsf{T}_i^\mathcal{J}\}$;
4. define accordingly interpretations of concepts evaluating the circuit and $\mathsf{Test}_1^\mathcal{J}$, $\mathsf{Test}_2^\mathcal{J}$;
5. set $s_C^\mathcal{J} = \{(e, a) \mid e \in ((\mathsf{Test}_1 \sqcap C_1) \sqcup (\mathsf{Test}_2 \sqcap C_2))^\mathcal{J}\}$;
6. define accordingly $\mathsf{GoodCol}^\mathcal{J}$ and $\mathsf{Goal}^\mathcal{J}$.

In particular, Equation 109 and Step 4 in the above ensure $e_1, e_2 \in \mathsf{Test}_1^\mathcal{J}$. Therefore Step 5 above yields $(e_1, a) \in s_{C_1}^\mathcal{J}$ and $(e_2, a) \in s_{C_2}^\mathcal{J}$, which triggers Equation 114 in Step 6 and ensures $\mathsf{GoodCol}^\mathcal{J} = \emptyset$. This yields the desired contradiction $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. Thus, $\mathsf{GoodCol}^\mathcal{I} = \emptyset$, in particular Equation 115 yields $a \in \mathsf{Goal}^\mathcal{I}$.

□

We work towards a proof of Theorem 14. Assume given a model $\mathcal{I}$ of a DL-Lite$^\mathcal{H}_{\mathsf{horn}}$ cKB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. We "forget" about some parts of $\mathcal{I}$, only retaining which roles may be needed due to a combination of fixed predicates in the forgotten part. To this end, we define the fixed-type $\mathsf{ftp}_\mathcal{I}(e)$ of an element $e \in \Delta^\mathcal{I}$ as the set of DL-Lite concepts C such that $\mathcal{T} \models (\bigsqcap_{e \in F^\mathcal{I},\, F \in \mathsf{F}} F) \sqsubseteq C$. Given a role $r \in \mathsf{N_R}^{\pm}$, we say that r is *forced* in $\mathcal{I}$ if there exists an element $f_r \in \Delta^\mathcal{I}$ such that $\exists r \in \mathsf{ftp}_\mathcal{I}(f_r)$. For each forced role in $\mathcal{I}$, we assume chosen such an element f_r. Similarly, for each role $r \in \mathsf{N_R}^{\pm}$, we assume chosen an element $w_r \in (\exists r^-)^\mathcal{I}$ if $(\exists r^-)^\mathcal{I} \neq \emptyset$. We now construct an interpretation $\mathcal{I}_0$ whose domain $\Delta^{\mathcal{I}_0}$ consists in $\mathsf{ind}(\mathcal{A})$, all chosen f_r and all chosen w_r. The interpretation $\mathcal{I}_0$ is now defined as:

$$\begin{aligned}
A^{\mathcal{I}_0} &= A^\mathcal{I} \cap \Delta^{\mathcal{I}_0} \\
r^{\mathcal{I}_0} &= r^\mathcal{I} \cap (\mathsf{ind}(\mathcal{A}) \times \mathsf{ind}(\mathcal{A})) \\
&\quad \cup \{(e, w_s) \mid e \in (\exists s)^\mathcal{I} \cap \Delta^{\mathcal{I}_0}, \mathcal{T} \models s \sqsubseteq r\} \\
&\quad \cup \{(w_s, e) \mid e \in (\exists s)^\mathcal{I} \cap \Delta^{\mathcal{I}_0}, \mathcal{T} \models s \sqsubseteq r^-\}
\end{aligned}$$

Remark 5. *Notice $\mathcal{I}_0$ is a special case of the models $\mathcal{I}_\mathcal{P}$ considered in Section 4.1, and that $|\mathcal{I}_0| \leq |\mathcal{A}| + 2|\mathcal{T}|$.*

The key-result is now the following:

Lemma 25. *Let q be an AQ. If $\mathcal{I}$ is a countermodel for q over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, then so is $\mathcal{I}_0$.*

Proof. Assume $\mathcal{I}$ is a countermodel for q over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Setting $\mathcal{P} = \{f_r \mid r \text{ is forced in } \mathcal{I}\}$, we have $\mathcal{I}_\mathcal{P} = \mathcal{I}_0$ (see Section 4.1) and thus Lemma 22 ensures $\mathcal{I}_0 \models \mathcal{K}$. Since concept interpretations in $\mathcal{I}_0$ are inherited from $\mathcal{I}$, it is clear $\mathcal{I}_0$ does not entail q. It remains to prove $\mathcal{I}_0$ also complies with CP. By contradiction assume there exists $\mathcal{J}_0$ a model of $\mathcal{K}$ with $\mathcal{J}_0 <_{\mathsf{CP}} \mathcal{I}_0$. Notice that, for each forced role r in $\mathcal{I}$, there must exists an element $w'_r \in (\exists r^-)^{\mathcal{J}_0}$ as we kept element $f_r \in \Delta^{\mathcal{I}_0}$ and r is a consequence of the fixed predicates on f_r that must have been preserved in $\mathcal{J}_0$ (see Condition 2 from the definition of $<_{\mathsf{CP}}$). We assume chosen such an element w'_r per forced role and build an interpretation $\mathcal{J}$:

$$\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$$
$$A^{\mathcal{J}} = A^{\mathcal{J}_0} \cup \{e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{J}_0} \mid A \in \mathsf{ftp}_{\mathcal{I}}(e)\}$$
$$r^{\mathcal{J}} = r^{\mathcal{J}_0} \cup \left\{ (e, w'_s) \;\middle|\; \begin{array}{l} e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{J}_0} \\ \exists s \in \mathsf{ftp}_{\mathcal{I}}(e), \mathcal{T} \models s \sqsubseteq r \end{array} \right\}$$
$$\cup \left\{ (w'_s, e) \;\middle|\; \begin{array}{l} e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{J}_0} \\ \exists s \in \mathsf{ftp}_{\mathcal{I}}(e), \mathcal{T} \models s \sqsubseteq r^- \end{array} \right\}$$

It is easily verified that $\mathcal{J}$ is a model of $\mathcal{K}$, and we now prove $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, which will contradict $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. We first notice a useful property: for all concept name A, we have $A^{\mathcal{J}} \cap (\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_0}) \subseteq A^{\mathcal{I}} \cap (\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_0})$ ($\star$). Indeed, if $e \in A^{\mathcal{J}} \cap (\Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_0})$, then $A \in \mathsf{ftp}_{\mathcal{I}}(e)$, and therefore $e \in A^{\mathcal{I}}$. The converse does not holds in general. We now check that all four conditions from the definition of $<_{\mathsf{CP}}$ are satisfied:

1. By definition, we have $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$.
2. Let $A \in \mathsf{F}$. Definitions of $\mathcal{I}_0$ and $\mathcal{J}$ ensure $A^{\mathcal{I}} \cap \Delta^{\mathcal{I}_0} = A^{\mathcal{I}_0}$ and $A^{\mathcal{J}} \cap \Delta^{\mathcal{I}_0} = A^{\mathcal{J}_0}$. From $\mathcal{J}_0 <_{\mathsf{CP}} \mathcal{I}_0$, we get $A^{\mathcal{I}_0} = A^{\mathcal{J}_0}$, which yields $A^{\mathcal{I}} \cap \Delta^{\mathcal{I}_0} = A^{\mathcal{J}} \cap \Delta^{\mathcal{I}_0}$. For an element $e \in \Delta^{\mathcal{I}} \setminus \Delta^{\mathcal{I}_0}$, we remark: $e \in A^{\mathcal{I}}$ iff $A \in \mathsf{ftp}_{\mathcal{I}}(e)$ iff $e \in A^{\mathcal{J}}$. Altogether, we obtain $A^{\mathcal{I}} = A^{\mathcal{J}}$.
3. Let $A \in \mathsf{M}$ such that $A^{\mathcal{J}} \not\subseteq A^{\mathcal{I}}$. Let thus $e \in A^{\mathcal{J}} \setminus A^{\mathcal{I}}$. Notice e must belong to $\Delta^{\mathcal{I}_0}$, as otherwise $\star$ yields $e \in A^{\mathcal{I}}$. It follows that $e \in A^{\mathcal{J}_0} \not\subseteq A^{\mathcal{I}_0}$. Since $\mathcal{J}_0 <_{\mathsf{CP}} \mathcal{I}_0$, there exists $B \prec A$ s.t. $B^{\mathcal{J}_0} \subsetneq B^{\mathcal{I}_0}$. By $\star$, this extends into $B^{\mathcal{J}} \subsetneq B^{\mathcal{I}}$ and we found B as desired.
4. From $\mathcal{J}_0 <_{\mathsf{CP}} \mathcal{I}_0$, there exists $A \in \mathsf{M}$ s.t. $A^{\mathcal{J}_0} \subsetneq A^{\mathcal{I}_0}$ and for all $B \prec A$, $B^{\mathcal{J}_0} = B^{\mathcal{I}_0}$. Joint with $\star$, it follows that $A^{\mathcal{J}} \subsetneq A^{\mathcal{I}}$ and for all $B \prec A$, $B^{\mathcal{J}} \subseteq B^{\mathcal{I}}$. If ever there exists such a $B \prec A$, such that additionally $B^{\mathcal{J}} \subsetneq B^{\mathcal{I}}$, then we select a minimal such B w.r.t. $\prec$, which provides the desired minimized concept. Otherwise, A fits.

It $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, contradicting $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ as desired. □

Theorem 14. *AQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{horn}}$ KBs is in Π^P_2 w.r.t. combined complexity.*

Proof. Guess a candidate interpretation $\mathcal{I}$ with size $|\mathcal{A}| + 2|\mathcal{T}|$. Check whether it is a model of $\mathcal{K}$ and if the AQ of interest is *not* satisfied in $\mathcal{I}$. Using a NP oracle, check whether $\mathcal{I}$ is a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. If all above tests succeed, then accept, otherwise reject. From Lemma 25, it is immediate that there exists a accepting run if there exists a countermodel for the AQ over $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Conversely, if a run accepts, then the guessed model $\mathcal{I}$ is a countermodel. □

Lemma 11. $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ *iff* $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0(a_0)$.

Proof. "⇒". Assume $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models A_0(a_0)$ and let $\mathcal{I}'$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$.

We extend $\mathcal{I}'$ into an interpretation $\mathcal{I}$ whose domain is $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}'} \cup (\mathsf{ind}(\mathcal{A}) \setminus \mathsf{ind}(\mathcal{A}'))$. To define a suitable interpretation of concepts on an individual a from $\Delta = \mathsf{ind}(\mathcal{A}) \setminus \mathsf{ind}(\mathcal{A}')$, we consider its ABox-type $t = \mathsf{tp}_{\mathcal{A}}(a)$. Notice $m_t = 4^{|\mathcal{T}|} + 1$, otherwise a would have been kept in W. Consider now the types in $\mathcal{I}$ of elements $a_{t,1}, \dots a_{t,4^{|\mathcal{T}|}+1}$. Since there are $4^{|\mathcal{T}|} + 1$ such elements for only $2^{|\mathcal{T}|}$ possible types, there exists a type t' realized in $\mathcal{I}$ at least $2^{|\mathcal{T}|}$ times. We now chose an element $\mathsf{ref}(t) \in \Delta^{\mathcal{I}'}$ s.t. $\mathsf{tp}_{\mathcal{A}}(\mathsf{ref}(t)) = t$ and $\mathsf{tp}_{\mathcal{I}'}(\mathsf{ref}(t)) = t'$. Such an element $\mathsf{ref}(t)$ serves as a reference to interpret concepts and roles on individuals $a \in \Delta$ with ABox type t, as the following construction shows:

$$A^{\mathcal{I}} = A^{\mathcal{I}'} \cup \{a \mid a \in \Delta, \mathsf{ref}(\mathsf{tp}_{\mathcal{A}}(a)) \in A^{\mathcal{I}'}\}$$
$$p^{\mathcal{I}} = p^{\mathcal{I}'} \cup \{(a, b) \mid \mathcal{K} \models p(a, b)\}$$
$$\cup \{(a, e) \mid a \in \Delta, (\mathsf{ref}(\mathsf{tp}_{\mathcal{A}}(a), e) \in p^{\mathcal{I}'}\}$$
$$\cup \{(e, a) \mid a \in \Delta, (e, \mathsf{ref}(\mathsf{tp}_{\mathcal{A}}(a)) \in p^{\mathcal{I}'}\}.$$

It is easily verified that $\mathcal{I}$ is a model of $\mathcal{K}$, and in particular that it satisfies role assertions from $\mathcal{A}$. We now prove $\mathcal{I} \models \mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$, which, by hypothesis, gives $\mathcal{I} \models A_0(a_0)$, and thus $\mathcal{I}' \models A_0(a_0)$ by definition of $A_0^{\mathcal{I}}$. Assume by contradiction one can find a model $\mathcal{J}$ of $\mathcal{K}$ s.t. $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$. We construct $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}'$ which will contradict $\mathcal{I}'$ being a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$. Intuitively, we proceed as in Lemma 2 and simulate $\mathcal{J}$ in $\mathcal{J}'$. For each ABox type t s.t. there exists $a \in \Delta$ with $\mathsf{tp}_{\mathcal{A}}(a) = t$, we set:

$$D_t = \{d \in \Delta^{\mathcal{I}'} \mid \mathsf{tp}_{\mathcal{A}}(d) = t, \mathsf{tp}_{\mathcal{I}'}(d) = \mathsf{tp}_{\mathcal{I}'}(\mathsf{ref}(t))\}$$
$$S_t = \{\mathsf{tp}_{\mathcal{J}}(d) \mid \mathsf{tp}_{\mathcal{A}}(d) = t, \mathsf{tp}_{\mathcal{I}'}(d) = \mathsf{tp}_{\mathcal{I}'}(\mathsf{ref}(t))\}$$

From the definition of $\mathsf{ref}(t)$, we have $|D_t| \geq 2^{|\mathcal{T}|}$, while, clearly, $|S_t| \leq 2^{|\mathcal{T}|}$. Since $S_t \neq \emptyset$ (it contains e.g. $\mathsf{tp}_{\mathcal{J}}(\mathsf{ref}(t))$), we can find a surjective function $\pi_t : D_t \to S_t$. We consider π the union of all such π_t, that is with t s.t. there exists $a \in \Delta$ with $\mathsf{tp}_{\mathcal{A}}(a) = t$. We further extend the domain of definition of π to $\Delta^{\mathcal{I}'}$ by setting $\pi(d) = \mathsf{tp}_{\mathcal{J}}(d)$ for the remaining elements of $\Delta^{\mathcal{I}'}$. We now define the interpretation $\mathcal{J}'$ as:

$$\Delta^{\mathcal{J}'} = \Delta^{\mathcal{I}'}$$
$$A^{\mathcal{J}'} = \{d \in \Delta^{\mathcal{I}'} \mid A \in \pi(d)\}$$
$$r^{\mathcal{J}'} = \{(d, e) \in \Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}'} \mid \pi(d) \rightsquigarrow_r \pi(e)\}.$$

It can now be verified that $\mathcal{J}'$ is a model of $\mathcal{K}'$ such that $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}'$.

"⇐". Assume $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}') \models A_0(a_0)$ and let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Here again, we denote $\Delta = \mathsf{ind}(\mathcal{A}) \setminus \mathsf{ind}(\mathcal{A}')$. By definition of W, all elements from Δ have an ABox type t that is already realized at least $4^{|\mathcal{T}|} + 1$ times in W. Therefore, we can define a permutation σ on $\Delta^{\mathcal{I}}$ such that for all $d \in \Delta^{\mathcal{I}}$, the following conditions are respected:

1. $\sigma(a_0) = a_0$;
2. $\mathsf{tp}_{\mathcal{A}}(\sigma(d)) = \mathsf{tp}_{\mathcal{A}}(d)$;
3. If $d \in \Delta$, then there exists an element $\mathsf{ref}(d) \in W$ with $\mathsf{tp}_{\mathcal{A}}(\mathsf{ref}(d)) = \mathsf{tp}_{\mathcal{A}}(d)$ and $\mathsf{tp}_{\mathcal{I}}(\mathsf{ref}(d)) = \mathsf{tp}_{\mathcal{I}}(d)$.

We can then define an interpretation $\mathcal{I}_\sigma$ as follows:

$$\Delta^{\mathcal{I}_\sigma} = \Delta^{\mathcal{I}}$$
$$A^{\mathcal{I}_\sigma} = \sigma(A^{\mathcal{I}})$$
$$r^{\mathcal{I}_\sigma} = \{(d, e) \mid \mathcal{K} \models r(d, e)\} \cup (\sigma \times \sigma)(r^{\mathcal{I}}).$$

It is straightforward that $\mathcal{I}_\sigma$ models $\mathcal{K}$ (notably relying on Condition 2), that preserves interpretation of concepts on the individual a_0 of interest (from Condition 1). Additionally, $\mathcal{I}_\sigma$ models $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ as otherwise a model $\mathcal{J}$ of $\mathcal{K}$ with $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}_\sigma$ immediately yields a model $\mathcal{J}_{\sigma^{-1}}$ of $\mathcal{K}$ with $\mathcal{J}_{\sigma^{-1}} <_{\mathsf{CP}} \mathcal{I}$, contradicting $\mathcal{I}$ being a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$.

We extend the mapping ref from Condition 3 to the identity for elements from $\Delta^{\mathcal{I}} \setminus \Delta$, and define an interpretation $\mathcal{I}'$ as follows:

$$\Delta^{\mathcal{I}'} = \Delta^{\mathcal{I}} \setminus \Delta$$
$$A^{\mathcal{I}'} = A^{\mathcal{I}_\sigma} \cap \Delta^{\mathcal{I}'}$$
$$r^{\mathcal{I}'} = (\mathsf{ref} \times \mathsf{ref})(r^{\mathcal{I}_\sigma} \cap (\Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}'})).$$

It can be verified that $\mathcal{I}'$ is a model of $\mathcal{K}'$. We now prove it is also a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}')$, which, by hypothesis, gives $\mathcal{I}' \models A_0(a_0)$, and thus $\mathcal{I} \models A_0(a_0)$ as unfolding definitions gives $a_0 \in A_0^{\mathcal{I}}$ iff $a_0 \in A_0^{\mathcal{I}_\sigma}$ iff $a_0 \in A_0^{\mathcal{I}'}$ (notably via Condition 1 on σ). Assume by contradiction one can find a model $\mathcal{J}'$ of $\mathcal{K}'$ with $\mathcal{J}' <_{\mathsf{CP}} \mathcal{I}'$. We now construct $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ which will contradict $\mathcal{I}$ being a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. Relying on the mapping ref, we set:

$$\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$$
$$A^{\mathcal{J}} = \mathsf{ref}^{-1}(A^{\mathcal{J}'})$$
$$r^{\mathcal{J}} = \{(d, e) \mid \mathcal{K} \models r(d, e)\} \cup (\mathsf{ref}^{-1} \times \mathsf{ref}^{-1})(r^{\mathcal{J}'}).$$

It is easily checked that $\mathcal{J}$ models $\mathcal{K}$. From Condition 3 on σ, it then follows $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$, ie the desired contradiction. □

H Proofs for Section 5.3

Theorem 16. *AQ evaluation on circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{core}}$ KBs with negative role inclusions is Π^P_2-hard. This holds even without fixed concept names and with a single negative role inclusion.*

Proof. We give a polynomial time reduction from ∀∃3SAT, c.f. the proof of Theorem 4. Let a ∀∃-3CNF sentence $\forall \bar{x} \exists \bar{y}\, \varphi$ be given where $\bar{x} = x_1 \cdots x_m$, $\bar{y} = y_1 \cdots y_n$, and $\varphi = \bigwedge_{i=1}^{\ell} \bigvee_{j=1}^{3} L_{ij}$ with $L_{ij} = v$ or $L_{ij} = \neg v$ for some $v \in \{x_1, \ldots x_m, y_1, \ldots, y_n\}$. We construct a circumscribed DL-Lite$^{\mathcal{H}}_{\mathsf{core}}$ KB $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$ and an atomic query q such that $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K}) \models q$ iff $\forall \bar{x} \exists \bar{y}\, \varphi$ is true.

The circumscription pattern CP involves four minimized concept names with the preference

$$\mathsf{VVal} \prec \mathsf{XVal} \prec \mathsf{CVal} \prec \mathsf{FVal},$$

and no fixed concept names. We now describe the construction of the KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, not strictly separating $\mathcal{T}$ from $\mathcal{A}$. We first introduce an ABox individual for each variable, marking the existential variables with the concept name XVar and the universal ones with YVar:

$$\mathsf{XVar}(x) \quad \text{for all } x \in \bar{x} \tag{116}$$
$$\mathsf{YVar}(y) \quad \text{for all } y \in \bar{y}. \tag{117}$$

To choose truth values for variables, we use the minimized concept name VVal. We introduce two instances of VVal for each variable, one representing true and the other false:

$$\mathsf{VVal}(t_v) \quad \text{for all } v \in \bar{x} \cup \bar{y} \tag{118}$$
$$\mathsf{VVal}(f_v) \quad \text{for all } v \in \bar{x} \cup \bar{y}. \tag{119}$$

Each variable must choose an instance of VVal via the role names $\mathsf{eval_X}$ an $\mathsf{eval_Y}$, depending on whether it is existential or universal:

$$\mathsf{XVar} \sqsubseteq \exists \mathsf{eval_X} \tag{120}$$
$$\mathsf{YVar} \sqsubseteq \exists \mathsf{eval_Y} \tag{121}$$
$$\exists \mathsf{eval_X}^- \sqsubseteq \mathsf{VVal} \tag{122}$$
$$\exists \mathsf{eval_Y}^- \sqsubseteq \mathsf{VVal}. \tag{123}$$

There is no guarantee yet, however, that a variable v chooses one of the instances t_v and f_v of VVal reserved for it. To ensure this, we first install the following role inclusions:

$$\mathsf{eval_X} \sqsubseteq \mathsf{eval} \tag{124}$$
$$\mathsf{eval_Y} \sqsubseteq \mathsf{eval} \tag{125}$$
$$\mathsf{eval} \sqsubseteq \neg \overline{\mathsf{eval}}. \tag{126}$$

The negative role inclusion allows us to control in a flexible way the targets for the existential restrictions in CIs (120) and (121), as follows:

$$\overline{\mathsf{eval}}(v, t_{v'}) \quad \text{for all } v, v' \in \bar{x} \cup \bar{y} \text{ with } v \neq v' \tag{127}$$
$$\overline{\mathsf{eval}}(v, f_{v'}) \quad \text{for all } v, v' \in \bar{x} \cup \bar{y} \text{ with } v \neq v'. \tag{128}$$

To force the truth values of the variables in $\bar{x}$ to be identical in all models that are smaller w.r.t. '$<_{\mathsf{CP}}$', we mark those truth values with the minimized concept name XVal:

$$\exists \mathsf{eval_X}^- \sqsubseteq \mathsf{XVal}. \tag{129}$$

We next introduce an individual for each clause:

$$\mathsf{Clause}(c_i) \quad \text{for } 1 \leq i \leq \ell. \tag{130}$$

We also assign a truth value to every clause, via the role name $\mathsf{eval_C}$, another subrole of eval. Every element with an incoming $\mathsf{eval_C}$-edge must make the concept name CVal true which is minimized, but with lower priority than VVal and XVal. This reflects the fact that the truth values of variables determine the truth values of clauses. We put:

$$\mathsf{Clause} \sqsubseteq \exists \mathsf{eval_C} \tag{131}$$
$$\mathsf{eval_C} \sqsubseteq \mathsf{eval} \tag{132}$$
$$\exists \mathsf{eval_C}^- \sqsubseteq \mathsf{CVal} \tag{133}$$

The truth values of clauses are represented by the same individuals as the truth values of variables:

$$\mathsf{CVal}(t_v) \quad \text{for all } v \in \bar{x} \cup \bar{y} \tag{134}$$
$$\mathsf{CVal}(f_v) \quad \text{for all } v \in \bar{x} \cup \bar{y} \tag{135}$$

We again use the role $\overline{\mathsf{eval}}$ to control which instance of Clause can be used as truth values for which clause. This is based on the literals that occur in the clause. If a variable v occurs positively in the i^{th} clause, then c_i has access to f_v, while if v occurs negatively, then c_i has access to t_v. And those are the only instances of VVal to which c_i has access: for $1 \leq i \leq \ell$ and all $v \in \bar{x} \cup \bar{y}$, put

$$\overline{\mathsf{eval}}(c_i, t_v) \quad \text{if } \neg v \notin \{L_{i,1}, L_{i,2}, L_{i,3}\} \tag{136}$$
$$\overline{\mathsf{eval}}(c_i, f_v) \quad \text{if } v \notin \{L_{i,1}, L_{i,2}, L_{i,3}\}. \tag{137}$$

Note that clause individual c_i has access via $\mathsf{eval_C}$ to t_v if making v *false* leads to satisfaction of the i^{th} clause, and likewise for f_v and making v *true*. Moreover, v being made true is represented by an $\mathsf{eval_X}$- or $\mathsf{eval_Y}$-edge from v to t_v and likewise for v being made false and f_v. Also recall that t_v and f_v cannot have such incoming edges from elsewhere. Consequently, the i^{th} clause evaluates to true if we can find a target for the $\mathsf{eval_C}$-edge that does *not* have an incoming $\mathsf{eval_X}$- or $\mathsf{eval_Y}$-edge:

$$\exists \mathsf{eval_X}^- \sqsubseteq \neg \exists \mathsf{eval_C}^- \tag{138}$$
$$\exists \mathsf{eval_Y}^- \sqsubseteq \neg \exists \mathsf{eval_C}^-. \tag{139}$$

In the case that a clause evaluates to false, it cannot reuse one of the admitted instances of CVal. We introduce an extra instance f of CVal that can be used instead:

$$\mathsf{CVal}(f). \tag{140}$$

To make sure that f is indeed used as an $\mathsf{eval_C}$-target only if at least one clause is falsified, we use the fourth minimized concept name FVal. Note that it is minimized with least priority. We make sure that f must have an outgoing $\mathsf{eval_F}$-edge leading to an instance of FVal, that f itself is an instance of FVal, and that having an incoming $\mathsf{eval_F}$-edge precludes having an incoming $\mathsf{eval_C}$-edge. If all clauses evaluate to true, then there is no need to use f as a target for $\mathsf{eval_C}$ and f can use itself as the $\mathsf{eval_F}$-target. But if a clause evaluates to false, then we must use f as a last resort $\mathsf{eval_C}$-target and cannot use it as an $\mathsf{eval_F}$-target. Since FVal is minimized with least priority, the latter will simply lead to a fresh instance of $\mathsf{eval_F}$ to be created. In summary, all clauses evaluate to true if and only of f has an incoming $\mathsf{eval_F}$-edge. We put:

$$\mathsf{Formula}(f) \qquad \mathsf{FVal}(f) \tag{141}$$
$$\mathsf{Formula} \sqsubseteq \exists \mathsf{eval_F} \tag{142}$$
$$\exists \mathsf{eval_F}^- \sqsubseteq \mathsf{FVal} \tag{143}$$
$$\exists \mathsf{eval_C}^- \sqsubseteq \neg \exists \mathsf{eval_F}^-. \tag{144}$$

By what was said above, to finish the reduction it suffices to add

$$\exists \mathsf{eval_F}^- \sqsubseteq \mathsf{Goal}. \tag{145}$$

to choose as the query $q = \mathsf{Goal}(x)$, and to ask whether f is an answer.

To prove correctness, we thus have to show the following.

Claim. $\mathsf{Circ_{CP}}(\mathcal{K}) \models q(f)$ iff $\forall \bar{x} \exists \bar{y}\, \varphi$ is true.

To prepare for the proof of the claim, we first observe that every valuation V for $\bar{x} \cup \bar{y}$ gives rise to a corresponding model $\mathcal{I}_V$ of $\mathcal{K}$. We use domain

$$\Delta^{\mathcal{I}_V} = \mathsf{ind}(\mathcal{A})$$

and set

$$A^{\mathcal{I}_V} = \{a \mid A(a) \in \mathcal{A}\}$$

for all concept names

$$A \in \{\mathsf{XVar}, \mathsf{YVar}, \mathsf{VVal}, \mathsf{CVal}, \mathsf{Clause}, \mathsf{Formula}\}.$$

We interpret $\mathsf{eval_X}$ and $\mathsf{eval_Y}$ according to V, that is

$$\begin{aligned} \mathsf{eval_X}^{\mathcal{I}_V} \quad = \quad & \{(x, t_v) \mid x \in \bar{x} \text{ and } V(x) = 1\} \cup \\ & \{(x, f_v) \mid x \in \bar{x} \text{ and } V(x) = 0\} \end{aligned}$$

and analogously for $\mathsf{eval_Y}$. Next we put

$$\mathsf{XVal}^{\mathcal{I}_V} = \{e \mid (e, d) \in \mathsf{eval_X}^{\mathcal{I}_V}\}.$$

If V makes the i^{th} clause true, then there is a literal in it that is true. Choose such a literal L and set $w_i = t_v$ if $L = \neg v$ an $w_i = f_v$ if $L = v$. If V makes the i^{th} clause false, then set $w_i = f$. We proceed with the definition of $\mathcal{I}_V$:

$$\begin{aligned} \mathsf{eval_C}^{\mathcal{I}_V} &= \{(c_i, w_i) \mid 1 \leq i \leq \ell\} \\ \mathsf{eval}^{\mathcal{I}_V} &= \mathsf{eval_X}^{\mathcal{I}_V} \cup \mathsf{eval_Y}^{\mathcal{I}_V} \cup \mathsf{eval_C}^{\mathcal{I}_V} \\ \overline{\mathsf{eval}}^{\mathcal{I}_V} &= \{(a, b) \mid \overline{\mathsf{eval}}(a, b) \in \mathcal{A}\} \\ \mathsf{FVal}^{\mathcal{I}_V} &= \{f, c_1\} \\ \mathsf{eval_F}^{\mathcal{I}_V} &= \{(f, c_1 0)\} \\ \mathsf{Goal}^{\mathcal{I}_V} &= \{c_1\}. \end{aligned}$$

The choice of c_1 is somewhat arbitrary in the last three statements, any other individual without an incoming $\mathsf{eval_C}$-edge would also do. It is straightforward to verify that $\mathcal{I}_V$ is a model of $\mathcal{K}$ with $\mathcal{I}_V \not\models q(f)$.

We can define a variation $\mathcal{I}'_V$ of $\mathcal{I}_V$ that is still a model of $\mathcal{K}$ provided that all clauses are satisfied by V (which guarantees that f has no incoming $\mathsf{eval_C}$-edge), and that satisfies $\mathcal{I}'_V \models q(f)$. To achieve this, we replace the last three lines from the definition of $\mathcal{I}_V$ with

$$\begin{aligned} \mathsf{FVal}^{\mathcal{I}_V} &= \{f\} \\ \mathsf{eval_F}^{\mathcal{I}_V} &= \{(f, f)\} \\ \mathsf{Goal}^{\mathcal{I}_V} &= \{f\}. \end{aligned}$$

Now for the actual proof of the claim

"⇒". Assume that $\forall \bar{x} \exists \bar{y}\, \varphi$ is false, and thus $\exists \bar{x} \forall \bar{y}\, \neg\varphi$ is true and there is a valuation $V_{\bar{x}} : \bar{x} \rightarrow \{0, 1\}$ such that $\forall \bar{y}\, \neg\varphi'$ holds where φ' is obtained from φ by replacing every variable $x \in \bar{x}$ with the truth constant $V_{\bar{x}}(x)$. Consider any extension V of $V_{\bar{x}}$ to the variables in $\bar{y}$ and the interpretation $\mathcal{I}_V$. Since $\mathcal{I}_V$ is a model of $\mathcal{K}$ with $\mathcal{I}_V \not\models q(f)$, to show that $\mathsf{Circ_{CP}}(\mathcal{K}) \not\models q(f)$ it remains to prove that there is no model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$.

Assume to the contrary that there is such a $\mathcal{J}$. We first observe that $A^{\mathcal{J}} = A^{\mathcal{I}_V}$ for all $A \in \{\mathsf{VVal}, \mathsf{XVal}, \mathsf{CVal}\}$. This holds for VVal since $\mathsf{VVal}^{\mathcal{J}} \subsetneq \mathsf{VVal}^{\mathcal{I}_V}$ means that $\mathcal{J}$ does not satisfy $\mathcal{A}$, and likewise for CVal. It also holds for XVal since every model of $\mathcal{K}$ must make XVal true at at least one of t_x and f_x for all $x \in \bar{x}$. This is because of CIs (120), (122), (124), and (129), the role disjointness (126), the assertions (127) and (128), and the fact that minimization of VVal is preferred over minimization of XVal. But no subset of $\mathsf{XVal}^{\mathcal{I}_V}$ satisfies this condition, so we must have $\mathsf{XVal}^{\mathcal{I}_V} \subseteq \mathsf{XVal}^{\mathcal{J}}$, consequently $\mathsf{XVal}^{\mathcal{J}} = \mathsf{XVal}^{\mathcal{I}_V}$.

The above and $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ means that $\mathsf{FVal}^{\mathcal{J}} \subsetneq \mathsf{FVal}^{\mathcal{I}_V}$. Since $\mathcal{J}$ is a model of $\mathcal{K}$, this implies $\mathsf{FVal}^{\mathcal{J}} = \{f\}$.

Since $\mathsf{XVal}^{\mathcal{J}} = \mathsf{XVal}^{\mathcal{I}_V}$ and $\mathcal{J}$ is a model of $\mathcal{K}$, $V(x) = 1$ implies $(x, t_x) \in \mathsf{eval}^{\mathcal{J}}$ and $V(x) = 0$ implies $(x, f_x) \in \mathsf{eval}^{\mathcal{J}}$. Moreover, since $\mathsf{VVal}^{\mathcal{J}} = \mathsf{VVal}^{\mathcal{I}_V}$ and $\mathcal{J}$ is a model of $\mathcal{K}$, for every $y \in \bar{y}$ we have $(y, t_y) \in \mathsf{eval}^{\mathcal{J}}$ or $(y, f_y) \in \mathsf{eval}^{\mathcal{J}}$. We thus find an extension V' of $V_{\bar{x}}$ to $\bar{x} \cup \bar{y}$ that is *compatible with* $\mathcal{J}$ in the sense that for all $v \in \bar{x} \cup \bar{y}$,

- $V'(v) = 1$ implies $(v, t_v) \in \mathsf{eval}^{\mathcal{J}}$ and
- $V'(v) = 0$ implies $(v, f_v) \in \mathsf{eval}^{\mathcal{J}}$.

Choose some such V'. Since $\forall \bar{y}\, \neg\varphi'$ holds, at least one of the clauses must be made false by V'. As explained alongside the construction of $\mathcal{K}$, that clause can only have an $\mathsf{eval}_{\mathsf{C}}$-edge to f. By (144), the outgoing $\mathsf{eval}_{\mathsf{F}}$-edge from f cannot end at f. But $\mathsf{FVal}^{\mathcal{J}} = \{f\}$ and thus (131) implies that f is the only point where that edge may end. We have arrived at a contradiction.

"$\Leftarrow$". Assume that $\forall \bar{x} \exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ is true and let $\mathcal{I}$ be a model of $\mathsf{Circ}_{\mathsf{CP}}(\mathcal{K})$. We have to show that $\mathcal{I} \models q(f)$. To this end, we first observe that

$$\mathsf{VVal}^{\mathcal{I}} = \{a \mid \mathsf{VVal}(a) \in \mathcal{A}\}. \qquad (*)$$

If this is not the case, in fact, then we find a model $\mathcal{J} <_{\mathsf{CP}} \mathcal{I}$ of $\mathcal{K}$, contradicting the minimality of $\mathcal{J}$. This model $\mathcal{J}$ is essentially $\mathcal{I}_V$ except that we might have to extend the domain with additional elements that do not occur in the extension of any concept or role name, to make sure that the domains of $\mathcal{I}$ and of $\mathcal{I}_V$ are identical.

It follows from $(*)$ and the fact that $\mathcal{I}$ is a model of $\mathcal{K}$ that for every $x \in \bar{x}$, we have $(x, t_x) \in \mathsf{eval}^{\mathcal{J}}$ or $(x, f_x) \in \mathsf{eval}^{\mathcal{J}}$. Consequently, we find a valuation $V_{\bar{x}}$ for $\bar{x}$ that is compatible with $\mathcal{I}$ as defined in the "$\Rightarrow$" direction of the proof. Since $\forall \bar{x} \exists \bar{y}\, \varphi(\bar{x}, \bar{y})$ is true, we can extend $V_{\bar{x}}$ to a valuation V for $\bar{x} \cup \bar{y}$ that satisfies φ. Consider the model $\mathcal{I}'_V$, extended to domain $\Delta^{\mathcal{I}}$. By construction of $\mathcal{I}'_V$ and $(*)$, we have $\mathsf{VVal}^{\mathcal{I}} = \mathsf{VVal}^{\mathcal{I}'_V}$. Using the facts that $\mathcal{I}$ is a minimal model of $\mathcal{K}$ and the construction of $\mathcal{I}'_V$, it can now be observed that $\mathsf{XVal}^{\mathcal{I}} = \mathsf{XVal}^{\mathcal{I}'_V}$. Moreover, by construction $\mathcal{I}'_V$ interprets CVal and FVal in a the minimal possible way among all models of $\mathcal{K}$. Since $\mathcal{I}$ is a minimal model, this implies $\mathsf{CVal}^{\mathcal{I}} = \mathsf{CVal}^{\mathcal{I}'_V}$ and $\mathsf{FVal}^{\mathcal{I}} = \mathsf{FVal}^{\mathcal{I}'_V}$. In particular, $\mathsf{FVal}^{\mathcal{I}} = \{f\}$. But this implies $(f, f) \in \mathsf{eval}_{\mathsf{F}}^{\mathcal{I}}$ as all objects with an incoming $\mathsf{eval}_{\mathsf{F}}$-edge must satisfy FVal. It follows that $f \in \mathsf{Goal}^{\mathcal{I}}$ and thus $\mathcal{I} \models q(f)$, as desired. $\square$