Cardinality Queries over DL-Lite Ontologies

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Abstract

Ontology-mediated query answering (OMQA) employs structured knowledge and automated reasoning in order to facilitate access to incomplete and possibly heterogeneous data. While most research on OMQA adopts (unions of) conjunctive queries as the query language, there has been recent interest in handling queries that involve counting. In this paper, we advance this line of research by investigating cardinality queries (which correspond to Boolean atomic counting queries) coupled with DL-Lite ontologies. Despite its apparent simplicity, we show that such an OMQA setting gives rise to rich and complex behaviour. While we prove that cardinality query answering is tractable (TC^0) in data complexity when the ontology is formulated in DL-Lite$_{core}$, the problem becomes coNP-hard as soon as role inclusions are allowed. For DL-Lite$^{\mathcal{H}}_{pos}$ (which allows only positive axioms), we establish a P-coNP dichotomy and pinpoint the TC^0 cases; for DL-Lite$^{\mathcal{H}}_{core}$ (allowing also negative axioms), we identify new sources of coNP complexity and also exhibit L-complete cases. Interestingly, and in contrast to related tractability results, we observe that the canonical model may not give the optimal count value in the tractable cases, which led us to develop an entirely new approach based upon exploring a space of strategies to determine the minimum possible number of query matches.

1 Introduction

In ontology-mediated query answering (OMQA) [Poggi *et al.*, 2008; Bienvenu and Ortiz, 2015; Xiao *et al.*, 2018], data is enriched with an ontology, which serves both to provide a user-friendly vocabulary for query formulation and to capture domain knowledge that is exploited at query time to obtain a more complete set of answers. While the OMQA approach offers many advantages, it also makes the query answering task more challenging than 'plain' query evaluation. Indeed, instead of having to evaluate the query over the single explicitly given data instance, one must identify the *certain answers*, i.e. those holding in all possible situations (models) compatible with the data and the ontology.

A major topic in OMQA research has thus been to understand the complexity of OMQA and identify tractable settings. Nowadays, for the most commonly considered query language, namely, conjunctive queries (CQs), we have an almost complete picture of the complexity landscape for ontologies formulated in a wide range of different description logics (DLs) [Baader *et al.*, 2017] and rule-based languages [Baget *et al.*, 2011; Calì *et al.*, 2012]. In particular, it has been shown that CQ answering is tractable in data complexity for ontologies expressed in the most commonly considered dialects of the DL-Lite family [Calvanese *et al.*, 2007; Artale *et al.*, 2009], which are often employed in OMQA. A well-known and frequently used property of such DL-Lite dialects and other Horn DLs is that they admit a *canonical model*, which is a single (possibly infinite) model that, by virtue of being homomorphically embeddable into every model, is guaranteed to give the correct answers to all CQs.

While CQs are a natural and well-studied class of queries, there are many other relevant forms of database queries that could be potentially be employed in OMQA. In the present paper, our focus will be on counting queries, which together with other forms of aggregate queries, are widely used for data analysis, yet still not well understood in the context of OMQA. A natural way to equip CQs with counting is to count the number of distinct query matches for each answer. As the count value may differ between models, Kostylev and Reutter (2015) advocated a form of certain answer semantics that considers lower and upper bounds on the count value across different models. Their work provided the first investigation of the complexity of answering counting CQs in the presence of ontologies, revealing such queries to be much more challenging to handle than plain CQs: coNP-complete in data complexity for the well-known DL-Lite$_{core}$ and DL-Lite$^{\mathcal{H}}_{core}$ dialects. A recent work by Bienvenu *et al.* (2020) refined and generalized the complexity results from Kostylev and Reutter to a wider class of counting queries and identified a restricted scenario with very low (TC^0-complete) data complexity: rooted CQs coupled with DL-Lite$_{core}$ ontologies. A similar tractability result for connected rooted CQs was proven independently by Calvanese *et al.* (2020a), who also initiated a study of the impact of other restrictions on query shape and developed the first query rewriting procedure for counting CQs. Notably, both the aforementioned TC^0 result and the rewriting procedure crucially relied upon showing that the canonical model gives the right answers under the considered restrictions. We briefly mention two alternative approaches to counting queries: an epistemic semantics for aggregate queries (which only counts query matches over the data constants, ignoring unnamed elements) was explored by Calvanese *et al.* (2008), while another very recent study by Feier *et al.* (2021) classifies the complexity of counting the number of certain answers (rather than the number of ways a certain answer is obtained) for guarded existential rules.

While recent studies have improved our understanding of the complexity of counting CQs, there nevertheless remain many unanswered questions. In this paper, we focus on Boolean atomic counting queries of the form $\exists z.\mathrm{A}(z)$ and $\exists z_1, z_2.\mathrm{R}(z_1, z_2)$, which we term *cardinality queries* as they correspond to the natural task of determining (bounds on) the cardinality of a given concept or role name. The data complexity of answering such basic counting queries remains completely open for DL-Lite$_{core}$ ontologies, whilst for DL-Lite$^{\mathcal{H}}_{core}$, the problem is known to be P-hard and in coNP [Calvanese *et al.*, 2020a]. The main results of our investigation are displayed in Table 1. We show that when ontologies are expressed in DL-Lite$_{core}$, cardinality query answering is tractable in data complexity and enjoys the lower possible complexity (TC^0-complete). For cardinality queries based upon a concept atom, TC^0 membership holds even for the fragment of DL-Lite$^{\mathcal{H}}_{core}$ obtained by disallowing negative role inclusions. By contrast, for role cardinality queries, we show that coNP-hard situations arise in DL-Lite$^{\mathcal{H}}_{pos}$, which allows only positive concept and role inclusions. In fact, we obtain a complete data complexity classification for DL-Lite$^{\mathcal{H}}_{pos}$, showing that every ontology-mediated query is either TC^0-complete, coNP-complete, or is in P and logspace-equivalent to the complement of PERFECT MATCHING (whose precise complexity is a longstanding open problem). The preceding classification does not extend to DL-Lite$^{\mathcal{H}}_{core}$: we identify new sources of coNP-hardness and further exhibit L-complete cases. We find it intriguing that such complex behaviour arises in what appears at first glance to be a simple OMQA setting. Moreover, in all of the tractable cases we identify, the canonical model may not yield the minimum cardinality, and query answering involves solving non-trivial optimization problems. This led us to devise an entirely new approach based upon exploring a space of strategies to find the optimal way of merging witnesses for existential axioms.

The paper is organized as follows. Section 2 recalls relevant background material and presents the considered OMQA setting. Section 3 introduces strategies and uses them to establish TC^0 membership. Our complexity classification for DL-Lite$^{\mathcal{H}}_{pos}$ is the topic of Section 4, while Section 5 presents our results for DL-Lite$^{\mathcal{H}}_{core}$. Section 6 concludes with a brief discussion of related and future work.

An appendix with full proofs can be found in the long version of this paper, available on arXiv.

2 Preliminaries

We recall standard definitions and notation for OMQA in DL-Lite and introduce the particular setting studied in this paper.

	Concept	Role
DL-Lite$_{core}$	TC^0-c	TC^0-c
DL-Lite$^{\mathcal{H}}_{pos}$	TC^0-c†	TC^0-c | co-PM-c | coNP-c
DL-Lite$^{\mathcal{H}}_{core}$	TC^0-c | L-c | coNP-c | ?	TC^0-c | L-c | co-PM-c | coNP-c | ?

Table 1: Data complexity of cardinality queries based upon concept and role atoms for various DL-Lite dialects. †: upper bound holds for all DL-Lite$^{\mathcal{H}}_{core}$ ontologies without negative role inclusions.

Knowledge Bases. We assume mutually disjoint sets $\mathsf{N_C}$ of concept names (unary predicates), $\mathsf{N_R}$ of role names (binary predicates), and $\mathsf{N_I}$ of individual names (constants). We denote by $\mathsf{N_R^\pm}$ the set $\mathsf{N_R} \cup \{\mathrm{R}^- \mid \mathrm{R} \in \mathsf{N_R}\}$ of role names and their inverses. A *knowledge base (KB)* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ consists of an ABox (dataset) $\mathcal{A}$ and a TBox (ontology) $\mathcal{T}$. An *ABox* is a finite set of *concept assertions* $\mathrm{A}(\mathsf{b})$ (with $\mathrm{A} \in \mathsf{N_C}$, $b \in \mathsf{N_I}$) and *role assertions* $\mathrm{P}(\mathsf{a}, \mathsf{b})$ (with $\mathrm{P} \in \mathsf{N_R}$, $\mathsf{a}, \mathsf{b} \in \mathsf{N_I}$), while the TBox consists of a finite set of axioms, whose forms are dictated by the considered description logic.

In this paper, our focus will be on DL-Lite$^{\mathcal{H}}_{core}$ (alternatively referred to as DL-Lite$_{\mathcal{R}}$), which is the logic underlying the OWL 2 QL profile. DL-Lite$^{\mathcal{H}}_{core}$ TBoxes contain four types of axioms: *positive concept inclusions* $\mathrm{B}_1 \sqsubseteq \mathrm{B}_2$, *negative concept inclusions* $\mathrm{B}_1 \sqsubseteq \neg\mathrm{B}_2$, *positive role inclusions* $\mathrm{R}_1 \sqsubseteq \mathrm{R}_2$, and *negative role inclusions* $\mathrm{R}_1 \sqsubseteq \neg\mathrm{R}_2$, where the $\mathrm{B_i}$ and $\mathrm{R_i}$ are *positive concepts and roles* given by:

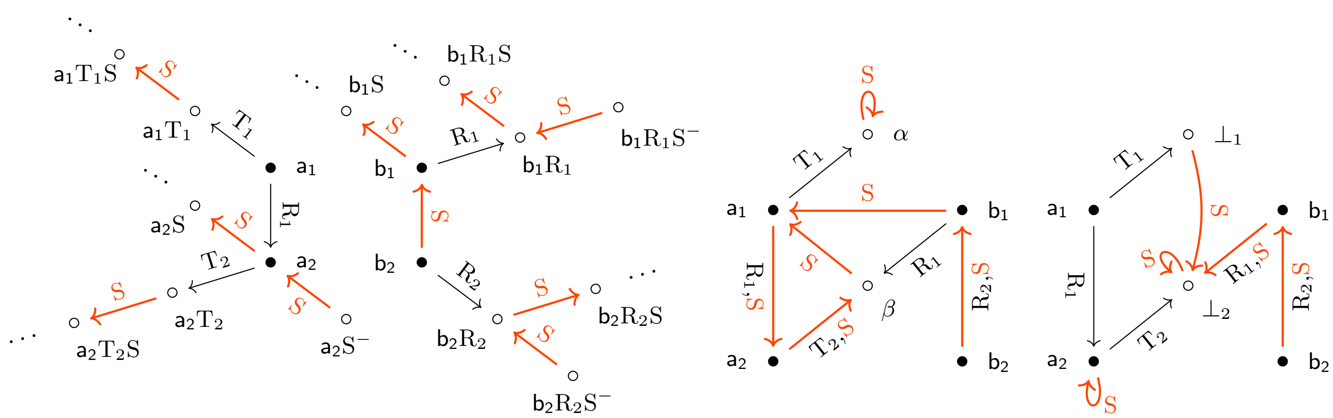
$$\mathrm{B_i} := \mathrm{A} \mid \exists \mathrm{R_i} \qquad \mathrm{R_i} := \mathrm{P} \mid \mathrm{P}^- \qquad (\mathrm{A} \in \mathsf{N_C},\ \mathrm{P} \in \mathsf{N_R})$$

The sublogic DL-Lite$_{core}$ allows only concept inclusions (which may be either positive or negative), while DL-Lite$^{\mathcal{H}}_{pos}$ is restricted to positive (concept and role) inclusions.

We denote by $\mathsf{Ind}(\mathcal{A})$ the set of individuals occurring in an ABox $\mathcal{A}$. A *signature* is a finite set of concept and role names. Given a signature Σ, we denote by $\Sigma^\pm_\mathsf{C}$ (resp. $\Sigma^\pm_\mathsf{R}$) the set of positive concepts (resp. roles) built from Σ. The signature of a TBox $\mathcal{T}$ (resp. ABox $\mathcal{A}$) is the set of concept and role names it contains, denoted $\mathsf{sig}(\mathcal{T})$ (resp. $\mathsf{sig}(\mathcal{A})$). To simplify the presentation, we will assume w.l.o.g. that $\mathsf{sig}(\mathcal{A}) \subseteq \mathsf{sig}(\mathcal{T})$.

Semantics of KBs. An interpretation takes the form $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a non-empty set (called the domain) and $\cdot^\mathcal{I}$ is the interpretation function that maps each $\mathrm{A} \in \mathsf{N_C}$ to $\mathrm{A}^\mathcal{I} \subseteq \Delta^\mathcal{I}$, each $\mathrm{P} \in \mathsf{N_R}$ to $\mathrm{P}^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$, and each $\mathsf{a} \in \mathsf{N_I}$ to $\mathsf{a}^\mathcal{I}$. In this paper, we will make the *Standard Names Assumption* by setting $\mathsf{a}^\mathcal{I} = \mathsf{a}$. Note however that our results only rely upon the weaker *Unique Names Assumption* (UNA), which stipulates that $\mathsf{a}^\mathcal{I} \neq \mathsf{b}^\mathcal{I}$ whenever $\mathsf{a} \neq \mathsf{b}$. The UNA is commonly adopted for DL-Lite KBs and enables more interesting reasoning in the context of counting queries.

The function $\cdot^\mathcal{I}$ is extended to general concepts and roles as follows: $(\mathrm{P}^-)^\mathcal{I} = \{(e, d) \mid (d, e) \in \mathrm{P}^\mathcal{I}\}$, $(\exists\mathrm{R})^\mathcal{I} = \{d \mid (d, e) \in \mathrm{R}^\mathcal{I}\}$, and $(\neg\mathrm{G})^\mathcal{I} = \Delta^\mathcal{I} \setminus \mathrm{G}^\mathcal{I}$. An inclusion $\mathrm{G} \sqsubseteq \mathrm{H}$ is satisfied in $\mathcal{I}$ if $\mathrm{G}^\mathcal{I} \subseteq \mathrm{H}^\mathcal{I}$; an assertion $\mathrm{A}(\mathsf{b})$ (resp. $\mathrm{P}(\mathsf{a}, \mathsf{b})$) is satisfied in $\mathcal{I}$ if $\mathsf{b} \in \mathrm{A}^\mathcal{I}$ (resp. $(\mathsf{a}, \mathsf{b}) \in \mathrm{P}^\mathcal{I}$). An interpretation is a model of a TBox $\mathcal{T}$ (resp. ABox $\mathcal{A}$)) if it satisfies all axioms in $\mathcal{T}$ (resp. assertions in $\mathcal{A}$), and it is a *model of a KB* $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ if it is a model of both $\mathcal{T}$ and $\mathcal{A}$. A KB



(a) Initial portion of the canonical model of $\mathcal{K}_e$. (b) Another model of $\mathcal{K}_e$. (c) Interpretation of strategy σ_e.

Figure 1: Models of the example KB $\mathcal{K}_e$. For readability, we have omitted concepts and highlighted the role S from the cardinality query.

is *satisfiable* if it has at least one model. An inclusion (resp. assertion) Φ is *entailed* from $\mathcal{T}$ (resp. $\mathcal{K}$), written $\mathcal{T} \models \Phi$ (resp. $\mathcal{K} \models \Phi$), if Φ is satisfied in every model of $\mathcal{T}$ (resp. $\mathcal{K}$). We use $\mathcal{K} \models \exists \mathrm{R}(\mathsf{a})$ (resp. $\mathcal{K} \models \mathrm{R}(\mathsf{a}, \mathsf{b})$ with $\mathrm{R} \in \mathsf{N}_\mathsf{R}^\pm$) to indicate $\mathsf{a} \in \exists \mathrm{R}^\mathcal{I}$ (resp. $(\mathsf{a}, \mathsf{b}) \in \mathrm{R}^\mathcal{I}$) for every model $\mathcal{I}$ of $\mathcal{K}$.

Example 1. *As a running example, we will consider the KB* $\mathcal{K}_e = (\mathcal{T}_e, \mathcal{A}_e)$ *whose TBox contains the following inclusions*

$$\begin{array}{llll} \mathrm{A}_1 \sqsubseteq \exists \mathrm{T}_1 & \mathrm{A}_2 \sqsubseteq \exists \mathrm{T}_2 & \exists \mathrm{T}_1^- \sqsubseteq \exists \mathrm{S} & \exists \mathrm{R}_1^- \sqsubseteq \neg \exists \mathrm{R}_2^- \\ \mathrm{B}_1 \sqsubseteq \exists \mathrm{R}_1 & \mathrm{B}_2 \sqsubseteq \exists \mathrm{R}_2 & \exists \mathrm{R}_1^- \sqsubseteq \exists \mathrm{S}^- & \exists \mathrm{R}_1^- \sqsubseteq \neg \exists \mathrm{T}_1^- \\ \exists \mathrm{T}_2^- \sqsubseteq \exists \mathrm{S} & \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S} & \exists \mathrm{R}_2^- \sqsubseteq \exists \mathrm{S}^- & \end{array}$$

and whose ABox contains the assertions

$$\{\mathrm{A}_1(\mathsf{a}_1), \mathrm{A}_2(\mathsf{a}_2), \mathrm{B}_1(\mathsf{b}_1), \mathrm{B}_2(\mathsf{b}_2), \mathrm{R}_1(\mathsf{a}_1, \mathsf{a}_2), \mathrm{S}(\mathsf{b}_2, \mathsf{b}_1)\}$$

Two finite models of $\mathcal{K}_e$ *are displayed in Figures 1b and 1c.*

Canonical Model. Every satisfiable DL-Lite$_{core}^\mathcal{H}$ KB $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ has a *canonical model* $\mathcal{C}_\mathcal{K}$, defined as follows. The domain of $\mathcal{C}_\mathcal{K}$ contains $\mathsf{Ind}(\mathcal{A})$ and all words $\mathsf{a}\mathrm{R}_1 \ldots \mathrm{R}_n$, with $\mathsf{a} \in \mathsf{Ind}(\mathcal{A})$, $\mathrm{R}_i \in \mathsf{N}_\mathsf{R}^\pm$, and $n \geqslant 1$, such that:

- $\mathcal{K} \models \exists \mathrm{R}_1(\mathsf{a})$ and there is no $\mathrm{R}_1(\mathsf{a}, \mathsf{b}) \in \mathcal{A}$;
- for $1 \leq i < n$, $\mathcal{T} \models \exists \mathrm{R}_i^- \sqsubseteq \exists \mathrm{R}_{i+1}$ and $\mathrm{R}_i^- \neq \mathrm{R}_{i+1}$.

Concept and role names are interpreted as follows:

$$\begin{aligned} \mathrm{A}^{\mathcal{C}_\mathcal{K}} = & \{\mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \mid \mathcal{K} \models \mathrm{A}(a)\} \\ & \cup \{\mathsf{a}\mathrm{R}_1 \ldots \mathrm{R}_n \in \Delta^{\mathcal{C}_\mathcal{K}} \setminus \mathsf{Ind}(\mathcal{A}) \mid \mathcal{T} \models \exists \mathrm{R}_n^- \sqsubseteq \mathrm{A}\} \\ \mathrm{P}^{\mathcal{C}_\mathcal{K}} = & \{(\mathsf{a}, \mathsf{b}) \mid \mathrm{P}(\mathsf{a}, \mathsf{b}) \in \mathcal{A}\} \\ & \cup \{(e_1, e_2) \mid e_2 = e_1\mathrm{R} \text{ and } \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{P}\} \\ & \cup \{(e_2, e_1) \mid e_2 = e_1\mathrm{R} \text{ and } \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{P}^-\} \end{aligned}$$

We use $\mathsf{gen}_\mathcal{K}$ to refer to the set of *generated roles*, i.e. those $\mathrm{R} \in \mathsf{N}_\mathsf{R}^\pm$ such that $\Delta^{\mathcal{C}_\mathcal{K}}$ contains an element $w\mathrm{R}$.

Example 2. *An initial portion of (the infinite) canonical model of* $\mathcal{K}_e$ *is displayed in Figure 1a. Observe that* $\mathsf{gen}_\mathcal{K} = \{\mathrm{S}, \mathrm{S}^-, \mathrm{R}_1, \mathrm{R}_2, \mathrm{T}_1, \mathrm{T}_2\}$.

It is well known (see e.g. [Calvanese *et al.*, 2007]) that, for every model $\mathcal{I}$ of $\mathcal{K}$, there is a *homomorphism* from $\mathcal{C}_\mathcal{K}$ to $\mathcal{I}$, i.e. a function $f : \Delta^{\mathcal{C}_\mathcal{K}} \to \Delta^\mathcal{I}$ such that (i) $f(\mathsf{a}) = \mathsf{a}$ for all $\mathsf{a} \in \mathsf{Ind}$, (ii) $e \in \mathrm{A}^{\mathcal{C}_\mathcal{K}}$ implies $f(e) \in \mathrm{A}^\mathcal{I}$, and (iii) $(d, e) \in \mathrm{P}^{\mathcal{C}_\mathcal{K}}$ implies $(f(d), f(e)) \in \mathrm{P}^\mathcal{I}$.

Cardinality Queries. A *cardinality query* is either a *concept cardinality query* $\exists z.\mathrm{C}(z)$ or a *role cardinality query* $\exists z_1, z_2.\mathrm{S}(z_1, z_2)$. Throughout the paper, we use q_C (resp. q_S) as a shorthand for the cardinality query based upon C (resp. S). A *match* for a cardinality query q_C (resp. q_S) in an interpretation $\mathcal{I}$ is an element of $\mathrm{C}^\mathcal{I}$ (resp. $\mathrm{S}^\mathcal{I}$). We define the *answer to a cardinality query* q *in an interpretation* $\mathcal{I}$, denoted $q^\mathcal{I}$, as the number of matches of q in $\mathcal{I}$, or equivalently, as the cardinality of $\mathrm{F}^\mathcal{I}$, with F the concept or role name in q. A *certain answer* to q w.r.t. $\mathcal{K}$ is an interval $[m, M] \in \mathbb{N} \times \mathbb{N}$ such that $q^\mathcal{I} \in [m, M]$ for every model $\mathcal{I}$ of $\mathcal{K}$.

Example 3. *Consider the role cardinality query* q_S. *The answer to* q_S *is* $+\infty$ *in* $\mathcal{C}_{\mathcal{K}_e}$, 6 *in the model from Figure 1b, and* 5 *in the model from Figure 1c. The latter implies that* $[6, +\infty]$ *is* not *a certain answer. We leave it is an exercise to find a model with 3 matches and show there is no model with fewer matches, which means that* $[m, +\infty]$ *is a certain answer to* q_S *over* $\mathcal{K}_e$ *if and only if* $m \leq 3$.

Cardinality queries as defined above correspond to a special case of the counting queries considered in [Kostylev and Reutter, 2015; Bienvenu *et al.*, 2020; Calvanese *et al.*, 2020a].

Observe that since DL-Lite$_{core}^\mathcal{H}$ cannot restrict the size of models, the value M in a certain answer $[m, M]$ must be $+\infty$ whenever the query predicate F is satisfiable w.r.t. $\mathcal{T}$ (i.e. there is a model $\mathcal{I}$ of $\mathcal{T}$ such that $\mathrm{F}^\mathcal{I} \neq \emptyset$). For this reason, we assume the latter condition holds and focus on identifying certain answers of the form $[m, +\infty]$.

Complexity. We will be interested in classifying the complexity of the following problem:

OMQA$(q, \mathcal{T})$: Given $\mathcal{A}$ and an integer $m \geq 1$ (in binary), decide whether $[m, +\infty]$ is a certain answer to q w.r.t. $\mathcal{K}$.

where $(q, \mathcal{T})$ is an *ontology-mediated query (OMQ)* based upon a cardinality query q and a TBox $\mathcal{T}$ formulated in DL-Lite$_{core}^\mathcal{H}$ or one of its sublogics. Note that we are adopting the *data complexity* measure as $(q, \mathcal{T})$ is fixed.

Beyond well-known complexity classes such as P and coNP, we will refer to the following classes: TC^0 is the class of problems solvable by families of constant-depth polynomial-size circuits based upon AND, OR, NOT, and threshold gates, and L (resp. NL) is the class of problems solvable in deterministic (resp. nondeterministic) logarithmic space. It is known that: $\mathsf{TC}^0 \subseteq \mathsf{L} \subseteq \mathsf{NL} \subseteq ... \subseteq \mathsf{P} \subseteq \mathsf{coNP}$.

3 Tractable Cases

In this section, we identify two settings in which cardinality queries can be answered with the lowest possible complexity:

Theorem 1. *$\mathrm{OMQA}(q, \mathcal{T})$ is TC^0-complete if either (i) q is a role cardinality query and $\mathcal{T}$ a DL-Lite$_{core}$ TBox, or (ii) q is a concept cardinality query and $\mathcal{T}$ is a DL-Lite$^{\mathcal{H}}_{core}$ TBox without negative role inclusions.*

The remainder of this section is devoted to establishing TC^0 membership for case (i) where our query is $q_{\mathrm{S}} = \exists z_1, z_2.\mathrm{S}(z_1, z_2)$. A similar but simpler argument can be used for the membership half of case (ii), while TC^0-hardness is easily shown by reduction from the TC^0-complete NUMONES problem [Aehlig *et al.*, 2007] asking, given a binary string X and $k \geq 1$, whether X contains at least k 1-bits.

Existing proofs of sub-polynomial data complexity for restricted classes of counting queries rely on the canonical model minimizing the number of matches [Bienvenu *et al.*, 2020; Calvanese *et al.*, 2020a]. However, for the class of cardinality queries, the canonical model may not yield the minimum value. Therefore, we develop a different approach based upon a systematic exploration of a set of models that is guaranteed to contain an optimal model and whose size depends only on the TBox. This special set of models will be induced from strategies that dictate how to merge elements of the canonical model. To show such models contain the optimal value, we show that if we extract a strategy σ from an arbitrary model $\mathcal{I}$ and consider any model $\mathcal{J}$ induced by σ, then $\mathcal{J}$ has at most as many matches as the initial model $\mathcal{I}$.

We now formalize this approach. In order to abstract from specific ABox individuals, we introduce types.

Definition 1. *A* type *for a TBox $\mathcal{T}$ is a subset of $\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{C}}$. The set of all types is $\Theta_{\mathcal{T}} = 2^{\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{C}}}$. We denote by $\theta_{\mathcal{K}}(d)$ the* type of a domain element d w.r.t. $\mathcal{K}$ *and define it by: $\theta_{\mathcal{K}}(d) = \{\mathrm{B} \in \mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{C}} \mid \mathcal{K} \models \mathrm{B}(d)\}$ if $d \in \mathsf{Ind}(\mathcal{A})$, else $\theta_{\mathcal{K}}(d) = \emptyset$.*

Example 4. *In our running example, $\theta_{\mathcal{K}_e}(\mathsf{a}_1) = \{\mathrm{A}_1, \exists\mathrm{R}_1, \exists\mathrm{T}_1\}$ and $\theta_{\mathcal{K}_e}(\alpha) = \emptyset$ (since $\alpha \notin \mathsf{Ind}(\mathcal{A}_e)$).*

Strategies indicate for each generated role R the type onto which all elements $w\mathrm{R}$ should merge. Several copies of a type might be required to comply with negative inclusions (e.g. R_1 and R_2 associated to the same type but $\mathcal{T} \models \exists\mathrm{R}_1^- \sqsubseteq \neg\exists\mathrm{R}_2^-$).

Definition 2. *A* strategy *σ for the KB $\mathcal{K}$ is a function from $\mathsf{gen}_{\mathcal{K}}$ to $\Theta_{\mathcal{T}} \times \{1, \ldots, |\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}|\}$, that satisfies:*

1. $\forall \mathrm{R} \in \mathsf{gen}_{\mathcal{K}} :\ \sigma(\mathrm{R}) = (\mathsf{t}, i) \wedge \mathrm{B} \in \mathsf{t} \Rightarrow \mathcal{T} \not\models \exists\mathrm{R}^- \sqsubseteq \neg\mathrm{B}$.

2. $\forall \mathrm{R}_1, \mathrm{R}_2 \in \mathsf{gen}_{\mathcal{K}} : \sigma(\mathrm{R}_1) = \sigma(\mathrm{R}_2) \Rightarrow \mathcal{T} \not\models \exists\mathrm{R}_1^- \sqsubseteq \neg\exists\mathrm{R}_2^-$.

3. $\forall \mathsf{t} \in \Theta_{\mathcal{T}}$, if $\mathsf{t} \neq \emptyset$, then $|\{i \mid \exists\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}, \sigma(\mathrm{R}) = (\mathsf{t}, i)\}| \leq |\{\mathsf{a} \mid \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \wedge \theta_{\mathcal{K}}(\mathsf{a}) = \mathsf{t}\}|$.

Conditions 1 and 2 ensure that merging will not violate any negative inclusions. Condition 3 ensures the ABox provides at least as many individuals of a non-empty type as the strategy requires copies of this type.

Example 5. *The following mapping σ_e is a strategy for $\mathcal{K}_e$:*

$$\begin{array}{llllll} \mathrm{T}_1 & \mapsto & (\emptyset, 1) & \mathrm{R}_2 & \mapsto & (\{\mathrm{B}_1, \exists\mathrm{R}_1, \exists\mathrm{S}, \exists\mathrm{S}^-\}, 1) \\ \mathrm{T}_2 & \mapsto & (\emptyset, 2) & \mathrm{S} & \mapsto & (\emptyset, 2) \\ \mathrm{R}_1 & \mapsto & (\emptyset, 2) & \mathrm{S}^- & \mapsto & (\{\mathrm{A}_1, \exists\mathrm{R}_1, \exists\mathrm{T}_1\}, 1) \end{array}$$

To construct a model from a strategy σ, the basic idea is to merge elements $w\mathrm{R}$ with an element of type $\sigma(\mathrm{R})$, with the latter selected according to a *choice of well-typed elements*:

Definition 3. *A mapping $\mathsf{ch} : \mathsf{gen}_{\mathcal{K}} \to \mathsf{Ind}(\mathcal{A}) \uplus \{\bot_i \mid i = 1, \ldots, |\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}|\}$, is a* choice of well-typed elements for σ over $\mathcal{K}$ *if it satisfies the following conditions:*

1. $\forall \mathrm{R} \in \mathsf{gen}_{\mathcal{K}}, \exists i$ such that $\sigma(\mathrm{R}) = (\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R})), i)$

2. $\forall \mathrm{R}_1, \mathrm{R}_2 \in \mathsf{gen}_{\mathcal{K}},\ \mathsf{ch}(\mathrm{R}_1) = \mathsf{ch}(\mathrm{R}_2) \Leftrightarrow \sigma(\mathrm{R}_1) = \sigma(\mathrm{R}_2)$.

Example 6. *The function ch_e, defined as below, is a choice of well-typed elements for σ_e over $\mathcal{K}_e$:*

$$\begin{array}{lllllllll} \mathrm{T}_1 & \mapsto & \bot_1 & \mathrm{T}_2 & \mapsto & \bot_2 & \mathrm{R}_1 & \mapsto & \bot_2 \\ \mathrm{R}_2 & \mapsto & \mathsf{b}_1 & \mathrm{S} & \mapsto & \bot_2 & \mathrm{S}^- & \mapsto & \mathsf{a}_1 \end{array}$$

It turns out however that when $\mathrm{R} = \mathrm{S}$ or $\mathrm{R} = \mathrm{S}^-$, it is useful to depart from this guideline in order to reduce the number of query matches, as this stand-alone example illustrates:

Example 7. *Consider $\mathcal{T} = \{\mathrm{A} \sqsubseteq \exists\mathrm{S}, \mathrm{B} \sqsubseteq \exists\mathrm{S}^-\}$ and $\mathcal{A} = \{\mathrm{A}(\mathsf{a}_1), \mathrm{A}(\mathsf{a}_2), \mathrm{B}(\mathsf{b}_1), \mathrm{B}(\mathsf{b}_2)\}$. If we merge $\mathsf{a}_1\mathrm{S}$ with $\mathsf{a}_2\mathrm{S}$, and $\mathsf{b}_1\mathrm{S}^-$ with $\mathsf{b}_2\mathrm{S}^-$, then there will be at least three matches of q_{S}, no matter which further merges are performed. However, by 'pairing' a_1 with b_1 and a_2 with b_2, we can obtain a model with only two matches: $(\mathsf{a}_1, \mathsf{b}_1), (\mathsf{a}_2, \mathsf{b}_2)$.*

The next three definitions serve to identify the *critical elements* for which such a pairing operation is useful.

Definition 4. *We set $\mathcal{D}^+_{\mathcal{K}} = \{\mathsf{a} \mid \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \wedge \mathsf{a}\mathrm{S} \in \Delta^{\mathcal{C}_{\mathcal{K}}}\}$ and $\mathcal{D}^-_{\mathcal{K}} = \{\mathsf{a} \mid \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \wedge \mathsf{a}\mathrm{S}^- \in \Delta^{\mathcal{C}_{\mathcal{K}}}\}$.*

Definition 5. *Given a strategy σ, we set $\mathcal{D}^+_{\sigma} = \{\mathrm{R} \mid \mathrm{R} \in \mathsf{dom}(\sigma) \setminus \{\mathrm{S}, \mathrm{S}^-\} \wedge \mathcal{T} \models \exists\mathrm{R}^- \sqsubseteq \exists\mathrm{S} \wedge \exists\mathrm{S} \notin \mathsf{t}$ if $\sigma(\mathrm{R}) = (\mathsf{t}, k)\}$ and $\mathcal{D}^-_{\sigma} = \{\mathrm{R} \mid \mathrm{R} \in \mathsf{dom}(\sigma) \setminus \{\mathrm{S}, \mathrm{S}^-\} \wedge \mathcal{T} \models \exists\mathrm{R}^- \sqsubseteq \exists\mathrm{S}^- \wedge \exists\mathrm{S}^- \notin \mathsf{t}$ if $\sigma(\mathrm{R}) = (\mathsf{t}, k)\}$.*

Definition 6. *Let ch be a choice of well-typed elements for σ. We set $\mathsf{crit}^+ = \mathcal{D}^+_{\mathcal{K}} \cup \mathsf{ch}(\mathcal{D}^+_{\sigma})$ and $\mathsf{crit}^- = \mathcal{D}^-_{\mathcal{K}} \cup \mathsf{ch}(\mathcal{D}^-_{\sigma})$ and use* critical elements *to refer to the elements of these sets.*

Example 8. *For σ_e and ch_e as defined in Examples 5 and 6, we have $\mathsf{crit}^+ = \{\mathsf{a}_2, \mathsf{b}_1, \bot_1, \bot_2\}$ and $\mathsf{crit}^- = \{\mathsf{a}_2, \bot_2\}$.*

Intuitively, a pairing matches critical elements from crit^+ (which require an outgoing S) with those from crit^- (which require an incoming S).

Definition 7. *A* pairing for ch and σ *consists of two partial functions $\mathsf{p}^+ : \mathsf{crit}^+ \to \mathsf{crit}^-$ and $\mathsf{p}^- : \mathsf{crit}^- \to \mathsf{crit}^+$ such that one of the functions is total and injective, and the other is its partial inverse.*

Example 9. *A pairing for ch_e and σ_e is given by $\mathsf{p}^+_e = \{\mathsf{a}_2 \mapsto \mathsf{a}_2, \mathsf{b}_1 \mapsto \bot_2\}$ and $\mathsf{p}^-_e = \{\mathsf{a}_2 \mapsto \mathsf{a}_2, \bot_2 \mapsto \mathsf{b}_1\}$.*

We are now ready to define the interpretation of a strategy.

Definition 8. *Consider a strategy σ, choice of well-typed elements ch, and pairing $(\mathsf{p}^+, \mathsf{p}^-)$ for ch. For every $\mathrm{R} \in \mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}$, pick a function s_{R} that maps every individual in*

$\{\mathsf{a} \mid \mathcal{K} \models \mathrm{R}(\mathsf{a},\mathsf{b})$ *for some* $\mathsf{b} \in \mathsf{N_I}\}$ *to an individual* $\mathsf{s}_\mathrm{R}(\mathsf{a})$ *such that* $\mathcal{K} \models \mathrm{R}(\mathsf{a}, \mathsf{s}_\mathrm{R}(\mathsf{a}))$. *Define function* χ *as follows:*

$$\begin{array}{rcl} \Delta^{\mathcal{C}_\mathcal{K}} & \to & \mathsf{Ind}(\mathcal{A}) \cup \{\bot_i \mid i = 1, \dots, |\mathsf{sig}(\mathcal{T})^\pm_\mathsf{R}|\} \\ \mathsf{a} & \mapsto & \mathsf{a} \\ w\mathrm{S} & \mapsto & \begin{cases} \mathsf{s}_\mathrm{S}(\chi(w)) & \text{if } \mathsf{s}_\mathrm{S}(\chi(w)) \text{ is defined} \\ \mathsf{p}^+(\chi(w)) & \text{else if } \mathsf{p}^+(\chi(w)) \text{ is defined} \\ \mathsf{ch}(\mathrm{S}) & \text{otherwise} \end{cases} \\ w\mathrm{S}^- & \mapsto & \begin{cases} \mathsf{s}_{\mathrm{S}^-}(\chi(w)) & \text{if } \mathsf{s}_{\mathrm{S}^-}(\chi(w)) \text{ is defined} \\ \mathsf{p}^-(\chi(w)) & \text{else if } \mathsf{p}^-(\chi(w)) \text{ is defined} \\ \mathsf{ch}(\mathrm{S}^-) & \text{otherwise} \end{cases} \\ w\mathrm{R} & \mapsto & \mathsf{ch}(\mathrm{R}) \end{array}$$

The interpretation of σ *(according to* $\mathsf{ch}, (\mathsf{p}^+, \mathsf{p}^-)$ *and the* s_R*) has domain* $\chi(\Delta^{\mathcal{C}_\mathcal{K}})$ *and interpretation function* $\chi \circ \cdot^{\mathcal{C}_\mathcal{K}}$.

Example 10. *With choice* ch_e *and pairing* $(\mathsf{p}^+_e, \mathsf{p}^-_e)$*, we get* $\chi(\mathsf{b}_2\mathrm{R}_2) = \mathsf{ch}(\mathrm{R}_2) = \mathsf{b}_1$, $\chi(\mathsf{b}_2\mathrm{R}_2\mathrm{S}) = \mathsf{p}^+_e(\mathsf{b}_1) = \bot_2$*, and* $\chi(\mathsf{b}_2\mathrm{R}_2\mathrm{S}^-) = \mathsf{s}_{\mathrm{S}^-}(\mathsf{b}_1) = \mathsf{b}_2$ *(observe that on our example, the function* $\mathsf{s}_{\mathrm{S}^-}$ *is uniquely defined, and the same is true for the other roles). Figure 1c displays the interpretation of* σ_e.

Observe that the interpretation of a strategy σ depends not only on σ but also on the functions $\mathsf{ch}, \mathsf{p}^+, \mathsf{p}^-, \mathsf{s}_\mathrm{R}$. Importantly, however, the key property of such interpretations (stated in Lemma 1 later in this section) holds for *any* particular choice of these functions.

It remains to prove that a model minimizing the number of matches can be found among the interpretations of strategies. The first step is to to extract a strategy from a model.

Definition 9. *Let* $\mathcal{I}$ *be a model of* $\mathcal{K}$, $f : \mathcal{C}_\mathcal{K} \to \mathcal{I}$ *be a homomorphism, and* repr *be a function mapping each role* $\mathrm{R} \in \mathsf{gen}_\mathcal{K}$ *to an element with shape* $w\mathrm{R}$ *from* $\Delta^{\mathcal{C}_\mathcal{K}}$. *Then* $\mathcal{P} = \{P_1, \dots, P_k\}$*, defined by*

$$\{P_1, \dots, P_k\} = \{(f \circ \mathsf{repr})^{-1}(w) \mid w \in \Delta^\mathcal{I}\} \setminus \{\emptyset\}$$

is a partition of $\mathsf{gen}_\mathcal{K}$. *The strategy extracted from* $\mathcal{I}$ *(for* f *and* repr*) is defined as:*

$$\begin{array}{rcl} \mathsf{gen}_\mathcal{K} & \to & \Theta_\mathcal{T} \times \{1, \dots, |\mathsf{sig}(\mathcal{T})^\pm_\mathsf{R}|\} \\ \mathrm{R} & \mapsto & ((\theta_\mathcal{K} \circ f \circ \mathsf{repr})(\mathrm{R}), i) \text{ with } \mathrm{R} \in P_i \end{array}$$

Example 11. *In our running example, there is a unique homomorphism* f_e *from* $\mathcal{C}_{\mathcal{K}_e}$ *to the model displayed in Figure 1b. Let* repr_e *be:*

$$\begin{array}{lllllllll} \mathrm{T}_1 & \mapsto & \mathsf{a}_1\mathrm{T}_1 & \mathrm{R}_2 & \mapsto & \mathsf{b}_2\mathrm{R}_2 & \mathrm{T}_2 & \mapsto & \mathsf{a}_2\mathrm{T}_2 \\ \mathrm{S} & \mapsto & \mathsf{b}_1\mathrm{SSS} & \mathrm{R}_1 & \mapsto & \mathsf{b}_1\mathrm{R}_1 & \mathrm{S}^- & \mapsto & \mathsf{a}_2\mathrm{S}^- \end{array}$$

The strategy extracted from this model (for f_e *and* repr_e*) is the strategy provided in Example 5.*

By applying the next lemma to a model $\mathcal{I}$ having the fewest possible number of matches, we obtain the desired conclusion: there is a model minimizing the number of matches among the models obtained by interpreting a strategy.

Lemma 1. *Let* $\mathcal{I}$ *be a model of* $\mathcal{K}$*, and* $\mathcal{J}$ *an interpretation of a strategy extracted from* $\mathcal{I}$. $\mathcal{J}$ *is a model of* $\mathcal{K}$ *and* $q_\mathrm{S}^\mathcal{J} \leq q_\mathrm{S}^\mathcal{I}$.

We now sketch how to construct a family of TC^0 circuits (one for each size of ABox) to decide $\mathrm{OMQA}(q_\mathrm{S}, \mathcal{T})$. Each such circuit first computes the set $\mathsf{gen}_\mathcal{K}$ and the type of each ABox individual. Next, for each function $\varrho : \mathsf{gen}_\mathcal{K} \to \Theta_\mathcal{T} \times \{1, \dots, |\mathsf{sig}(\mathcal{T})^\pm_\mathsf{R}|\}$ satisfying Conditions 1 and 2 of Definition 2, the circuit decides whether ϱ is a strategy for $\mathcal{K}$ (i.e. Condition 3 holds), and if so, computes the number of matches of q_S in interpretations induced by ϱ. Importantly, this can be done without actually building interpretations: in the appendix we give an explicit formula for this number and show it can be computed with a TC^0 circuit. Moreover, the number of strategies depends only on $|\mathcal{T}|$, so is constant w.r.t. data complexity. Finally, the circuit computes the minimum value across strategies and compares it with the input number.

4 Complexity Classification for DL-Lite$^\mathcal{H}_{pos}$

In this section, we consider DL-Lite$^\mathcal{H}_{pos}$ TBoxes. We show that coNP-hard OMQs exist and prove a complexity trichotomy which precisely delineates the tractability boundary.

We begin by exhibiting a coNP-complete[1] situation.

Example 12. $\mathrm{OMQA}(q_\mathrm{S}, \{\mathrm{B} \sqsubseteq \exists\mathrm{R}_1, \mathrm{R}_1 \sqsubseteq \mathrm{S}, \exists\mathrm{R}_1^- \sqsubseteq \exists\mathrm{R}_2, \mathrm{R}_2 \sqsubseteq \mathrm{S}\})$ *is* coNP*-complete. We consider the* NP*-complete* SET COVER *problem: given a set* $\mathcal{U}$*, set of subsets* $\mathcal{S} \subseteq 2^\mathcal{U}$ *whose union is* $\mathcal{U}$*, and number* k*, decide whether there exists a* k*-cover, i.e. a subset* $\mathcal{C}$ *of* $\mathcal{S}$ *with* $|\mathcal{C}| \leq k$ *whose union is* $\mathcal{U}$. *We prove that there exists a* k*-cover iff* $[\Sigma_{s \in \mathcal{S}}|s| + k + 1, +\infty]$ *is* not *a certain answer on the following ABox:* $\{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{U}\} \cup \{\mathrm{S}(\mathsf{u}, \mathsf{s}) \mid u \in s, s \in \mathcal{S}\}$. *Intuitively, from a* k*-cover* $\mathcal{C}$*, we obtain a countermodel in which role* R_1 *contains pairs* (u, s) *such that* $u \in s$ *and* $s \in \mathcal{C}$*, and there is one outgoing* R_2 *role from each* $s \in \mathcal{C}$.

The following definition abstracts the preceding example.

Definition 10. *A TBox* $\mathcal{T}$ *admits a* propagation of role W *by a concept* $\mathrm{B} \in \mathsf{sig}(\mathcal{T})^\pm_\mathsf{C}$ *and roles* $\mathrm{R}_1, \mathrm{R}_2$ *if* $\mathcal{T}$ *entails* $\{\mathrm{B} \sqsubseteq \exists\mathrm{R}_1, \mathrm{R}_1 \sqsubseteq \mathrm{W}, \exists\mathrm{R}_1^- \sqsubseteq \exists\mathrm{R}_2, \mathrm{R}_2 \sqsubseteq \mathrm{W}\}$.

A propagation of S (or S^-) is not sufficient to ensure coNP-hardness: the reduction sketched in Example 12 will fail in the presence of 'interferences', which can be of three types.

Definition 11. *A role* U interferes with the propagation of W by $\mathrm{B}, \mathrm{R}_1, \mathrm{R}_2$ *if it satisfies one of the following conditions:*

1. $\mathcal{T} \models \{\mathrm{B} \sqsubseteq \exists\mathrm{U}, \mathrm{U} \sqsubseteq \mathrm{W}, \mathrm{U} \sqsubseteq \mathrm{W}^-\}$*;*
2. $\mathcal{T} \models \{\exists\mathrm{W}^- \sqsubseteq \exists\mathrm{U}, \mathrm{U} \sqsubseteq \mathrm{W}\}$ *and either* $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{W}^-$ *or* $\mathcal{T} \not\models \mathrm{R}_2 \sqsubseteq \mathrm{W}^-$*;*
3. *if* $\mathrm{B} = \exists\mathrm{T}$ *and* $\mathrm{T} \sqsubseteq \mathrm{W}$*, then* $\mathcal{T} \models \{\exists\mathrm{T}^- \sqsubseteq \exists\mathrm{U}, \mathrm{U} \sqsubseteq \mathrm{W}\}$ *and either* $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{W}^-$ *or* $\mathcal{T} \not\models \mathrm{R}_2 \sqsubseteq \mathrm{W}^-$.

Remarkably, the existence of a propagation without any interfering role (which we call a *non-trivial propagation*) ensures coNP-hardness, while its absence ensures that $\mathrm{OMQA}(q_\mathrm{S}, \mathcal{T})$ is in P. We further distinguish two tractable cases, depending on the existence of a *non-trivial pairing*.

Definition 12. *A TBox* $\mathcal{T}$ *admits a* non-trivial pairing of S *if there exist* $\mathrm{B} \in \mathsf{sig}(\mathcal{T})^\pm_\mathsf{C}$ *and* $\mathrm{R} \in \mathsf{sig}(\mathcal{T})^\pm_\mathsf{R}$ *such that*

$$\mathcal{T} \models \mathrm{B} \sqsubseteq \exists\mathrm{R} \quad \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S} \quad \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^- \quad \mathcal{T} \not\models \mathrm{S} \sqsubseteq \mathrm{S}^-$$

and if $\mathrm{B} = \exists\mathrm{T}$*, then either* $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}$ *or* $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}^-$.

[1]A P upper bound for atomic counting queries in DL-Lite$^\mathcal{H}_{pos}$ erroneously appears in Table 1 of [Calvanese *et al.*, 2020a], but was corrected in a later arXiv version [Calvanese *et al.*, 2020b].

To formulate our trichotomy result, we recall that a *matching* in a graph $(\mathcal{V}, \mathcal{E})$ is a set of edges that are pairwise vertex-disjoint. The PERFECT MATCHING problem (abbreviated to PM) asks whether there exists a matching such that every vertex is incident to one of its edges. Despite being the focus of intensive research, its exact complexity remains open: in P [Edmonds, 1965] and NL-hard [Chandra *et al.*, 1984].

Theorem 2. *Let $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{pos}$ TBox.* $\mathrm{OMQA}(q_{\mathrm{S}}, \mathcal{T})$ *is* coNP*-complete if $\mathcal{T}$ admits a non-trivial propagation of either* S *or* S^-*, is* L*-equivalent to the complement of* PM *if it does not admit such a non-trivial propagation but admits a non-trivial pairing of* S*, and is in* TC^0 *otherwise.*

Proof sketch. The coNP-hardness proof generalizes the reduction sketched in Example 12. If there is a non-trivial pairing (but no non-trivial propagation), we show that, up to trivial cases solvable in TC^0, the existence of a model with few matches is equivalent to the existence of a large matching between critical individuals. This yields L-equivalence with the MAXIMUM MATCHING decision problem, which is L-equivalent to the better-known PM problem [Rabin and Vazirani, 1989]. TC^0 membership is proven by case analysis, where we exhibit for each case a model with an optimal (and easily computable) number of matches. □

5 First Look at DL-Lite$^{\mathcal{H}}_{core}$

We now turn to DL-Lite$^{\mathcal{H}}_{core}$ and exhibit new situations that are not captured by the preceding complexity classification.

First, we observe that negative concept and role inclusions introduce two new sources of coNP-hardness.

Theorem 3. *For* $\mathcal{T} = \{\mathrm{B} \sqsubseteq \exists\mathrm{U},\ \mathrm{U} \sqsubseteq \mathrm{S},\ \mathrm{C} \sqsubseteq \exists\mathrm{V},\ \mathrm{V} \sqsubseteq \mathrm{S},\ \exists\mathrm{U}^- \sqsubseteq \neg\exists\mathrm{V}^-\ \}$*,* $\mathrm{OMQA}(q_{\mathrm{S}}, \mathcal{T})$ *is* coNP*-complete.*

Proof sketch. Let $(\mathcal{U}, \mathcal{S}, k)$ be an instance of SET COVER, and consider the ABox $\mathcal{A} = \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{U}\} \cup \{\mathrm{S}(\mathsf{u}, \mathsf{s}^*) \mid u \in s, s \in \mathcal{S}\} \cup \{\mathrm{C}(\mathsf{s}) \mid s \in \mathcal{S}\} \cup \{\mathrm{S}(\mathsf{s}, \mathsf{s}^*) \mid s \in \mathcal{S}\}$. It can be shown that *no* k-cover exists iff every model of $(\mathcal{T}, \mathcal{A})$ has at least $|\mathcal{S}| + \sum_{s\in\mathcal{S}} |s| + k + 1$ matches. □

Theorem 4. *For* $\mathcal{T} = \{\ \mathrm{B} \sqsubseteq \exists\mathrm{U},\ \mathrm{U} \sqsubseteq \mathrm{S},\ \exists\mathrm{U}^- \sqsubseteq \exists\mathrm{V},\ \mathrm{V} \sqsubseteq \mathrm{S}^-,\ \mathrm{V} \sqsubseteq \neg\mathrm{W}\ \}$*,* $\mathrm{OMQA}(q_{\mathrm{S}}, \mathcal{T})$ *is* coNP*-complete.*

Perhaps more surprising, we show that there exist coNP-hard OMQs based upon concept cardinality queries.

Theorem 5. *For* $\mathcal{T} = \{\mathrm{A} \sqsubseteq \exists\mathrm{U},\ \exists\mathrm{U}^- \sqsubseteq \mathrm{C},\ \mathrm{U} \sqsubseteq \neg\mathrm{U}',\ \mathrm{B} \sqsubseteq \exists\mathrm{V},\ \exists\mathrm{V}^- \sqsubseteq \mathrm{C},\ \mathrm{V} \sqsubseteq \neg\mathrm{V}',\ \exists\mathrm{U}^- \sqsubseteq \neg\exists\mathrm{V}^-\}$*,* $\mathrm{OMQA}(q_{\mathrm{C}}, \mathcal{T})$ *is* coNP*-complete.*

Proof sketch. Hardness is shown by reducing the tautology problem. Three individuals are introduced per propositional variable (one for the variable itself with concept A, two for its possible truth values), as well as one individual per clause (with concept B). Each variable should have a truth value given by U (whose possible values in the ABox are restricted through the use of U′), and each clause should have a falsified literal given by V (whose possibles values in the ABox are restricted, according to the input formula, with V′). The input formula is a tautology iff every model introduces a new element marked C (as a witness for either ∃U or ∃V). □

Moreover, we further show that L-complete OMQs exist. The next result employs a role cardinality query, but a similar result can be obtained using a concept cardinality query.

Theorem 6. *For* $\mathcal{T} = \{\ \mathrm{B} \sqsubseteq \exists\mathrm{R},\ \mathrm{R} \sqsubseteq \mathrm{S},\ \mathrm{R} \sqsubseteq \neg\mathrm{R}^-\ \}$*,* $\mathrm{OMQA}(q_{\mathrm{S}}, \mathcal{T})$ *is* L*-complete.*

Proof. Hardness is by reduction from the L-complete problem UNDIRECTED FOREST ACCESSIBILITY (UFA) [Cook and McKenzie, 1987], which takes as input an undirected acyclic graph $(\mathcal{V}, \mathcal{E})$ with two connected components, vertices $s, t \in \mathcal{V}$, and asks if t is reachable from s. We set $\mathcal{A} = \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{V}\} \cup \{\mathrm{S}(\mathsf{u}, \mathsf{v}) \mid \{u, v\} \in \mathcal{E}\} \cup \{\mathrm{S}(\mathsf{s}, \mathsf{v}^*), \mathrm{S}(\mathsf{t}, \mathsf{v}^*)\} \cup \{\mathrm{R}(\mathsf{s}, \mathsf{v}^*), \mathrm{R}(\mathsf{t}, \mathsf{v}^*)\}$ and observe that $((\mathcal{V}, \mathcal{E}), s, t) \in$ UFA iff $[2|\mathcal{E}| + 3, +\infty]$ is a certain answer. Indeed, there are $2|\mathcal{E}| + 2$ matches in the ABox, and a further match arises if we add R-atoms to satisfy $\mathrm{B} \sqsubseteq \exists\mathrm{R}$ in a connected component that contains neither s nor t (such a match can be avoided if it contains s or t). For the upper bound, we characterize the minimum number of matches based upon the graph structure of the ABox and show it can be computed in L, by using an oracle for undirected reachability. □

Our results imply that, under standard complexity-theoretic assumptions, at least four different complexities are possible for cardinality queries coupled with DL-Lite$^{\mathcal{H}}_{core}$ ontologies.

6 Conclusion

In this paper, we investigated the complexity of answering cardinality queries in the presence of DL-Lite ontologies. Our study provides several novel insights into the challenge of adopting counting queries in OMQA. On the one hand, we identified new sources of coNP-hardness, showing that even single-atom counting queries can be difficult to handle (which closes some questions about restricted forms of counting queries left open in [Calvanese *et al.*, 2020a]). On the other hand, we exhibited several settings in which cardinality queries can be answered with (sub-)polynomial data complexity; in particular, the problem is in TC^0 when the ontology is formulated in DL-Lite$_{core}$. Interestingly, our tractability results do not rely on the canonical model yielding the minimum number of matches, but instead involve a sophisticated analysis of how to best merge witnesses for existential axioms. Differently from [Kostylev and Reutter, 2015; Calvanese *et al.*, 2020a; Bienvenu *et al.*, 2020], we conducted our complexity analysis on the level of ontology-mediated queries, and notably obtained a full classification of the complexity of OMQs based upon DL-Lite$^{\mathcal{H}}_{pos}$ ontologies.

We find it promising that very low data complexity can be obtained even for settings in which non-trivial optimization is required, and we plan to explore how to extend and adapt our techniques to identify further tractability results for counting queries. Another important topic for future work is to transform our TC^0 procedures into more practical algorithms that are suitable for implementation on top of database systems.

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A Proof of Theorem 1.

Theorem 1 will be proven in stages. First, we will give the proof of TC^0 membership for the case where q is a role cardinality query and $\mathcal{T}$ a DL-Lite$_{core}$ TBox (Appendix A.1), and show how to adapt it to the case where q is a concept cardinality query and $\mathcal{T}$ is a DL-Lite$^{\mathcal{H}}_{core}$ TBox without negative role inclusions (Appendix A.2). The TC^0 lower bound is given in Appendix A.3. To prepare for the membership proofs in A.1 and A.2, we give some additional technical preliminaries and insights that are relevant for both proofs.

We first slightly modify the notion of strategy as defined in Definition 2, by splitting it into a more ABox-independent notion of strategy (Definition 13), coupled with a notion of *legal strategy for a particular* $\mathcal{K}$ (Definition 14). This enables us to speak about the underlying set of abstract strategies we will explore (which depends only on the TBox), among which only some strategies are relevant, that is *legal*, for a particular ABox.

Definition 13 (Strategy)**.** *A strategy σ for the TBox $\mathcal{T}$ is a partial function from $\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}$ to $\Theta_{\mathcal{T}} \times \{1, \dots \left|\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}\right|\}$, satisfying the following two conditions:*

1. *Type compatibility:* $\forall \mathrm{R} \in \mathsf{dom}(\sigma) : \ \sigma(\mathrm{R}) = (\mathsf{t}, i) \wedge \mathrm{B} \in \mathsf{t} \Rightarrow \mathcal{T} \not\models \exists \mathrm{R}^- \sqsubseteq \neg \mathrm{B}$.
2. *Pseudo-injectivity:* $\forall \mathrm{R}_1, \mathrm{R}_2 \in \mathsf{dom}(\sigma), \ \sigma(\mathrm{R}_1) = \sigma(\mathrm{R}_2) \ \Rightarrow \ \mathcal{T} \not\models \exists \mathrm{R}_1^- \sqsubseteq \neg \exists \mathrm{R}_2^-$.

We denote by $\mathsf{dom}(\sigma)$ the subset of $\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}$ on which σ is defined.

Definition 14 (Legal strategy)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. A strategy σ for $\mathcal{T}$ is* legal *for $\mathcal{K}$ if it satisfies the following two conditions:*

1. *Coverage:* $\mathsf{gen}_{\mathcal{K}} = \mathsf{dom}(\sigma)$
2. *Availability:* $\forall \mathsf{t} \in \Theta_{\mathcal{T}} \setminus \{\emptyset\}, \ |\{i \mid \exists \mathrm{R} \in \mathsf{gen}_{\mathcal{K}}, \sigma(\mathrm{R}) = (\mathsf{t}, i)\}| \leq \left| \left\{ \mathsf{a} \;\middle|\; \begin{array}{l} \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \\ \theta_{\mathcal{K}}(\mathsf{a}) = \mathsf{t} \end{array} \right\} \right|$.

Note that legal strategies from the preceding definition correspond precisely to the strategies of Definition 2.

We further introduce the following useful lemma, stating that a choice of well-typed elements for a strategy σ extracted from a model provides, as one would expect, elements with the same type as those used in the first place to extract the strategy σ.

Lemma 2 (Properties of extracted strategies)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let $\mathsf{repr}_{\mathcal{K}}$ be a function mapping each role $\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}$ to an element with shape $w\mathrm{R}$ from $\Delta^{\mathcal{C}_{\mathcal{K}}}$. Let $\mathcal{I}$ be a model of $\mathcal{K}$, and let $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$ be a homomorphism. Let ch be a choice of well-typed elements for $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}$ over $\mathcal{A}$. The strategy $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}$ extracted from $\mathcal{I}$ (for f and $\mathsf{repr}_{\mathcal{K}}$) preserves both:*

1. $\forall \mathrm{R} \in \mathsf{gen}_{\mathcal{K}}, \ \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R})) = \theta_{\mathcal{K}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})))$
2. $\forall \mathrm{R}, \mathrm{T} \in \mathsf{gen}_{\mathcal{K}}, \ \mathsf{ch}(\mathrm{R}) = \mathsf{ch}(\mathrm{T}) \ \Leftrightarrow \ f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{T}))$

Proof. 1. Let $\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}$. By definition of $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}$, there exists $i \in \{1, \dots |\mathsf{sig}(\mathcal{T})|\}$ such that $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{R}) = ((\theta_{\mathcal{K}} \circ f \circ \mathsf{repr}_{\mathcal{K}})(\mathrm{R}), i)$ with $\mathrm{R} \in P_i$. From Condition 1 of Definition 3, we get $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{R}) = (\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R})), i)$, which gives the desired equality of types.

2. From Condition 2 of Definition 3, $\mathsf{ch}(\mathrm{R}) = \mathsf{ch}(\mathrm{T})$ iff $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{R}) = \sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{T})$. If $f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{T}))$, then by the definition of the extracted strategy, we have $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{R}) = \sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{T})$, so we are done. Conversely, if $\mathsf{ch}(\mathrm{R}) = \mathsf{ch}(\mathrm{T})$, then $\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{R}) = \sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}(\mathrm{T})$. This implies in particular that R and T belong to the same P_i, hence $f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{T}))$. □

We extend the functions s_{R} from Definition 8 as follows and introduce terminology to speak about such functions.

Definition 15 (Certain successor preference, successor preference)**.** *The family of functions $(\mathsf{s}_{\mathrm{R}})_{\mathrm{R}}$ from Definition 8 is called a* certain successor preference *and will also be denoted $(\mathsf{succ}^{\mathcal{K}}_{\mathrm{R}})_{\mathrm{R}}$. For a given interpretation $\mathcal{I}$, a family of functions $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{R}})_{\mathrm{R}}$ mapping an element $d \in (\exists \mathrm{R})^{\mathcal{I}}$ to an element $e \in \Delta^{\mathcal{I}}$ such that $(d, e) \in \mathrm{R}^{\mathcal{I}}$ is* a successor preference in $\mathcal{I}$.

Finally, before diving into the details of the proof, we give an idea for why this OMQA setting requires such an involved construction. Recall that our construction involves exploring a set of strategies, whose size is constant w.r.t data complexity. In the end, we prove that for each KB, there exists some best strategies among this set, which provide the minimum amount of matches. One might ask, starting from our input KB, why don't we simply exhibit one of these best strategies, instead of exploring all possible ones? First, this is due to the set of legal strategies varying according to the ABoxes: deciding if a given strategy is legal already requires computing statistics on the input ABox (see both conditions from Definition 14). Furthermore, even with a fixed ABox, deciding if a legal strategy is "the best" is in general a coNP-complete problem w.r.t the TBox. This is formalized by the following two results which concern single-individual ABoxes.

Theorem 7. *Given a DL-Lite$_{core}$ TBox $\mathcal{T}$ and an integer m, deciding if $[m, +\infty]$ is a certain answer to $q := \exists z_1, z_1.\mathrm{S}(z_1, z_2)$ over $\mathcal{K} := (\mathcal{T}, \{\mathrm{A}(\mathsf{a})\})$ is coNP-complete.*

Proof. Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be an undirected graph. Consider the TBox given by:

$$\mathcal{T}_\mathcal{G} \;:=\; \bigcup_{v\in\mathcal{V}}\{\mathrm{A} \sqsubseteq \exists \mathrm{V}, \exists \mathrm{V}^- \sqsubseteq \exists \mathrm{S}\} \;\cup \bigcup_{\{v_1,v_2\}\in\mathcal{E}} \{\exists \mathrm{V}_1^- \sqsubseteq \neg\exists \mathrm{V}_2^-\}.$$

It is easily verified that $\mathcal{G} \in \mathsf{3COL} \iff [4, +\infty] \notin q^{\mathcal{K}_\mathcal{G}}$ for $\mathcal{K}_\mathcal{G} := (\mathcal{T}_\mathcal{G}, \{\mathrm{A}(\mathsf{a})\})$, where 3COL is the well-known NP-complete problem of determining if a graph has a 3-colouring. □

An analogous statement holds for concept cardinality queries.

Theorem 8. *Given a DL-Lite$_{core}$ TBox $\mathcal{T}$ and an integer m, deciding if $[m, +\infty]$ is a certain answer for $q := \exists z.\mathrm{C}(z)$ over $\mathcal{K} := (\mathcal{T}, \{\mathrm{A}(\mathsf{a})\})$ is* coNP*-complete.*

Proof. Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be an undirected graph. Consider the TBox given by:

$$\mathcal{T}_\mathcal{G} \;:=\; \bigcup_{v\in\mathcal{V}}\{\mathrm{A} \sqsubseteq \exists \mathrm{V}, \exists \mathrm{V}^- \sqsubseteq \mathrm{C}\} \;\cup \bigcup_{\{v_1,v_2\}\in\mathcal{E}} \{\exists \mathrm{V}_1^- \sqsubseteq \neg\exists \mathrm{V}_2^-\}.$$

We have $\mathcal{G} \in \mathsf{3COL} \iff [4, +\infty] \notin q^{\mathcal{K}_\mathcal{G}}$ for $\mathcal{K}_\mathcal{G} := (\mathcal{T}_\mathcal{G}, \{\mathrm{A}(\mathsf{a})\})$. □

A.1 Proof of Theorem 1.i: Role Cardinality Queries and DL-Lite$_{core}$.

We prove TC^0 membership for OMQA based upon a DL-Lite$_{core}$ TBox $\mathcal{T}$ and a role cardinality query q_S.

Proof of Lemma 1.

We first prove the first point of Lemma 1, stating the interpretation of a strategy extracted from a model is also a model, in the following stronger form, which does not require the strategy to be extracted from a model in the first place.

Lemma 3. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ a satisfiable KB. Let $(\mathsf{succ}^\mathcal{K}_\mathrm{R})_\mathrm{R}$ be a certain successor preference. Let σ be a legal strategy for $\mathcal{K}$. Let* ch *be a choice of well-typed elements for σ over $\mathcal{K}$. Let $\mathsf{p} := (\mathsf{p}^+, \mathsf{p}^-)$ be a pairing for* ch *and σ. Then the interpretation $\mathcal{J}$ of σ (according to* ch, p, *and $(\mathsf{succ}^\mathcal{K}_\mathrm{R})_\mathrm{R}$) is a model.*

Proof. Assertions from the ABox and positive inclusions from $\mathcal{T}$ are satisfied since the interpretation $\mathcal{J}$ is built from $\mathcal{C}_\mathcal{K}$. Indeed, suppose that $\mathrm{B} \sqsubseteq \mathrm{C} \in \mathcal{T}$ and $d \in \mathrm{B}^\mathcal{J}$. Then from the definition of $\mathcal{J}$, there exists $w \in \Delta^{\mathcal{C}_\mathcal{K}}$ such that $w \in \mathrm{B}^{\mathcal{C}_\mathcal{K}}$ and $d = \chi(w)$. Since $\mathcal{C}_\mathcal{K}$ satisfies $\mathrm{B} \sqsubseteq \mathrm{C}$, we have $w \in \mathrm{C}^{\mathcal{C}_\mathcal{K}}$. If $\mathrm{C} \in \mathsf{N_C}$, this immediately gives $d \in \mathrm{C}^\mathcal{J}$. If $\mathrm{C} = \exists\mathrm{R}$, there exists w' such that $(w, w') \in \mathrm{R}^{\mathcal{C}_\mathcal{K}}$, and hence we will have $(\chi(w), \chi(w')) \in R^\mathcal{J}$, which yields $w \in \mathrm{C}^{\mathcal{C}_\mathcal{K}}$.

Consider now a negative axiom of the form $\mathrm{B} \sqsubseteq \neg\mathrm{C}$. By contradiction assume there is an element d such that $d \in \mathrm{B}^\mathcal{J} \cap \mathrm{C}^\mathcal{J}$. In what follows, χ is the function used in the definition of $\mathcal{J}$.

1. If $\mathcal{K} \models \mathrm{B}(d)$ and $\mathcal{K} \models \mathrm{C}(d)$, then this contradicts $\mathcal{K}$ being satisfiable.
2. If $\mathcal{K} \models \mathrm{B}(d)$ and $\mathcal{K} \not\models \mathrm{C}(d)$, then $d = \chi(w\mathrm{T})$ with $\mathrm{T} \in \mathsf{gen}_\mathcal{K}$ and $\mathcal{T} \models \exists\mathrm{T}^- \sqsubseteq \mathrm{C}$. Indeed $\mathcal{K} \not\models \mathrm{C}(d)$ ensures d is not the image through χ of some individual certainly satisfiying C. Nevertheless, since $d \in \mathrm{C}^\mathcal{J}$, it must be that d is the image through χ of some anonymous element, say $w\mathrm{T}$, such that $w\mathrm{T} \in \mathrm{C}^{\mathcal{C}_\mathcal{K}}$. By definition of $\mathrm{C}^{\mathcal{C}_\mathcal{K}}$, it yields $\mathcal{T} \models \exists\mathrm{T}^- \sqsubseteq \mathrm{C}$.
   (a) If $d = \mathsf{succ}^\mathcal{K}_\mathrm{T}(\chi(w))$ with $\mathrm{T} \in \{\mathrm{S}, \mathrm{S}^-\}$, then $\mathrm{T}(\chi(w), d) \in \mathcal{A}$, contradicting $\mathcal{K} \not\models \mathrm{C}(d)$.
   (b) If $d = \mathsf{p}^+(\chi(w))$ with $\mathrm{T} = \mathrm{S}$, then in particular $d \in \mathsf{crit}^-_\mathsf{ch}$.
      - If $d \in \mathcal{D}^-_\mathcal{K}$, then in particular $\mathcal{K} \models \exists\mathrm{S}^-(d)$, contradicting $\mathcal{K} \not\models \mathrm{C}(d)$.
      - If $d = \mathsf{ch}(\mathrm{T}_0)$ with $\mathrm{T}_0 \in \mathcal{D}^-_\sigma$, then in particular $\mathcal{T} \models \exists\mathrm{T}_0^- \sqsubseteq \exists\mathrm{S}^-$. Hence $\mathcal{T} \models \exists\mathrm{T}_0^- \sqsubseteq \neg\mathrm{B}$. Condition 1 in the definition of a choice of well-typed elements ensures $\sigma(\mathrm{T}_0) = (\theta_\mathcal{K}(d), i)$. Condition 1 in the definition of a strategy ensures $\mathrm{B} \notin \theta_\mathcal{K}(d)$, contradicting $\mathcal{K} \models \mathrm{B}(d)$.
   (c) If $d = \mathsf{p}^-(\chi(w))$ with $\mathrm{T} = \mathrm{S}^-$. Same argument as in Case 2.b, based on $d \in \mathsf{crit}^+_\mathsf{ch}$.
   (d) If $d = \mathsf{ch}(\mathrm{T})$. Condition 1 from the definition of choice of well-typed elements ensures $\sigma(\mathrm{T}) = (\theta_\mathcal{K}(d), i)$. Condition 1 from the definition of a strategy ensures $\mathrm{B} \notin \theta_\mathcal{K}(d)$, contradicting $\mathcal{K} \models \mathrm{B}(d)$.
3. If $\mathcal{K} \not\models \mathrm{B}(d)$ and $\mathcal{K} \models \mathrm{C}(d)$. Symmetric to Case 2.
4. If $\mathcal{K} \not\models \mathrm{B}(d)$ and $\mathcal{K} \not\models \mathrm{C}(d)$, then $d = \chi(w\mathrm{R}) = \chi(w'\mathrm{T})$ with $\mathrm{R}, \mathrm{T} \in \mathsf{gen}_\mathcal{K}$ such that $\mathcal{T} \models \exists\mathrm{R}^- \sqsubseteq \mathrm{B}$ and $\mathcal{T} \models \exists\mathrm{T}^- \sqsubseteq \mathrm{C}$, due to the same reason than in Case 2, applied here to both concepts B and C.
   (a) If $d = \mathsf{succ}^\mathcal{K}_\mathrm{R}(\chi(w))$ with $\mathrm{R} \in \{\mathrm{S}, \mathrm{S}^-\}$, then $\mathrm{R}(\chi(w), d) \in \mathcal{A}$, contradicting $\mathcal{K} \not\models \mathrm{B}(d)$.
   (b) If $d = \mathsf{p}^+(\chi(w))$ with $\mathrm{R} = \mathrm{S}$, then in particular $d \in \mathsf{crit}^-_\mathsf{ch}$.
      - If $d \in \mathcal{D}^-_\mathcal{K}$, then it contradicts $\mathcal{K} \not\models \mathrm{B}(d)$.
      - If $d = \mathsf{ch}(\mathrm{R}_0)$ with $\mathrm{R}_0 \in \mathcal{D}^-_\sigma$, then in particular $\mathcal{T} \models \exists\mathrm{R}_0^- \sqsubseteq \exists\mathrm{S}^-$.
      i. If $d = \mathsf{succ}^\mathcal{K}_\mathrm{T}(\chi(w'))$ with $\mathrm{T} \in \{\mathrm{S}, \mathrm{S}^-\}$, then it contradicts $\mathcal{K} \not\models \mathrm{C}(d)$.

ii. If $d = \mathsf{p}^+(\chi(w'))$ with $\mathrm{T} = \mathrm{S}$, then $\mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \neg\exists \mathrm{S}^-$. Contradiction.

iii. If $d = \mathsf{p}^-(\chi(w'))$ with $\mathrm{T} = \mathrm{S}^-$, in particular $d \in \mathsf{crit}^+_{\mathsf{ch}}$.

- If $d \in \mathcal{D}^+_{\mathcal{K}}$ then it contradicts $\mathcal{K} \not\models \mathrm{C}(d)$.
- If $d = \mathsf{ch}(\mathrm{T}_0)$ with $\mathrm{T}_0 \in \mathcal{D}^+_\sigma$, then in particular $\mathcal{T} \models \exists \mathrm{T}_0^- \sqsubseteq \exists \mathrm{S}$. Condition 2 in the definition of the choice of well-typed elements ensures: $\sigma(\mathrm{R}_0) = \sigma(\mathrm{T}_0)$. Condition 2 in the definition of a strategy ensures: $\mathcal{T} \not\models \exists \mathrm{R}_0^- \sqsubseteq \neg\exists \mathrm{T}_0^-$, contradicting $\mathcal{T} \models \mathrm{B} \sqsubseteq \neg\mathrm{C}$.

iv. If $d = \mathsf{ch}(\mathrm{T})$. Condition 2 in the definition of the choice of well-typed elements ensures: $\sigma(\mathrm{R}_0) = \sigma(\mathrm{T})$. Condition 2 in the definition of a strategy ensures: $\mathcal{T} \not\models \exists \mathrm{R}_0^- \sqsubseteq \neg\exists \mathrm{T}^-$, contradicting $\mathcal{T} \models \mathrm{B} \sqsubseteq \neg\mathrm{C}$.

(c) If $d = \mathsf{p}^-(\chi(w))$ with $\mathrm{R} = \mathrm{S}^-$. Analogous argument to Case 4.b.

(d) If $d = \mathsf{ch}(\mathrm{R})$.

i. If $d = \mathsf{succ}^{\mathcal{K}}_{\mathrm{T}}(\chi(w'))$ with $\mathrm{T} \in \{\mathrm{S}, \mathrm{S}^-\}$, then it contradicts $\mathcal{K} \not\models \mathrm{C}(d)$.

ii. If $d = \mathsf{p}^+(\chi(w'))$ with $\mathrm{T} = \mathrm{S}$. Symmetric to Case 4.b.iv.

iii. If $d = \mathsf{p}^-(\chi(w'))$ with $\mathrm{T} = \mathrm{S}^-$. Symmetric to Case 4.c.iv.

iv. If $d = \mathsf{ch}(\mathrm{T})$. Condition 2 in the definition of the choice of well-typed elements ensures: $\sigma(\mathrm{R}) = \sigma(\mathrm{T})$. Condition 2 in the definition of a strategy ensures: $\mathcal{T} \not\models \exists \mathrm{R}^- \sqsubseteq \neg\exists \mathrm{T}^-$, contradicting $\mathcal{T} \models \mathrm{B} \sqsubseteq \neg\mathrm{C}$. ☐

In order to prove the second point of Lemma 2, stating that the interpretation $\mathcal{J}$ of the strategy extracted from a model $\mathcal{I}$ has at most as many matches as the initial model $\mathcal{I}$, we need to understand which pairs appear in the role of interest S in an interpretation of our strategy. This is is the purpose of the following result.

Lemma 4 (Matches in a model interpreting a strategy)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let $(\mathsf{succ}^{\mathcal{K}}_{\mathrm{R}})_{\mathrm{R}}$ be a certain successor preference. Let σ be a legal strategy for $\mathcal{K}$. Let* ch *be a choice of well-typed elements for σ over $\mathcal{K}$. Let $\mathsf{p} := (\mathsf{p}^+, \mathsf{p}^-)$ be a pairing for* ch*. Denote by $\mathcal{J}$ the interpretation of σ (according to* ch, p, *and $(\mathsf{succ}^{\mathcal{K}}_{\mathrm{R}})_{\mathrm{R}}$)*

$$
\begin{aligned}
\mathrm{S}^{\mathcal{J}} = \; & \{(\mathsf{a},\mathsf{b}) \mid \mathcal{K} \models \mathrm{S}(\mathsf{a},\mathsf{b})\} && \textit{(Shape 1)}\\
\cup \; & \left\{(x,y) \;\middle|\; \begin{array}{l} (x,y) \in \mathsf{crit}^+_{\mathsf{ch}} \times \mathsf{crit}^-_{\mathsf{ch}} \\ \mathsf{p}^+(x) = y \end{array}\right\} && \textit{(Shape 2)}\\
\cup \; & \{(x, \mathsf{ch}(\mathrm{S})) \mid x \in \mathsf{crit}^+_{\mathsf{ch}} \setminus \mathsf{dom}(\mathsf{p}^+)\} && \textit{(Shape 3}^+\textit{)}\\
\cup \; & \{(\mathsf{ch}(\mathrm{S}^-), y) \mid y \in \mathsf{crit}^-_{\mathsf{ch}} \setminus \mathsf{dom}(\mathsf{p}^-)\} && \textit{(Shape 3}^-\textit{)}\\
\cup \; & \left\{(\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S})) \;\middle|\; \begin{array}{l} |\mathsf{crit}^+_{\mathsf{ch}}| > |\mathsf{crit}^-_{\mathsf{ch}}| \\ \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S} \\ \exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S})) \\ \mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma) \end{array}\right\} && \textit{(Shape 4}^+\textit{)}\\
\cup \; & \left\{(\mathsf{ch}(\mathrm{S}^-), \mathsf{ch}(\mathrm{S}^-)) \;\middle|\; \begin{array}{l} |\mathsf{crit}^-_{\mathsf{ch}}| > |\mathsf{crit}^+_{\mathsf{ch}}| \\ \mathcal{T} \models \exists \mathrm{S} \sqsubseteq \exists \mathrm{S}^- \\ \exists \mathrm{S}^- \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S}^-)) \\ \mathsf{ch}(\mathrm{S}^-) \notin \mathsf{ch}(\mathcal{D}^-_\sigma) \end{array}\right\} && \textit{(Shape 4}^-\textit{)}
\end{aligned}
$$

Notice there can be no overlap between two distinct shapes and that shapes with opposite superscripts cannot coexist.

Proof. The first inclusion ($\subseteq$) is rather straightforward. We focus on the other way around.

($\supseteq$) We consider each of the shapes in turn and show their elements belong to $\mathrm{S}^{\mathcal{J}}$.

1. Let (a,b) such that $\mathcal{K} \models \mathrm{S}(\mathsf{a},\mathsf{b})$.
   Therefore $(\mathsf{a},\mathsf{b}) \in \mathcal{C}_{\mathcal{K}}$. By definition: $\chi(\mathsf{a}) = \mathsf{a}$ and $\chi(\mathsf{b}) = \mathsf{b}$, hence $(\mathsf{a},\mathsf{b}) \in \mathrm{S}^{\mathcal{J}}$.
2. Let (x,y) such that $(x,y) \in \mathsf{crit}^+_{\mathsf{ch}} \times \mathsf{crit}^-_{\mathsf{ch}}$ and $\mathsf{p}^+(x) = y$.
   Distinguish two cases based on $x \in \mathsf{crit}^+_{\mathsf{ch}}$:
   - If $x \in \mathcal{D}^+_{\mathcal{K}}$. By definition, we must have $x \in \mathsf{Ind}(\mathcal{A})$, so $\chi(x) = x$. Moreover, $x\mathrm{S} \in \mathcal{C}_{\mathcal{K}}$, hence $\mathsf{succ}^{\mathcal{K}}_{\mathrm{S}}(x)$ is not defined. Together with $x \in \mathsf{dom}(\mathsf{p}^+)$, this gives $\chi(x\mathrm{S}) = \mathsf{p}^+(x)$. Since $(x, x\mathrm{S}) \in \mathrm{S}^{\mathcal{C}_{\mathcal{K}}}$, we have $(x, \mathsf{p}^+(x)) \in \mathrm{S}^{\mathcal{J}}$.
   - If $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_\sigma$. By definition of $\mathsf{gen}_{\mathcal{K}}$, there exists $w\mathrm{R} \in \mathcal{C}_{\mathcal{K}}$. Since $\mathrm{R} \notin \{\mathrm{S}, \mathrm{S}^-\}$, we have $\chi(w\mathrm{R}) = \mathsf{ch}(\mathrm{R})$. From $\mathrm{R} \in \mathcal{D}^+_\sigma$, we know that $\mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \exists \mathrm{S}$, which ensures $w\mathrm{RS} \in \Delta^{\mathcal{C}_{\mathcal{K}}}$. The definition of $\mathcal{D}^+_\sigma$ further tells us that $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R}))$. As $\chi(w\mathrm{R}) = x \in \mathsf{dom}(\mathsf{p}^+)$, we must have $\chi(w\mathrm{RS}) = \mathsf{p}^+(x)$. Finally $(w\mathrm{R}, w\mathrm{RS}) \in \mathrm{S}^{\mathcal{C}_{\mathcal{K}}}$ ensures $(x, \mathsf{p}^+(x)) \in \mathrm{S}^{\mathcal{J}}$.

3^+. Let $(x, \mathsf{ch}(\mathrm{S}))$ such that $x \in \mathsf{crit}^+_{\mathsf{ch}} \setminus \mathsf{dom}(\mathsf{p}^+)$.
Distinguish two cases based on $x \in \mathsf{crit}^+_{\mathsf{ch}}$:

- If $x \in \mathcal{D}^+_\mathcal{K}$. By definition $x \in \mathsf{Ind}(\mathcal{A})$, so $\chi(x) = x$. Moreover, $x\mathrm{S} \in \mathcal{C}_\mathcal{K}$, hence $\mathsf{succ}^\mathcal{K}_\mathrm{S}(x)$ is not defined. Combined with $x \notin \mathsf{dom}(\mathsf{p}^+)$, we obtain $\chi(x\mathrm{S}) = \mathsf{ch}(\mathrm{S})$. Since $(x, x\mathrm{S}) \in \mathrm{S}^{\mathcal{C}_\mathcal{K}}$, we have $(x, \mathsf{ch}(\mathrm{S})) \in \mathrm{S}^\mathcal{J}$.
- If $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_\sigma$. By definition of $\mathsf{gen}_\mathcal{K}$, there exists $w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$. Since $\mathrm{R} \notin \{\mathrm{S}, \mathrm{S}^-\}$, it gives $\chi(w\mathrm{R}) = \mathsf{ch}(\mathrm{R})$. The hypothesis $\mathcal{T} \models \exists\mathrm{R}^- \sqsubseteq \exists\mathrm{S}$ ensures $w\mathrm{RS} \in \Delta^{\mathcal{C}_\mathcal{K}}$. Since $\exists\mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{R}))$ and $\chi(w\mathrm{R}) = x \notin \mathsf{dom}(\mathsf{p}^+)$, it gives $\chi(w\mathrm{RS}) = \mathsf{ch}(\mathrm{S})$. Finally $(w\mathrm{R}, w\mathrm{RS}) \in \mathrm{S}^{\mathcal{C}_\mathcal{K}}$ ensures $(x, \mathsf{ch}(\mathrm{S})) \in \mathrm{S}^\mathcal{J}$.

3^-. Symmetric to Case 3^+.

4^+. Let $(\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S}))$ with $|\mathsf{crit}^+_\mathsf{ch}| > |\mathsf{crit}^-_\mathsf{ch}|$, $\mathcal{T} \models \exists\mathrm{S}^- \sqsubseteq \exists\mathrm{S}$, $\exists\mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}))$ and $\mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma)$. Because of $|\mathsf{crit}^+_\mathsf{ch}| > |\mathsf{crit}^-_\mathsf{ch}|$, we know that there exists some $x \in \mathsf{crit}^+_\mathsf{ch} \setminus \mathsf{dom}(\mathsf{p}^+)$. Distinguish two cases based on $x \in \mathsf{crit}^+_\mathsf{ch}$:

- If $x \in \mathcal{D}^+_\mathcal{K}$. By definition, $x\mathrm{S} \in \mathcal{C}_\mathcal{K}$, hence $\mathsf{succ}^\mathcal{K}_\mathrm{S}(x)$ is not defined. Moreover, we chose $x \in \mathsf{crit}^+_\mathsf{ch} \setminus \mathsf{dom}(\mathsf{p}^+)$, so $x \notin \mathsf{dom}(\mathsf{p}^+)$. It follows that $\chi(x\mathrm{S}) = \mathsf{ch}(\mathrm{S})$. Since $\mathcal{T} \models \exists\mathrm{S}^- \sqsubseteq \exists\mathrm{S}$, we have $x\mathrm{SS} \in \Delta^{\mathcal{C}_\mathcal{K}}$. Combined with our assumptions $\exists\mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}))$ and $\mathsf{ch}(\mathrm{S}) \notin \mathsf{crit}^+_\mathsf{ch}$, we obtain $\chi(x\mathrm{SS}) = \mathsf{ch}(\mathrm{S})$. Finally from $(x\mathrm{S}, x\mathrm{SS}) \in \mathrm{S}^{\mathcal{C}_\mathcal{K}}$, we can infer $(\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S})) \in \mathrm{S}^\mathcal{J}$.
- If $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_\sigma$. By definition of $\mathsf{gen}_\mathcal{K}$, there exists $w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$. From $\mathrm{R} \in \mathcal{D}^+_\sigma$, we have $\mathrm{R} \notin \{\mathrm{S}, \mathrm{S}^-\}$, which gives $\chi(w\mathrm{R}) = \mathsf{ch}(\mathrm{R})$. Moreover, we also have that $\mathcal{T} \models \exists\mathrm{R}^- \sqsubseteq \exists\mathrm{S}$, which ensures $w\mathrm{RS} \in \Delta^{\mathcal{C}_\mathcal{K}}$. Since $\exists\mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{R}))$ and $\chi(w\mathrm{R}) = x \notin \mathsf{dom}(\mathsf{p}^+)$, we have $\chi(w\mathrm{RS}) = \mathsf{ch}(\mathrm{S})$. Furthermore, we assumed $\exists\mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}))$ and $\mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma)$, ensuring in particular $\mathsf{ch}(\mathrm{S}) \notin \mathsf{crit}^+_\mathsf{ch}$. Hence $\chi(w\mathrm{RSS}) = \mathsf{ch}(\mathrm{S})$. We conclude by using $(w\mathrm{RS}, w\mathrm{RSS}) \in \mathrm{S}^{\mathcal{C}_\mathcal{K}}$ to infer $(\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S})) \in \mathrm{S}^\mathcal{J}$.

4^-. Symmetric to Case 4^+. □

We next need to understand how to relate the elements of $\mathcal{J}$ with the elements of the original model $\mathcal{I}$. In particular, we are interested in critical elements, which motivates the following definition.

Definition 16 (Origin of critical elements)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let $\mathsf{repr}_\mathcal{K}$ be function mapping each role $\mathrm{R} \in \mathsf{gen}_\mathcal{K}$ to an element with shape $w\mathrm{R}$ from $\Delta^{\mathcal{C}_\mathcal{K}}$. Let $\mathcal{I}$ be a model of $\mathcal{K}$ and $f : \mathcal{C}_\mathcal{K} \to \mathcal{I}$ be a homomorphism. Let σ be the strategy extracted from $\mathcal{I}$ (for f and $\mathsf{repr}_\mathcal{K}$). Let ch be a choice of well-typed elements for σ over $\mathcal{K}$. The* origins of critical elements *are given by:*

$$\begin{array}{rcl} \mathsf{ori}^+ : \mathsf{crit}^+_\mathsf{ch} & \to & \Delta^\mathcal{I} \\ x & \mapsto & \begin{cases} x & \text{if } x \in \mathcal{D}^+_\mathcal{K} \\ f(\mathsf{repr}_\mathcal{K}(\mathrm{R})) & \text{if } x = \mathsf{ch}(\mathrm{R}) \text{ with } \mathrm{R} \in \mathcal{D}^+_\sigma \end{cases} \\ \mathsf{ori}^- : \mathsf{crit}^-_\mathsf{ch} & \to & \Delta^\mathcal{I} \\ y & \mapsto & \begin{cases} y & \text{if } y \in \mathcal{D}^-_\mathcal{K} \\ f(\mathsf{repr}_\mathcal{K}(\mathrm{T})) & \text{if } y = \mathsf{ch}(\mathrm{T}) \text{ with } \mathrm{T} \in \mathcal{D}^-_\sigma \end{cases} \end{array}$$

Notice the second point in Lemma 2 ensures that ori^+, resp. ori^-, is well defined, that is, it does not depend on the choice of the role R, resp. T.

Observe that this way of associating critical elements with elements of the original model is injective.

Lemma 5. *The functions ori^+ and ori^- as defined in Definition 16, are injective.*

Proof. Let $x, x' \in \mathsf{crit}^+_\mathsf{ch}$ such that $\mathsf{ori}^+(x) = \mathsf{ori}^+(x')$. We consider the four possible cases.

1. Suppose $x \in \mathcal{D}^+_\mathcal{K}$.
   (a) Suppose $x' \in \mathcal{D}^+_\mathcal{K}$.
   Trivial: $x = \mathsf{ori}^+(x) = \mathsf{ori}^+(x') = x'$.
   (b) Suppose $x' = \mathsf{ch}(\mathrm{R}')$ with $\mathrm{R}' \in \mathcal{D}^+_\sigma$.
   On the one hand, statement 1 from Lemma 2 ensures $\theta_\mathcal{K}(x') = \theta_\mathcal{K}(f(\mathsf{repr}_\mathcal{K}(\mathrm{R}')))$. Since by our assumptions we have $x = \mathsf{ori}^+(x) = \mathsf{ori}^+(x') = f(\mathsf{repr}_\mathcal{K}(\mathrm{R}')$, we get $\theta_\mathcal{K}(x) = \theta_\mathcal{K}(f(\mathsf{repr}_\mathcal{K}(\mathrm{R}')))$, hence $\theta_\mathcal{K}(x') = \theta_\mathcal{K}(x)$. Since $x \in \mathcal{D}^+_\mathcal{K}$, this means in particular that $\exists\mathrm{S} \in \theta_\mathcal{K}(x')$.
   On the other hand, ch must satisfy Condition 1 of the definition of choice of well-typed elements, so $\sigma(\mathrm{R}') = (\theta_\mathcal{K}(x'), i)$ for some i. However, from $\mathrm{R}' \in \mathcal{D}^+_\sigma$, we have that $\exists\mathrm{S} \notin \theta_\mathcal{K}(x')$, a contradiction.
2. Suppose $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_\sigma$.
   (a) Suppose $x' \in \mathcal{D}^+_\mathcal{K}$.
   Symmetric to Case 1.b.
   (b) Suppose $x' = \mathsf{ch}(\mathrm{R}')$ with $\mathrm{R}' \in \mathcal{D}^+_\sigma$.
   Then $\theta_\mathcal{K}(f(\mathsf{repr}_\mathcal{K}(\mathrm{R})) = \theta_\mathcal{K}(f(\mathsf{repr}_\mathcal{K}(\mathrm{R}'))$. Statement 2 from Lemma 2 yields $\mathsf{ch}(\mathrm{R}) = \mathsf{ch}(\mathrm{R}')$, hence $x' = x$.

Therefore, ori^+ is injective. The argument for ori^- is symmetric. □

We are now ready to prove the second point of Lemma 1, which is formulated in full detail in the following statement.

Lemma 6. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let $\mathsf{repr}_{\mathcal{K}}$ be function mapping each role $\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}$ to an element with shape $w\mathrm{R}$ from $\Delta^{\mathcal{C}_{\mathcal{K}}}$. Let $\mathcal{I}$ be a model of $\mathcal{K}$ and $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$ be a homomorphism. Let σ be the strategy extracted from $\mathcal{I}$ (for f and $\mathsf{repr}_{\mathcal{K}}$). Let ch be a choice of well-typed elements for σ over $\mathcal{K}$. Let p be a pairing for ch. Denote by $\mathcal{J}$ the model resulting from interpreting the strategy σ (according to ch, p, and any certain successor preference). Then we have:*

$$q_S^{\mathcal{J}} \leq q_S^{\mathcal{I}}.$$

Proof. In the following, assume $|\mathsf{crit}^+_{\mathsf{ch}}| \geq |\mathsf{crit}^-_{\mathsf{ch}}|$ so that the only possible shapes for matches are 1, 2, 3^+ and 4^+. The case $|\mathsf{crit}^-_{\mathsf{ch}}| > |\mathsf{crit}^+_{\mathsf{ch}}|$ with possible shapes 1, 2, 3^- and 4^- is symmetrical.

Pick some successor preference $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{R}})_{\mathrm{R}}$ for $\mathcal{I}$ (refer back to Definition 15). We associate with each match π of q_S in $\mathcal{J}$ a match $\rho(\pi)$ in $\mathcal{I}$ depending on the shape of π:

$$\rho(\pi) : \begin{cases} (\mathsf{a}, \mathsf{b}) & \text{if } \pi = (\mathsf{a}, \mathsf{b}) \text{ is of Shape 1} \\ (\mathsf{ori}^+(x), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(\mathsf{ori}^+(x))) & \text{if } \pi = (x, y) \text{ is of Shape 2 or } 3^+ \\ (f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})))) & \text{if } \pi = (\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S})) \text{ is of Shape } 4^+ \end{cases}$$

Notice that in all cases $\rho(\pi)$ is indeed a match in $\mathcal{I}$. This is obvious if π is of Shape 1. When π of Shape 2 or 3^+, $\mathsf{ori}^+(x)$ is an element of $\Delta^{\mathcal{I}}$ that possesses an S-successor, so $\mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(\mathsf{ori}^+(x))$ is well defined, and we have $\rho(\pi) = (\mathsf{ori}^+(x), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(\mathsf{ori}^+(x))) \in \mathrm{S}^{\mathcal{I}}$. Finally, if π is of Shape 4^+, this means $\mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S}$, $\mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))))$ is well defined, and $(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})))) \in \mathrm{S}^{\mathcal{I}}$.

Now we prove that ρ is injective. Consider two matches π_1, π_2 of q_{S} in $\mathcal{J}$ such that $\rho(\pi_1) = \rho(\pi_2)$. We will use $\pi_1[1], \pi_1[2]$ to refer to the first and second arguments of π_1, and similarly for π_2. We consider all nine cases, showing in each case that either the situation cannot occur or that $\pi_1 = \pi_2$:

1\.
- 1\. When π_1, π_2 are both of Shape 1, we have $\pi_1 = \rho(\pi_1) = \rho(\pi_2) = \pi_2$.
- 2, 3^+. $\pi_1 = (\mathsf{a}, \mathsf{b})$ is of Shape 1, so $\pi_1 = \rho(\pi_1)$, while $\pi_2 = (x, y)$ is of Shape 2 or 3^+, which implies that $x \in \mathsf{crit}^+_{\mathsf{ch}}$.
  - If $x \in \mathcal{D}^+_{\mathcal{K}}$, then $\rho(\pi_2)[1] = \mathsf{ori}^+(x) = x$. It follows that $\rho(\pi_1)[1] = \mathsf{a} = x$. But $\mathsf{a} \notin \mathcal{D}^+_{\mathcal{K}}$ since $\mathrm{S}(\mathsf{a}, \mathsf{b}) \in \mathcal{A}$, which is a contradiction.
  - If $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_{\sigma}$, then in particular $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R}))$. Lemma 2 tells us that $\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R})) = \theta_{\mathcal{K}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})))$. We also have $\mathsf{ori}^+(x) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))$, so $\rho(\pi_2)[1] = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))$. From $\rho(\pi_1) = \rho(\pi_2)$ we get $\mathsf{a} = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))$. Putting this together, we get $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{a})$, which contradicts $\mathrm{S}(\mathsf{a}, \mathsf{b}) \in \mathcal{A}$.
- 4^+. In particular $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S}))$ and $\rho(\pi_2)[1] = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))$. Lemma 2 provides $\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S})) = \theta_{\mathcal{K}}(\rho(\pi_2)[1])$. Recall $\pi_1 = \rho(\pi_1) = (\mathsf{a}, \mathsf{b})$ and $\rho(\pi_1) = \rho(\pi_2)$, hence $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\pi_1[1])$. Contradiction with $\mathrm{S}(\mathsf{a}, \mathsf{b}) \in \mathcal{A}$.

2, 3^+.
- 1\. Symmetric to Case 1.(2, 3^+).
- 2, 3^+. As both π_1 and π_2 are of Shapes 2 / 3^+, we have $\mathsf{ori}^+(\pi_1[1]) = \mathsf{ori}^+(\pi_2[1])$. We can apply Lemma 5 to obtain $\pi_1[1] = \pi_2[1]$. By examining the conditions of Shapes 2 and 3^+, we can see that π_1 and π_2 must have the same shape, and moreover, their second arguments must coincide, yielding $\pi_1 = \pi_2$.
- 4^+. As $\pi_1 = (x, y)$ is of Shape 2 / 3^+, we have $x = \pi_1[1] \in \mathsf{crit}^+_{\mathsf{ch}}$. As $\pi_2 = (\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S}))$ is of Shape 4, we have $\mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S}$, $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S}))$, $\mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_{\sigma})$, and $\rho(\pi_2) = (f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))))$.
  - If $x \in \mathcal{D}^+_{\mathcal{K}}$, then $\rho(\pi_1)[1] = \mathsf{ori}^+(x) = x \in \mathsf{Ind}(\mathcal{A})$. From $\rho(\pi_1) = \rho(\pi_2)$ and above, we get $x = \rho(\pi_1)[1] = \rho(\pi_2)[1] = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))$. By statement 1 of Lemma 2, we have $\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S})) = \theta_{\mathcal{K}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S})))$, yielding $\theta_{\mathcal{K}}(x) = \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S}))$. Recall that $x \in \mathcal{D}^+_{\mathcal{K}}$ ensures in particular $\exists \mathrm{S} \in \theta_{\mathcal{K}}(x)$, it contradicts the assumption $\exists \mathrm{S} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{S}))$.
  - If $x = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}^+_{\sigma}$, then $\rho(\pi_1)[1] = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))$. As $\rho(\pi_1) = \rho(\pi_2)$ and $\rho(\pi_2)[1] = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))$, we have $f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{S}))$. The second statement of Lemma 2 gives us $\mathsf{ch}(\mathrm{R}) = \mathsf{ch}(\mathrm{S})$. Since $\mathrm{R} \in \mathcal{D}^+_{\sigma}$, we get a contradiction with $\mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_{\sigma})$.

4^+.
- 1\. Symmetric to Case 1.4^+.
- 2, 3^+. Symmetric to Case (2, 3^+).4^+.
- 4^+. By definition $\pi_1 = \pi_2 = (\mathsf{ch}(\mathrm{S}), \mathsf{ch}(\mathrm{S}))$. □

Explicit formula for number of matches in the interpretation of a strategy.

To avoid computing an actual model interpreting a strategy and then computing its number of matches, it is useful to observe that this amount is easily decided in advance and, in particular, is independent of the choice of well-typed elements and of the pairing. This is expressed by the following lemma.

Lemma 7. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let σ be a legal strategy over $\mathcal{K}$. Every model interpreting the strategy σ provides the following amount $\lambda_{\sigma/\mathcal{K}}$ of matches:*

$$\begin{array}{rcl} \lambda_{\sigma/\mathcal{K}} & := & |\{(\mathsf{a},\mathsf{b}) \mid \mathcal{K} \models \mathrm{S}(\mathsf{a},\mathsf{b})\}| \;+\; \max\left(\left|\mathcal{D}^+_\mathcal{K}\right| + |\sigma(\mathcal{D}^+_\sigma)|, \left|\mathcal{D}^-_\mathcal{K}\right| + |\sigma(\mathcal{D}^-_\sigma)|\right) \\ & & +1 \quad \textit{if} \quad \left| \begin{array}{l} \left|\mathcal{D}^+_\mathcal{K}\right| + |\sigma(\mathcal{D}^+_\sigma)| > \left|\mathcal{D}^-_\mathcal{K}\right| + |\sigma(\mathcal{D}^-_\sigma)| \\ \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S} \\ \exists \mathrm{S} \notin \mathsf{t} \textit{ if } \sigma(\mathrm{S}) = (\mathsf{t}, k) \\ \sigma(\mathrm{S}) \notin \sigma(\mathcal{D}^+_\sigma) \end{array} \right. \\ & & +1 \quad \textit{if} \quad \left| \begin{array}{l} \left|\mathcal{D}^-_\mathcal{K}\right| + |\sigma(\mathcal{D}^-_\sigma)| > \left|\mathcal{D}^+_\mathcal{K}\right| + |\sigma(\mathcal{D}^+_\sigma)| \\ \mathcal{T} \models \exists \mathrm{S} \sqsubseteq \exists \mathrm{S}^- \\ \exists \mathrm{S}^- \notin \mathsf{t} \textit{ if } \sigma(\mathrm{S}^-) = (\mathsf{t}, k) \\ \sigma(\mathrm{S}^-) \notin \sigma(\mathcal{D}^-_\sigma) \end{array} \right. \end{array}$$

Before giving the proof of the preceding lemma, it will helpful to first establish the relationship holding between the sizes of the sets $\mathcal{D}^+_\mathcal{K}, \mathcal{D}^+_\sigma, \mathcal{D}^-_\mathcal{K}, \mathcal{D}^-_\sigma$ and the sets of critical elements.

Lemma 8. *Let* ch *be a choice of well-typed elements for a legal strategy σ over $\mathcal{K}$. Then the sets* $\mathsf{crit}^+_\mathsf{ch}$ *and* $\mathsf{crit}^-_\mathsf{ch}$ *satisfy the following:*

$$\left|\mathsf{crit}^+_\mathsf{ch}\right| \;=\; \left|\mathcal{D}^+_\mathcal{K}\right| \;+\; \left|\sigma(\mathcal{D}^+_\sigma)\right| \qquad \left|\mathsf{crit}^-_\mathsf{ch}\right| \;=\; \left|\mathcal{D}^-_\mathcal{K}\right| \;+\; \left|\sigma(\mathcal{D}^-_\sigma)\right|.$$

In particular, the sizes of $\mathsf{crit}^+_\mathsf{ch}$ *and* $\mathsf{crit}^-_\mathsf{ch}$ *do not depend on* ch.

Proof. First we prove that $\mathcal{D}^+_\mathcal{K}$ and $\mathsf{ch}(\mathcal{D}^+_\sigma)$ are disjoint. Notice that if $\mathsf{a} \in \mathcal{D}^+_\mathcal{K}$, then $\exists \mathrm{S} \in \theta_\mathcal{K}(\mathsf{a})$. Therefore, if ever $\mathsf{ch}(\mathrm{R}) = \mathsf{a}$, then by Condition 1 from the definition of a choice of well-typed elements: $\sigma(\mathrm{R}) = (\theta_\mathcal{K}(\mathsf{a}), k)$, which would contradict $\mathrm{R} \in \mathcal{D}^+_\sigma$. Hence $\mathcal{D}^+_\mathcal{K} \cap \mathsf{ch}(\mathcal{D}^+_\sigma) = \emptyset$. We conclude by applying Condition 2 from the definition of a choice of well-typed elements, which ensures that $|\mathsf{ch}(\mathcal{D}^+_\sigma)| = |\sigma(\mathcal{D}^+_\sigma)|$. The case of $\mathsf{crit}^-_\mathsf{ch}$ is symmetric. □

We now return to the proof of Lemma 7:

Proof of Lemma 7. Let $\mathcal{J}$ be the interpretation of σ obtained according to a choice of well-typed element ch, a pairing p, and some certain successor preference. From Lemma 4, and recalling that distinct shapes are incompatible, we have:

$$\begin{array}{rcl} \left|\mathrm{S}^\mathcal{J}\right| & = & |\{(\mathsf{a},\mathsf{b}) \mid \mathcal{K} \models \mathrm{S}(\mathsf{a},\mathsf{b})\}| \\ & & + \min\left(\left|\mathsf{crit}^+_\mathsf{ch}\right|, \left|\mathsf{crit}^-_\mathsf{ch}\right|\right) \\ & & + \max\left(\left|\mathsf{crit}^+_\mathsf{ch}\right| - \left|\mathsf{crit}^-_\mathsf{ch}\right|, 0\right) + \max\left(\left|\mathsf{crit}^-_\mathsf{ch}\right| - \left|\mathsf{crit}^+_\mathsf{ch}\right|, 0\right) \\ & & +1 \;\text{ if } \left|\mathsf{crit}^+_\mathsf{ch}\right| > \left|\mathsf{crit}^-_\mathsf{ch}\right| \;\wedge\; \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S} \;\wedge\; \exists \mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S})) \;\wedge\; \mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma) \\ & & +1 \;\text{ if } \left|\mathsf{crit}^-_\mathsf{ch}\right| > \left|\mathsf{crit}^+_\mathsf{ch}\right| \;\wedge\; \mathcal{T} \models \exists \mathrm{S} \sqsubseteq \exists \mathrm{S}^- \;\wedge\; \exists \mathrm{S}^- \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}^-)) \;\wedge\; \mathsf{ch}(\mathrm{S}^-) \notin \mathsf{ch}(\mathcal{D}^-_\sigma) \\ & = & |\{(\mathsf{a},\mathsf{b}) \mid \mathcal{K} \models \mathrm{S}(\mathsf{a},\mathsf{b})\}| + \max\left(\left|\mathsf{crit}^+_\mathsf{ch}\right|, \left|\mathsf{crit}^-_\mathsf{ch}\right|\right) \\ & & +1 \;\text{ if } \left|\mathsf{crit}^+_\mathsf{ch}\right| > \left|\mathsf{crit}^-_\mathsf{ch}\right| \;\wedge\; \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{S} \;\wedge\; \exists \mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S})) \;\wedge\; \mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma) \\ & & +1 \;\text{ if } \left|\mathsf{crit}^-_\mathsf{ch}\right| > \left|\mathsf{crit}^+_\mathsf{ch}\right| \;\wedge\; \mathcal{T} \models \exists \mathrm{S} \sqsubseteq \exists \mathrm{S}^- \;\wedge\; \exists \mathrm{S}^- \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}^-)) \;\wedge\; \mathsf{ch}(\mathrm{S}^-) \notin \mathsf{ch}(\mathcal{D}^-_\sigma) \end{array}$$

We can then apply Lemma 8 to express $\mathsf{crit}^+_\mathsf{ch}$ and $\mathsf{crit}^-_\mathsf{ch}$ in terms of the sets $\mathcal{D}^+_\mathcal{K}, \mathcal{D}^+_\sigma, \mathcal{D}^-_\mathcal{K}, \mathcal{D}^-_\sigma$. We also need to use Condition 1 from the definition of a choice of well-typed elements in order to replace $\exists \mathrm{S} \notin \theta_\mathcal{K}(\mathsf{ch}(\mathrm{S}))$ by $\exists \mathrm{S} \notin \mathsf{t}$ if $\sigma(\mathrm{S}) = (\mathsf{t}, k)$, and Condition 2 to replace $\mathsf{ch}(\mathrm{S}) \notin \mathsf{ch}(\mathcal{D}^+_\sigma)$ by $\sigma(\mathrm{S}) \notin \sigma(\mathcal{D}^+_\sigma)$ (and similarly for S^-). It can be verified that this indeed yields the desired number $\lambda_{\sigma/\mathcal{K}}$. □

Proof of TC^0 membership for Theorem 1(i).

With Lemma 1 in hand, we can now describe how to construct a family of TC^0 circuits to decide our problem. We need a family of circuits in order to be able to handle ABoxes of different sizes. More precisely, we will create one circuit for each possible number ℓ of individual names. We can assume w.l.o.g. that the same set of individuals, denoted Ind_ℓ, is used for all of the ABoxes having ℓ individuals. In what follows, we introduce the different gates which are used for computing the various sets and values used in the construction and how they are connected to each other. We start by the input gates which show how we represent an input $(\mathcal{A}^*, m^*)$ to the circuit that handles ℓ-individual ABoxes. It can be verified that for each of the gates we introduce decides the statement or property occurring in its label (with $\mathcal{A}^*$, resp. $\mathcal{K}^* = (\mathcal{T}, \mathcal{A}^*)$ substituted for $\mathcal{A}$, resp. $\mathcal{K}$).

Input gates.

- Each atomic role P appearing in $\mathcal{T}$ is represented by input gates $\bigcirc_{\mathrm{P}(\mathsf{a},\mathsf{b})\in\mathcal{A}}$ for $\mathsf{a},\mathsf{b}\in\mathsf{Ind}_\ell$. The gate $\bigcirc_{\mathrm{P}(\mathsf{a},\mathsf{b})\in\mathcal{A}}$ is set to 1 iff $\mathrm{P}(\mathsf{a},\mathsf{b})\in\mathcal{A}^*$.
- Each atomic concept A appearing in $\mathcal{T}$ is represented by input gates $\bigcirc_{\mathrm{A}(\mathsf{a})\in\mathcal{A}}$ for $\mathsf{a}\in\mathsf{Ind}_\ell$. The gate $\bigcirc_{\mathrm{A}(\mathsf{a})\in\mathcal{A}}$ is set to 1 iff $\mathrm{A}(\mathsf{a})\in\mathcal{A}^*$.
- The integer m^* is represented in binary by input gates $\bigcirc_{b_k=1}$ for each $0\leq k<\log_2(|\mathsf{Ind}(\mathcal{A}^*)|+|\mathcal{T}|)^{|q|})$. The gate $\bigcirc_{b_k=1}$ is set to 1 iff the k^{th} bit of m^* is 1 (with 0^{th}-bit being the least significant bit).

Regarding the last point, we use the observation from [Kostylev and Reutter, 2015] that if m^* is a certain answer for q over $\mathcal{K}^*$, then m^* cannot exceed $(|\mathsf{Ind}(\mathcal{A}^*)|+|\mathcal{T}|)^{|q|}=(|\mathsf{Ind}_\ell|+|\mathcal{T}|)^{|q|}$. We denote K this upper bound. This is a direct consequence of the fact that every satisfiable DL-Lite$^{\mathcal{H}}_{core}$ KB $\mathcal{K}=(\mathcal{T},\mathcal{A})$ has a model with at most $|\mathsf{Ind}(\mathcal{A})|+|\mathcal{T}|$ elements.

Gates for computing available roles and entailed concepts for ABox individuals.

For each positive role R and each individual name $\mathsf{a}\in\mathsf{Ind}_\ell$, introduce a disjunctive gate $\textcircled{\vee}_{\exists\mathsf{b},\mathcal{K}\models\mathrm{R}(\mathsf{a},\mathsf{b})}$ taking as inputs:

- $\bigcirc_{\mathrm{R}(\mathsf{a},\mathsf{b})\in\mathcal{A}}$ for each $\mathsf{b}\in\mathsf{Ind}(\mathcal{A})$, if $\mathrm{R}\in\mathsf{N_R}$.
- $\bigcirc_{\mathrm{P}(\mathsf{b},\mathsf{a})\in\mathcal{A}}$ for each $\mathsf{b}\in\mathsf{Ind}(\mathcal{A})$, if $\mathrm{R}=\mathrm{P}^-$ with $\mathrm{P}\in\mathsf{N_R}$.

For each positive concept B and each individual name $\mathsf{a}\in\mathsf{Ind}_\ell$, introduce a disjunctive gate $\textcircled{\vee}_{\mathcal{K}\models\mathrm{B}(\mathsf{a})}$ taking as inputs:

- $\bigcirc_{\mathrm{A}(\mathsf{a})\in\mathcal{A}}$ for each atomic concept A such that $\mathcal{T}\models\mathrm{A}\sqsubseteq\mathrm{B}$.
- $\textcircled{\vee}_{\exists\mathsf{b},\mathcal{K}\models\mathrm{R}(\mathsf{a},\mathsf{b})}$ for all role $\mathrm{R}\in\mathsf{N}^{\pm}_{\mathsf{R}}$ such that $\mathcal{T}\models\exists\mathrm{R}\sqsubseteq\mathrm{B}$.

Computing types and counting number of occurring types.

For each type $\mathfrak{t}\in\Theta_\mathcal{T}$ and each individual name $\mathsf{a}\in\mathsf{Ind}_\ell$, introduce a conjunctive gate $\textcircled{\wedge}_{\theta_\mathcal{K}(\mathsf{a})=\mathfrak{t}}$ taking as inputs:

- $\textcircled{\vee}_{\mathcal{K}\models\mathrm{B}(\mathsf{a})}$ for each positive concept B such that $\mathrm{B}\in\mathfrak{t}$.
- the negation of $\textcircled{\vee}_{\mathcal{K}\models\mathrm{B}(\mathsf{a})}$ for each positive concept B such that $\mathrm{B}\notin\mathfrak{t}$.

For each type $\mathfrak{t}\in\Theta_\mathcal{T}$ and each $k\in\{0,\dots\left|\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}\right|\}$, introduce a threshold gate $\textcircled{\top_k}_{\exists_{\geq k}\text{ ind. of type }\mathfrak{t}}$ taking as inputs: $\textcircled{\wedge}_{\theta_\mathcal{K}(\mathsf{a})=\mathfrak{t}}$ for each individual name $\mathsf{a}\in\mathsf{Ind}_\ell$.

Remark: Notice here that k ranges up to $\left|\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}\right|$ as any strategy requires at most this many copies of a type (see availability condition from Definition 14). Notice also the label "$\exists_{\geq k}$ ind. of type $\mathfrak{t}$", which stands for $|\{\mathsf{a}\in\mathsf{Ind}_\ell\mid\theta_\mathcal{K}(\mathsf{a})=\mathfrak{t}\}|\geq k$.

Identifying generated roles.

For each individual name $\mathsf{a}\in\mathsf{Ind}_\ell$ and each positive role R, introduce a conjunctive gate $\textcircled{\wedge}_{\mathsf{a}\mathrm{R}\in\Delta^{\mathcal{C}_\mathcal{K}}}$ taking as inputs: $\textcircled{\vee}_{\mathcal{K}\models\exists\mathrm{R}(\mathsf{a})}$ and the negation of $\textcircled{\vee}_{\exists\mathsf{b},\mathcal{K}\models\mathrm{R}(\mathsf{a},\mathsf{b})}$.
For each positive role R, introduce a disjunctive gate $\textcircled{\vee}_{\mathrm{R}\in\mathsf{gen}_\mathcal{K}}$ taking as inputs: $\textcircled{\wedge}_{\mathsf{a}\mathrm{T}\in\Delta^{\mathcal{C}_\mathcal{K}}}$ for each positive role T such that $\mathcal{T}$ ensures that if $\mathsf{a}\mathrm{T}\in\Delta^{\mathcal{C}_\mathcal{K}}$, then there exists a word w starting with T and ending by R s.t $\mathsf{a}w\in\Delta^{\mathcal{C}_\mathcal{K}}$.

Identifying demanding individuals (see Definition 4).

For each $\mathsf{a}\in\mathsf{Ind}_\ell$, introduce a conjunctive gate $\textcircled{\wedge}_{\mathsf{a}\in\mathcal{D}^+_\mathcal{K}}$ taking as inputs: $\textcircled{\vee}_{\mathcal{K}\models\exists\mathrm{S}(\mathsf{a})}$ and the negation of $\textcircled{\vee}_{\exists\mathsf{b},\mathcal{K}\models\mathrm{S}(\mathsf{a},\mathsf{b})}$.
For each $\mathsf{a}\in\mathsf{Ind}_\ell$, introduce a conjunctive gate $\textcircled{\wedge}_{\mathsf{a}\in\mathcal{D}^-_\mathcal{K}}$ taking as inputs: $\textcircled{\vee}_{\mathcal{K}\models\exists\mathrm{S}^-(\mathsf{a})}$ and the negation of $\textcircled{\vee}_{\exists\mathsf{b},\mathcal{K}\models\mathrm{S}^-(\mathsf{a},\mathsf{b})}$.

Subcircuit for each strategy $\sigma \in (\Theta_{\mathcal{T}} \times \{1, \dots |\mathsf{N}^{\pm}_{\mathsf{R}}|\})^{\mathsf{N}^{\pm}_{\mathsf{R}}}$ (see Definition 13), to check legality w.r.t. input (Definition 14).

Introduce a conjunctive gate $\textcircled{\wedge}_{\text{coverage }\sigma}$ taking as inputs:

- $\textcircled{\vee}_{\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}}$ for each positive role $\mathrm{R} \in \mathsf{dom}(\sigma)$,
- the negation of $\textcircled{\vee}_{\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}}$ for each positive role $\mathrm{R} \notin \mathsf{dom}(\sigma)$.

Introduce a conjunctive gate $\textcircled{\wedge}_{\text{availability }\sigma}$ taking as inputs: $\textcircled{\top_k}_{\exists_{\geq k} \text{ ind. of type } \mathfrak{t}}$ for each type $\mathfrak{t}$ being required k times by σ.

Introduce a conjunctive gate $\textcircled{\wedge}_{\text{legal }\sigma}$ taking as inputs: $\textcircled{\wedge}_{\text{coverage }\sigma}$ and $\textcircled{\wedge}_{\text{availability }\sigma}$.

Computing $\lambda_{\sigma/\mathcal{K}}$ for each strategy σ (see Lemma 7).

A threshold gate $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k}$ for each $k \in \{0, \dots, K\}$ taking as inputs:

- $\bigcirc_{\mathrm{S}(\mathsf{a},\mathsf{b}) \in \mathcal{A}}$ for each $(\mathsf{a},\mathsf{b}) \in \mathsf{Ind}_\ell \times \mathsf{Ind}_\ell$,
- $\textcircled{\wedge}_{\mathsf{a} \in \mathcal{D}^+_{\mathcal{K}}}$ for each $\mathsf{a} \in \mathcal{D}^+_{\mathcal{K}}$,
- $|\sigma(\mathcal{D}^+_{\sigma})|$ copies of a true gate $\textcircled{1}$.

A threshold gate $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k}$ for each $k \in \{0, \dots, K\}$ taking as inputs:

- $\bigcirc_{\mathrm{S}(\mathsf{a},\mathsf{b}) \in \mathcal{A}}$ for each $(\mathsf{a},\mathsf{b}) \in \mathsf{Ind}_\ell \times \mathsf{Ind}_\ell$,
- $\textcircled{\wedge}_{\mathsf{a} \in \mathcal{D}^-_{\mathcal{K}}}$ for each $\mathsf{a} \in \mathcal{D}^-_{\mathcal{K}}$,
- $|\sigma(\mathcal{D}^-_{\sigma})|$ copies of a true gate $\textcircled{1}$.

Introduce $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|=|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-m_{\mathcal{A}}}$ for each $k \in \{0, \dots, K\}$ taking as inputs:

- $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k}$ and the negation of $\textcircled{\top_{k+1}}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k+1}$,
- $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k}$ and the negation of $\textcircled{\top_{k+1}}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k+1}$,
- $\textcircled{\wedge}_{\text{legal }\sigma}$.

Notice that for the latter and upcoming gates of this block, we omit "legal σ" from the labels for clarity.

Introduce $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|<|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-m_{\mathcal{A}}}$ for each $k \in \{0, \dots, K\}$ taking as inputs:

- the negation of $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k}$,
- $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k}$ and the negation of $\textcircled{\top_{k+1}}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k+1}$,
- $\textcircled{\wedge}_{\text{legal }\sigma}$.

Introduce $\textcircled{\wedge}_{|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|<|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|=k-m_{\mathcal{A}}}$ for each $k \in \{0, \dots, K\}$ taking as inputs:

- $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k}$ and the negation of $\textcircled{\top_{k+1}}_{m_{\mathcal{A}}+|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|\geq k+1}$,
- the negation of $\textcircled{\top_k}_{m_{\mathcal{A}}+|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|\geq k}$,
- $\textcircled{\wedge}_{\text{legal }\sigma}$.

For each $k \in \{0, \dots, K\}$, introduce a disjunctive gate $\textcircled{\vee}_{\lambda_{\sigma/\mathcal{K}}=k}$ taking as inputs:

- $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|=|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-m_{\mathcal{A}}}$,
- If $\mathcal{T} \models \exists\mathrm{S} \sqsubseteq \exists\mathrm{S}^-$, $\exists\mathrm{S}^- \notin \mathfrak{t}$ with $\sigma(\mathrm{S}^-) = (\mathfrak{t}, k)$ and $\sigma(\mathrm{S}^-) \notin \sigma(\mathcal{D}^-_{\sigma})$, then gate $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|<|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-1-m_{\mathcal{A}}}$, otherwise gate $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|<|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-m_{\mathcal{A}}}$,
- If $\mathcal{T} \models \exists\mathrm{S}^- \sqsubseteq \exists\mathrm{S}$, $\exists\mathrm{S} \notin \mathfrak{t}$ with $\sigma(\mathrm{S}) = (\mathfrak{t}, k)$, and $\sigma(\mathrm{S}) \notin \sigma(\mathcal{D}^+_{\sigma})$, then gate $\textcircled{\wedge}_{|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|<|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|=k-1-m_{\mathcal{A}}}$, otherwise gate $\textcircled{\wedge}_{|\mathcal{D}^+_{\mathcal{K}}|+|\sigma(\mathcal{D}^+_{\sigma})|<|\mathcal{D}^-_{\mathcal{K}}|+|\sigma(\mathcal{D}^-_{\sigma})|=k-m_{\mathcal{A}}}$.

We are now able to compute the minimal amount of matches given by legal strategies.

Final comparison with the input integer (see Lemma 1).

For each $k \in \{0, \dots, K\}$, introduce a disjunctive gate $\bigcirc\!\!\!\!\vee_{\min\limits_{\text{legal }\sigma} \lambda_{\sigma/\mathcal{K}} < k}$ taking as inputs: $\bigcirc\!\!\!\!\wedge_{\substack{\lambda_{\sigma/\mathcal{K}}=k' \\ \text{legal }\sigma}}$ for each strategy σ and each $k' < k$.
For each $k \in \{0, \dots, K\}$, introduce a conjunctive gate $\bigcirc\!\!\!\!\wedge_{m=k}$ taking as inputs:

- $\bigcirc_{b_j=1}$ such that the j^{th} bit of the binary encoding of k is 1,
- the negation of $\bigcirc_{b_j=1}$ such that the j^{th} bit of the binary encoding of k is 0.

For each $k \in \{0, \dots, K\}$, introduce a conjunctive gate $\bigcirc\!\!\!\!\wedge_{\min\limits_{\text{legal }\sigma} \lambda_{\sigma/\mathcal{K}} \geq m=k}$ taking as inputs: $\bigcirc\!\!\!\!\wedge_{m=k}$ and the negation of $\bigcirc\!\!\!\!\vee_{\min\limits_{\text{legal }\sigma} \lambda_{\sigma/\mathcal{K}} < k}$.
Introduce an **output** disjunctive gate $\bigcirc\!\!\!\!\vee_{\min\limits_{\text{legal }\sigma} \lambda_{\sigma/\mathcal{K}} \geq m}$ taking as inputs: $\bigcirc\!\!\!\!\wedge_{\min\limits_{\text{legal }\sigma} \lambda_{\sigma/\mathcal{K}} \geq m=k}$ for each $k \in \{0, \dots, K\}$.

To complete the proof, we observe that, since the TBox $\mathcal{T}$ is fixed, the number of gates is polynomial in the described family of circuits. Moreover, all circuits in the family have the same depth (13). Thus, the construction yields a TC^0 of circuits for deciding $\mathrm{OMQA}(q_S, \mathcal{T})$ and establishes membership in TC^0.

A.2 Proof of Theorem 1.ii: Concept Cardinality Queries and DL-Lite$^{\mathcal{H}}_{core}$ without negative role inclusions.

We now turn to the case where $\mathcal{T}$ is a DL-Lite$^{\mathcal{H}}_{core}$ TBox *without negative role inclusions* and q_{C} is the concept cardinality query: $\exists z\ \mathrm{C}(z)$.

Due to a simpler shape of the query, several notions simplify. In particular, distinguishing between positive and negative critical elements is no longer necessary and these notions can be unified as follows.

Definition 17 (Demanding roles, critical elements)**.** *Let σ be a strategy. Define* demanding roles $\mathcal{D}_\sigma$ *as:*

$$\mathcal{D}_\sigma := \left\{ \mathrm{R} \;\middle|\; \begin{array}{l} \mathrm{R} \in \mathsf{dom}(\sigma) \\ \mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \mathrm{C} \\ \mathrm{C} \notin \mathsf{t} \text{ if } \sigma(\mathrm{R}) = (\mathsf{t}, k) \end{array} \right\}$$

Let ch *be a choice of well-typed elements for σ over $\mathcal{K}$. Define the set of* critical elements *as:*

$$\mathsf{crit}_{\mathsf{ch}} = \mathsf{ch}(\mathcal{D}_\sigma)$$

Pairing is also no longer necessary, which means the interpretation of a strategy can be drastically simplified as follows.

Definition 18 (Interpretation of a strategy)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let σ be a legal strategy over $\mathcal{K}$. Let* ch *be a choice of well-typed elements for σ over $\mathcal{A}$. Consider the following mapping:*

$$\begin{array}{rrcl} \chi: & \Delta^{\mathcal{C}_\mathcal{K}} & \to & \mathsf{Ind}(\mathcal{A}) \cup \{\bot_i \mid i = 1, \dots |\mathsf{sig}(\mathcal{T})^{\pm}_{\mathsf{R}}|\} \\ & \mathsf{a} & \mapsto & \mathsf{a} \\ & w\mathrm{R} & \mapsto & \mathsf{ch}(\mathrm{R}) \end{array}$$

The interpretation $\mathcal{J}$ of σ w.r.t. ch *is defined as the image of $\cdot^{\mathcal{C}_\mathcal{K}}$ through χ: its domain is $\Delta^{\mathcal{J}} = \chi(\Delta^{\mathcal{C}_\mathcal{K}})$, and its interpretation function is $\cdot^{\mathcal{J}} = \chi \circ \cdot^{\mathcal{C}_\mathcal{K}}$.*

Under these updated definitions, notice the Lemma 1 still makes perfect sense, and we start by proving it, following closely the analogous proof for role cardinality queries.

Proof of Lemma 1 for concept cardinality queries

We first prove the first point of Lemma 1, stating that the interpretation of a strategy extracted from a model is also a model, in the following stronger form, not requiring the strategy to be extracted from a model in the first place.

Lemma 9. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ be a satisfiable KB. Let σ be a legal strategy over $\mathcal{K}$. Let* ch *be a choice of well-typed elements for σ over $\mathcal{A}$. The interpretation $\mathcal{J}$ of the strategy σ w.r.t.* ch *is a model.*

Proof. Assertions from the ABox and axioms without negation are satisfied since the interpretation $\mathcal{J}$ is built from $\mathcal{C}_\mathcal{K}$. Consider now a negative concept inclusion $\mathrm{B}_1 \sqsubseteq \neg \mathrm{B}_2$. Assume for a contradiction that there is an element d such that $d \in \mathrm{B}_1{}^{\mathcal{J}} \cap \mathrm{B}_2{}^{\mathcal{J}}$. There are four cases to consider:

1. If $\mathcal{K} \models \mathrm{B}_1(d)$ and $\mathcal{K} \models \mathrm{B}_2(d)$, then this contradicts $\mathcal{K}$ being satisfiable.

2. If $\mathcal{K} \models \mathrm{B}_1(d)$ and $\mathcal{K} \not\models \mathrm{B}_2(d)$, then $d = \chi(w\mathrm{R})$ with $\mathrm{R} \in \mathsf{gen}_{\mathcal{K}}$ and $\mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \mathrm{B}_2$. In particular, $d = \mathsf{ch}(\mathrm{R})$ and $\mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \neg \mathrm{B}_1$. Condition 1 from the definition of choice of well-typed elements ensures $\sigma(\mathrm{R}) = (\theta_{\mathcal{K}}(d), i)$ for some i. Condition 1 from the definition of a strategy implies that $\mathrm{B}_1 \notin \theta_{\mathcal{K}}(d)$, contradicting $\mathcal{K} \models \mathrm{B}_1(d)$.
3. If $\mathcal{K} \not\models \mathrm{B}_1(d)$ and $\mathcal{K} \models \mathrm{B}_2(d)$. Symmetric to Case 2.
4. If $\mathcal{K} \not\models \mathrm{B}_1(d)$ and $\mathcal{K} \not\models \mathrm{B}_2(d)$, then $d = \chi(w_1\mathrm{R}_1) = \chi(w_2\mathrm{R}_2)$ with $\mathrm{R}_1 \in \mathsf{gen}_{\mathcal{K}}$, $\mathrm{R}_2 \in \mathsf{gen}_{\mathcal{K}}$, $\mathcal{T} \models \exists \mathrm{R}_1^- \sqsubseteq \mathrm{B}_1$ and $\mathcal{T} \models \exists \mathrm{R}_2^- \sqsubseteq \mathrm{B}_2$. In particular, $d = \mathsf{ch}(\mathrm{R}_1) = \mathsf{ch}(\mathrm{R}_2)$. Condition 2 in the definition of the choice of well-typed elements ensures: $\sigma(\mathrm{R}_1) = \sigma(\mathrm{R}_2)$. Condition 2 in the definition of a strategy ensures: $\mathcal{T} \not\models \exists \mathrm{R}_1^- \sqsubseteq \neg \exists \mathrm{R}_2^-$, contradicting $\mathcal{T} \models \mathrm{B}_1 \sqsubseteq \neg \mathrm{B}_2$. □

In order to prove the second point of Lemma 2, stating an interpretation $\mathcal{J}$ of the strategy extracted from a model $\mathcal{I}$ has at most as matches as the original model $\mathcal{I}$, we need to better understand what kinds of matches of q_C can be found in $\mathcal{J}$. This is achieved by the following result which precisely characterizes $\mathrm{C}^{\mathcal{J}}$

Lemma 10 (Matches in a model interpreting a strategy)**.** *Let $\mathcal{A}$ be an ABox, and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ be satisfiable KB. Let σ be a legal strategy over $\mathcal{K}$, and let* ch *be a choice of well-typed elements for σ over $\mathcal{K}$. Denote by $\mathcal{J}$ the interpretation of σ w.r.t.* ch*. Then we have:*

$$\begin{aligned} \mathrm{C}^{\mathcal{J}} \;=\; & \{\mathsf{a} \mid \mathcal{K} \models \mathrm{C}(\mathsf{a})\} && \textit{(Shape 1)} \\ \cup\; & \{\mathsf{ch}(\mathrm{R}) \mid \mathrm{R} \in \mathcal{D}_\sigma\} && \textit{(Shape 2)} \end{aligned}$$

Furthermore, there is no overlap between these two distinct shapes.

Proof. The first inclusion ($\subseteq$) is rather straightforward. We therefore focus on proving the direction ($\supseteq$).

1. Let a be such that $\mathcal{K} \models \mathrm{C}(\mathsf{a})$, in particular $\mathsf{a} \in \mathsf{Ind}(\mathcal{A})$. By definition, $\chi(\mathsf{a}) = \mathsf{a}$, hence $\mathsf{a} \in \mathrm{C}^{\mathcal{J}}$.
2. Let $\mathrm{R} \in \mathcal{D}_\sigma$. By definition of $\mathsf{gen}_{\mathcal{K}}$, there exists $w\mathrm{R} \in \mathcal{C}_{\mathcal{K}}$. By definition of the interpretation of a strategy, $\chi(w\mathrm{R}) = \mathsf{ch}(\mathrm{R})$. Moreover, $\mathrm{R} \in \mathcal{D}_\sigma$ implies that $\mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \mathrm{C}$, which ensures $w\mathrm{R} \in \mathrm{C}^{\mathcal{C}_{\mathcal{K}}}$. Therefore $\mathsf{ch}(\mathrm{R}) \in \mathrm{C}^{\mathcal{J}}$. □

We can now prove the second point of Lemma 1, recalled in the following statement.

Lemma 11. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Let $\mathcal{I}$ be a model of $\mathcal{K}$. Let σ be the strategy extracted from $\mathcal{I}$. Let* ch *be a choice of well-typed elements for σ over $\mathcal{K}$. Denote $\mathcal{J}$ the resulting interpretation of σ. We have:*

$$q^{\mathcal{J}} \leq q^{\mathcal{I}}.$$

Proof. Associate each match π of q in $\mathcal{J}$ to a match $\rho(\pi)$ in $\mathcal{I}$ depending of the shape of π:

$$\rho(\pi) : \begin{cases} z \;\mapsto\; \pi(z) & \text{if } \pi \text{ has Shape 1} \\ z \;\mapsto\; f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})) & \text{if } \pi \text{ has Shape 2 with } \pi(z) = \mathsf{ch}(\mathrm{R}) \end{cases}$$

Notice $\rho(\pi)$ is indeed a match in $\mathcal{I}$. Now prove that ρ is injective. Let $\pi_1, \pi_2 : q \to \mathcal{J}_{\sigma_{f \circ \mathsf{repr}_{\mathcal{K}}}}$ be two matches such that $\rho(\pi_1) = \rho(\pi_2)$. We consider all four cases:

1.
   1. $\pi_1(z_1) = \rho(\pi_1)(z_1) = \rho(\pi_2)(z_1) = \pi_2(z_1)$ and $\pi_1(z_2) = \rho(\pi_1)(z_2) = \rho(\pi_2)(z_2) = \pi_2(z_2)$.
   2. We have $\pi_2(z) = \mathsf{ch}(\mathrm{R})$ with $\mathrm{R} \in \mathcal{D}_\sigma$. Therefore $\mathrm{C} \notin \theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R}))$. Lemma 2 provides $\theta_{\mathcal{K}}(\mathsf{ch}(\mathrm{R})) = \theta_{\mathcal{K}}(\rho(\pi_2)(z))$. Recall $\rho(\pi_1) = \rho(\pi_2)$, hence $\mathrm{C} \notin \theta_{\mathcal{K}}(\pi_1(z))$. Contradiction with $\mathcal{K} \models \mathrm{C}(\pi_1(z))$.
2.
   1. Symmetric to Case 1.2.
   2. We have $\pi_1(z) = \mathsf{ch}(\mathrm{R}_1)$ with $\mathrm{R}_1 \in \mathcal{D}_\sigma$ and $\pi_2(z) = \mathsf{ch}(\mathrm{R}_2)$ with $\mathrm{R}_2 \in \mathcal{D}_\sigma$. Therefore $f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}_1)) = f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}_2))$. Lemma 2 provides $\pi_1(z) = \pi_2(z)$,.

□

Explicit formula for number of matches in the interpretation of a strategy.
We will again avoid having to produce interpretations of strategies by showing that we can directly determine the number of matches occurring in such models. This is the purpose of the following lemma.

Theorem 9. *Let $\mathcal{A}$ be an ABox, and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ be a satisfiable KB. Let σ be a legal strategy over $\mathcal{K}$. Any interpretation $\mathcal{J}$ of the strategy σ has the following amount $\lambda_{\sigma/\mathcal{K}}$ of matches:*

$$\lambda_{\sigma/\mathcal{K}} \;=\; |\{\mathsf{a} \mid \mathcal{K} \models \mathrm{C}(\mathsf{a})\}| \;+\; |\sigma(\mathcal{D}_\sigma)|$$

Proof. The equation immediately follows from Lemma 10 and by noticing that $|\mathsf{crit}_{\mathsf{ch}}| = |\sigma(\mathcal{D}_\sigma)|$ due to second condition in the definition of a choice of well-typed elements. □

The family of circuits.

To complete the proof, we describe how to construct a family of TC^0 circuits that can be used to decide our problem. The construction is very similar to the one given for role cardinality queries, so we simply mention the updates required to adapt the family of circuits to concept cardinality queries.

- We need to introduce further gates in the second block to compute entailed role assertions.
- The circuits in the block "Deciding demanding individuals" are no longer required.
- Each block dedicated to a particular strategy simplifies as we no longer need to compare the size of positive vs negative critical elements: each strategy still comes with a specific amount of additional matches $|\sigma(\mathcal{D}_\sigma)|$ due to demanding roles, again introduced through constant gates, and counting ABox matches needs to be slightly updated from the role setting to the concept one.

A.3 TC^0-hardness.

To match our TC^0 membership results, it is natural to investigate the TC^0-hardness of our problem in these situations. Recall that we assumed, in order to exclude trivial cases, that our query predicate is satisfiable (Section 2, Cardinality Queries). We show that this assumption is sufficient to obtain TC^0-hardness (for any DL-Lite$^{\mathcal{H}}_{core}$ TBox), and it also necessary as the excluded situations can be decided within AC^0 (which we recall is the circuit complexity class obtained from TC^0 by disallowing threshold gates). We thus prove the following statement:

Theorem 10 (TC^0-hard / in AC^0). *Let q_{F} be a cardinality query and $\mathcal{T}$ be a DL-Lite$^{\mathcal{H}}_{core}$ TBox. If the query predicate* F *is satisfiable w.r.t. $\mathcal{T}$, then* $\mathrm{OMQA}(\mathcal{T}, q)$ *is TC^0-hard. Otherwise it is in AC^0.*

The argument for AC^0 membership is trivial: a cardinality query with an unsatisfiable predicate admits as certain answers precisely those intervals of the form $[0, M]$, since every model will contain 0 matches.

For both concept and role cardinality queries, we show TC^0-hardness by AC^0-reduction from the NUMONES problem, known to be TC^0-complete [Aehlig *et al.*, 2007].

	NUMONES
Input	Integer $k \geq 1$ (given in binary) and binary string X.
Output	Is the number of 1-bits in X at least k?

We note that we cannot reuse the TC^0-hardness proof given in [Bienvenu *et al.*, 2020], since that result used a rooted counting query coupled with an empty TBox. By contrast, we consider non-empty TBoxes which may include existential axioms, and our queries may match to unnamed elements.

Proof for concept cardinality queries. Let q_{C} be our concept cardinality query and assume C is satisfiable w.r.t. our TBox $\mathcal{T}$. Set $\mathcal{K}_{(\mathcal{T},q)} := (\mathcal{T}, \{\mathrm{C}(\mathsf{a})\})$. Our assumption ensures $\mathcal{K}_{(\mathcal{T},q)}$ is satisfiable hence its canonical interpretation (model) $\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}$ is indeed a model. Let (k, X) be an instance of NUMONES. Consider the following ABox:

$$\begin{aligned}\mathcal{A} =& \{\mathrm{A}(\mathsf{aux}_1) \mid \mathsf{a} \in \mathrm{A}^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}\} \cup \{\mathrm{R}'(\mathsf{aux}_1, \mathsf{aux}_\mathrm{R}) \mid \mathsf{a}\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\} \\ &\cup \{\mathrm{A}(\mathsf{b}) \mid \text{bit } b \text{ of } X \text{ is equal to } 1, \mathsf{a} \in \mathrm{A}^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}\} \cup \{\mathrm{R}'(\mathsf{b}, \mathsf{aux}_\mathrm{R}) \mid \text{bit } b \text{ of } X \text{ is equal to } 1, \mathsf{a}\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\} \\ &\cup \{\mathrm{A}(\mathsf{aux}_\mathrm{R}) \mid w\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}, \mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \mathrm{A}\} \cup \{\mathrm{R}'(\mathsf{aux}_\mathrm{T}, \mathsf{aux}_\mathrm{R}) \mid w\mathrm{TR} \in \Delta^{\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\}\end{aligned}$$

Note that in particular that $\mathcal{A}$ will contain $\mathrm{C}(\mathsf{b})$ for every 1-bit b of X, as well as $\mathrm{C}(\mathsf{aux}_1)$. The auxiliary individual aux_1 mimics a 1-bit from X in order to appropriately handle the case in which X doesn't contain any such bit. As the notation suggests, auxiliary individuals aux_R are intended to receive all needed outgoing roles R from other elements (so aux_R is intended to satisfy the concept $\exists\mathrm{R}^-$). Note that by construction the interpretation based upon $\mathcal{A}$ already satisfies all of the TBox axioms. In particular, this means that there exists a model of $(\mathcal{T}, \mathcal{A})$ all of whose matches are already present in $\mathcal{A}$. We can thus focus on counting the matches explicitly given in $\mathcal{A}$.

Observe that the number m of matches of q_{C} *among the auxiliary elements* only depends on the OMQ $(q_{\mathrm{C}}, \mathcal{T})$. In particular notice that aux_1 always provides a match, hence $m \geq 1$. It is straightforward to verify that $m + k$ is a certain answer for q over $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ iff $(k, X) \in$ NUMONES. Moreover, the input $(\mathcal{A}, m + k)$ to our OMQA problem can be computed from (k, X) by an AC^0 circuit (recall that binary integer addition is known to be computable in AC^0). □

The proof for role cardinality queries is a bit more involved but follows the same idea.

Proof for role cardinality queries. Let q_{S} be our concept cardinality query and assume S is satisfiable w.r.t. our TBox $\mathcal{T}$. Set $\mathcal{K}_{(\mathcal{T},q)} := (\mathcal{T}, \{\mathrm{S}(\mathsf{a}_1, \mathsf{a}_2)\})$. Our assumption ensures $\mathcal{K}_{(\mathcal{T},q)}$ is satisfiable hence its canonical interpretation $\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}$ is indeed a model. Let (k, X) be an instance of NUMONES. Consider the ABox $\mathcal{A}$ containing assertions:

$$\{\mathrm{R}(\mathsf{aux}_1, \mathsf{aux}_2) \mid (\mathsf{a}_1, \mathsf{a}_2) \in \mathrm{R}^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}\} \cup \{\mathrm{A}(\mathsf{aux}_j) \mid j \in \{1,2\}, \mathsf{a}_j \in \mathrm{A}^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}\}$$
$$\cup\, \{\mathrm{R}(\mathsf{b}_1, \mathsf{b}_2) \mid \text{bit } b \text{ of } X \text{ is equal to } 1, (\mathsf{a}_1, \mathsf{a}_2) \in \mathrm{R}^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}\} \cup \{\mathrm{A}(b_j) \mid j \in \{1,2\}, \text{bit } b \text{ of } X \text{ is equal to } 1, \mathsf{a}_j \in \mathrm{A}^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}\}$$
$$\cup\, \{\mathrm{R}'(\mathsf{aux_j}, \mathsf{aux}_{\mathrm{R}}) \mid j \in \{1,2\}, \mathsf{a}_j\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\}$$
$$\cup\, \{\mathrm{R}'(\mathsf{b_j}, \mathsf{aux}_{\mathrm{R}}) \mid j \in \{1,2\}, \text{bit } b \text{ of } X \text{ is equal to } 1, \mathsf{a}_j\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\}$$
$$\cup\, \{\mathrm{A}(\mathsf{aux}_{\mathrm{R}}) \mid w\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}, \mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \mathrm{A}\} \cup \{\mathrm{R}'(\mathsf{aux}_{\mathrm{T}}, \mathsf{aux}_{\mathrm{R}}) \mid w\mathrm{T}\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}(\mathcal{T},q)}}, \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{R}'\}$$

Here again we note that by construction all of the TBox axioms are already satisfied, so we only need to concern ourselves with matches explicitly given in the ABox. In order to compute this number, let us first denote by n the number of matches in $\mathcal{C}_{\mathcal{K}_{(\mathcal{T},q)}}$ that involve either a_1, a_2, or both individuals. Observe that n only depends on the OMQ $(q_{\mathrm{S}}, \mathcal{T})$ and that we have $n \geq 1$ due to the assertion $\mathrm{S}(\mathsf{a}_1, \mathsf{a}_2)$. Considering now the ABox $\mathcal{A}$, we will denote by m the number of matches of q_{S} in $\mathcal{A}$ which *only involve auxiliary elements*; this value depends solely on the OMQ $(q_{\mathrm{S}}, \mathcal{T})$. It is not hard to see that $m + k \times n$ is a certain answer for q over $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ iff $(k, X) \in$ NUMONES.

To complete the proof, we note that we can compute $(\mathcal{A}, m + k \times n)$ from (k, X) in AC^0 due to the fact that n is a constant. Indeed, $k \times n$ can be computed by performing a constant number of additions of k with itself, there is one more addition step to combine with m, and we can compute the addition of a constant number of binary numbers in AC^0. We point out that it is essential here that n be a constant value, since the problem of multiplying two arbitrary binary inputs is known to *not* be computable in AC^0. □

B Proof of Theorem 2.

The following definition introduces terminology for speaking about a specific kind of match that is guaranteed to be found in any model:

Definition 19 (ABox matches)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. An* ABox *match for q_{S} is a pair $(\mathsf{a}, \mathsf{b}) \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{K} \models \mathrm{S}(\mathsf{a}, \mathsf{b})$.*

B.1 Proof of coNP-hardness.

We begin by proving coNP-hardness of $\mathrm{OMQA}(q_{\mathrm{S}}, \mathcal{T})$ in the case where the input DL-Lite$^{\mathcal{H}}_{pos}$ TBox $\mathcal{T}$ admits a non-trivial propagation of either S or S^-. Note that membership in coNP is an immediate consequence of existing results on counting queries [Kostylev and Reutter, 2015].

Let us thus assume that $\mathcal{T}$ has a non-trivial propagation B, R_1, R_2 of S (the case of a non-trivial propagation of S^- being symmetrical). We proceed by reduction from the SET COVER problem, and distinguish two cases based on the shape of B.

Consider an instance $(\mathcal{U}, \mathcal{S}, k)$ of SET COVER: each element $u \in \mathcal{U}$ occurs in at least one subset of $\mathcal{S}$, we denote s_u by such a subset. We introduce an individual name u for each $u \in \mathcal{U}$, and an individual name s for each $s \in \mathcal{S}$. The individual introduced for the subset s_u is denoted $\mathsf{s_u}$. We further introduce auxiliary individuals a and b.

We now provide the reductions for the two cases.

Case 1: $\mathrm{B} \in \mathsf{N_C}$ **or** $\mathrm{B} = \exists\mathrm{T}$ **with** $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}$**.** Consider the ABox:

$$\begin{array}{lll}
\mathcal{A} := & \left| \begin{array}{ll} \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{U}\} & \text{if } \mathrm{B} \in \mathsf{N_C} \\ \{\mathrm{T}(\mathsf{u}, \mathsf{a}) \mid u \in \mathcal{U}\} & \text{else, with } \mathrm{B} = \exists\mathrm{T} \end{array} \right. & \text{(Representation of the elements)} \\
& \cup \{\mathrm{S}(\mathsf{u}, \mathsf{s}) \mid u \in s, s \in \mathcal{S}\} & \text{(Representation of the subsets)} \\
& \cup \left\{ \mathrm{U}(\mathsf{u}, \mathsf{a}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N_R^{\pm}}, u \in \mathcal{U} \\ \mathcal{T} \models \mathrm{B} \sqsubseteq \exists\mathrm{U} \\ \mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} & \text{(Introduction of non-relevant roles for elements)} \\
& \cup \left\{ \mathrm{U}(\mathsf{s}, \mathsf{b}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N_R^{\pm}}, s \in \mathcal{S} \\ \mathcal{T} \models \exists\mathrm{S}^- \sqsubseteq \exists\mathrm{U} \\ \mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} & \text{(Introduction of subroles of S for subsets)} \\
& \cup \left\{ \mathrm{U}(\mathsf{s}, \mathsf{a}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N_R^{\pm}}, s \in \mathcal{S} \\ \mathcal{T} \models \exists\mathrm{S}^- \sqsubseteq \exists\mathrm{U} \\ \mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} & \text{(Introduction of other roles for subsets)} \\
& \cup \{\mathrm{U}(\mathsf{a}, \mathsf{a}) \mid \mathrm{U} \in \mathsf{sig}(\mathcal{T})_{\mathsf{R}}\} & \text{(Saturation of the auxiliary individual a)} \\
& \cup \{\mathrm{U}(\mathsf{b}, \mathsf{b}) \mid \mathrm{U} \in \mathsf{sig}(\mathcal{T})_{\mathsf{R}}\} & \text{(Saturation of the auxiliary individual b)}
\end{array}$$

Consider the KB $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Notice that due to the many role assertions included in the ABox, all of the anonymous elements in $\mathcal{C}_{\mathcal{K}}$ are of the form $\mathsf{u}\mathrm{U}w$ with $u \in \mathcal{U}$, w some word, and $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$ but $\mathcal{T} \not\models \mathrm{U}^- \sqsubseteq \mathrm{S}$ (because we are considering a non-trivial propagation, the role U cannot satisfy Condition 1 of Definition 11). Notice also that (the negation of) Condition 2 from the same definition further ensures that there is no ABox match (b, s), with $s \in \mathcal{S}$.

Let us denote by $m_{\mathcal{A}}$ be the number of matches for q_{S} present in the ABox $\mathcal{A}$. In particular, $m_{\mathcal{A}} \geq \sum_{s \in \mathcal{S}} |s| + 1$, due to the representation of the subsets and the saturation of a. We claim:

$$[m_{\mathcal{A}} + k + 1, +\infty] \text{ is a certain answer for } q_{\mathrm{S}} \text{ w.r.t. } \mathcal{K} \;\Leftrightarrow\; (\mathcal{U}, \mathcal{S}, k) \notin \text{SET COVER}$$

$(\Rightarrow)$. Assume $(\mathcal{U}, \mathcal{S}, k) \in$ SET COVER. Take some k-cover $F \subseteq \mathcal{S}$ of $\mathcal{U}$. For each $u \in s$ with $s \in F$ and each positive role U such that $\mathcal{T} \models \mathrm{B} \sqsubseteq \exists\mathrm{U}$ and $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$, enrich the ABox $\mathcal{A}$ with the assertion $\mathrm{U}(\mathsf{u}, \mathsf{s})$. Saturate now the used subsets, that is, for each $s \in F$, add the assertions $\mathrm{U}(\mathsf{s}, \mathsf{s})$ for all $\mathrm{U} \in \mathsf{sig}(\mathcal{T})_{\mathsf{R}}$.

Up to introducing the entailed concepts, the resulting interpretation $\mathcal{I}_F$ (based upon the described enriched ABox) is a model, as we introduced the missing roles for the elements, the used subsets are now saturated, and the non-used subsets were already given their needed roles.

In addition to the $m_{\mathcal{A}}$ ABox matches, each used subset provides one additional match since the assertion $\mathrm{S}(\mathsf{s}, \mathsf{s})$ has been added. Recall Condition 1 from Definition 11 which ensures no match with shape $\mathrm{S}(\mathsf{s}, \mathsf{u})$ is introduced, hence the roles added between the elements and subset individuals only reuse pre-existing matches. We thus obtain a model with exactly $m_{\mathcal{A}} + k$ matches, and thus a countermodel for $[m_{\mathcal{A}} + k + 1, +\infty]$ being a certain answer.

($\Leftarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \notin$ SET COVER. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_\mathcal{K} \to \mathcal{I}$. For each $u \in \mathcal{U}$, we associate a subset $\rho(u) := s$ if $f(\mathsf{u}\mathrm{R}_1) = \mathsf{s}$ and $u \in s \in \mathcal{S}$, otherwise set $\rho(u) := s_u$. The image $\rho(\mathcal{U})$ is a covering of $\mathcal{U}$, hence $|\rho(\mathcal{U})| \geq k+1$. By definition, for each $s \in \rho(\mathcal{U})$ there exists $u \in \mathcal{U}$ such that: either $f(\mathsf{u}\mathrm{R}_1) = \mathsf{s}$ with $u \in s \in \mathcal{S}$, or $f(\mathsf{u}\mathrm{R}_1) \neq \mathsf{s}'$ for all $u \in s' \in \mathcal{S}$.

In the first case, since $\mathcal{T} \models \{\mathrm{R}_1 \sqsubseteq \mathrm{S}, \exists \mathrm{R}_1^- \sqsubseteq \exists \mathrm{R}_2, \mathrm{R}_2 \sqsubseteq \mathrm{S}\}$ (due to the assumed non-trivial propagation), we focus on the pair $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$. If $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$ is not already an ABox match, then we have found an additional match. Otherwise $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$ is an ABox match (i.e. $\mathcal{K} \models \mathrm{S}(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$. By construction of $\mathcal{A}$, this must be due to S propagating a subrole U of S (see 'Introduction of subroles of S for subsets' in the definition of the ABox), which means we have $f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2) = \mathsf{b}$. Condition 2 from Definition 11 applied with U provides $\mathcal{T} \models \mathrm{R}_2^- \sqsubseteq \mathrm{S}$, hence $(f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2), f(\mathsf{u}\mathrm{R}_1))$ is a new match (recall that (b, s) is not an ABox match!). In the second case, $(f(\mathsf{u}), f(\mathsf{u}\mathrm{R}_1))$ is a new match (in the case where $\mathrm{B} = \exists \mathrm{T}$, simply recall that $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}$). Therefore we can conclude that there are at least $m_\mathcal{A} + k + 1$ matches in $\mathcal{I}$.

Case 2: $\mathrm{B} = \exists \mathrm{T}$ **with** $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}$**.** Consider the ABox:

$$
\begin{aligned}
\mathcal{A} := \; & \{\mathrm{T}(\mathsf{u}, \mathsf{s}) \mid u \in s \in \mathcal{S}\} && \text{(Representation of the subsets)}\\
& \cup \left\{ \mathrm{U}(\mathsf{u}, \mathsf{a}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N}_\mathsf{R}^\pm, u \in \mathcal{U} \\ \mathcal{T} \models \exists \mathrm{T} \sqsubseteq \exists \mathrm{U} \\ \mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} && \text{(Introduction of non-relevant roles for elements)}\\
& \cup \left\{ \mathrm{U}(\mathsf{s}, \mathsf{b}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N}_\mathsf{R}^\pm, s \in \mathcal{S} \\ \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{U} \vee \mathcal{T} \models \exists \mathrm{T}^- \sqsubseteq \exists \mathrm{U} \\ \mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} && \text{(Introduction of subroles of S for subsets)}\\
& \cup \left\{ \mathrm{U}(\mathsf{s}, \mathsf{a}) \;\middle|\; \begin{array}{l} \mathrm{U} \in \mathsf{N}_\mathsf{R}^\pm, s \in \mathcal{S} \\ \mathcal{T} \models \exists \mathrm{S}^- \sqsubseteq \exists \mathrm{U} \\ \mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S} \end{array} \right\} && \text{(Introduction of other roles for subsets)}\\
& \cup \{\mathrm{U}(\mathsf{a}, \mathsf{a}) \mid \mathrm{U} \in \mathsf{sig}(\mathcal{T})_\mathsf{R}\} && \text{(Saturation of the auxiliary individual a)}\\
& \cup \{\mathrm{U}(\mathsf{b}, \mathsf{b}) \mid \mathrm{U} \in \mathsf{sig}(\mathcal{T})_\mathsf{R}\} && \text{(Saturation of the auxiliary individual b)}
\end{aligned}
$$

We again set $\mathcal{K} := (\mathcal{T}, \mathcal{A})$ and observe that the anonymous elements in $\mathcal{C}_\mathcal{K}$ are all of the form $\mathsf{u}\mathrm{U}w$ with $u \in \mathcal{U}$, $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$, and w a word. Notice again that *no* pair (b, s), with $s \in \mathcal{S}$, is an ABox match, which is due here to *both* Conditions 2 and 3 from Definition 11.

As before, we denote by $m_\mathcal{A}$ the number of matches for q_S in the ABox $\mathcal{A}$. In particular $m_\mathcal{A} \geq \sum_{s \in \mathcal{S}} |s| + 1$, due to the representation of the problem instance and the saturation of a. We claim:

$$[m_\mathcal{A} + k + 1, +\infty] \text{ is a certain answer for } q_\mathrm{S} \text{ w.r.t. } \mathcal{K} \iff (\mathcal{U}, \mathcal{S}, k) \notin \text{SET COVER}$$

($\Rightarrow$). The proof is essentially the same as for Case 1.

($\Leftarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \notin$ SET COVER. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_\mathcal{K} \to \mathcal{I}$. For each $u \in \mathcal{U}$, we associate a subset $\rho(u) := s$ if $f(\mathsf{u}\mathrm{R}_1) = \mathsf{s}$ and $u \in s \in \mathcal{S}$, otherwise set $\rho(u) := s_u$. The image $\rho(\mathcal{U})$ is a covering of $\mathcal{U}$, hence $|\rho(\mathcal{U})| \geq k+1$. By definition, for each $s \in \rho(\mathcal{U})$, there exists $u \in \mathcal{U}$ such that: either $f(\mathsf{u}\mathrm{R}_1) = \mathsf{s}$ with $u \in s \in \mathcal{S}$, or $f(\mathsf{u}\mathrm{R}_1) \neq \mathsf{s}'$ for all $u \in s' \in \mathcal{S}$.

In the first case, since $\mathcal{T} \models \{\mathrm{R}_1 \sqsubseteq \mathrm{S}, \exists \mathrm{R}_1^- \sqsubseteq \exists \mathrm{R}_2, \mathrm{R}_2 \sqsubseteq \mathrm{S}\}$ (due to the assumed non-trivial propagation), we focus on the pair $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$. If $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$ is not already an ABox match, then we are done. Otherwise $(f(\mathsf{u}\mathrm{R}_1), f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2))$ is an ABox match, then by construction of $\mathcal{A}$, it must be due to either S or T propagating a subrole U of S, in particular, we get $f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2) = \mathsf{b}$. Condition 2 (resp. Condition 3) from Definition 11 applied with U provides $\mathcal{T} \models \mathrm{R}_2^- \sqsubseteq \mathrm{S}$, hence $(f(\mathsf{u}\mathrm{R}_1\mathrm{R}_2), f(\mathsf{u}\mathrm{R}_1))$ is a new match (recall (b, s) is not an ABox match!). In the second case, $(f(\mathsf{u}), f(\mathsf{u}\mathrm{R}_1))$ is a new match. Therefore there are at least $m_\mathcal{A} + k + 1$ matches in $\mathcal{I}$.

B.2 Proof of reduction from co-Maximum-Matching.

We now turn to the second part of Theorem 2, which characterizes the complexity of $\text{OMQA}(\mathcal{T}, q_\mathrm{S})$ in the case in which the DL-Lite$^\mathcal{H}_{pos}$ TBox $\mathcal{T}$ admits a non-trivial pairing of S but does not have any non-trivial propagation of S or S^-. We start by proving a logspace reduction from the complement of MAXIMUM MATCHING to our problem $\text{OMQA}(\mathcal{T}, q_\mathrm{S})$. Let us first recall the definition of MAXIMUM MATCHING:

As explained in the main text, the latter problem is known to be equivalent, up to logspace reductions, to the better known PERFECT MATCHING problem [Rabin and Vazirani, 1989]. Thus, the reduction we give in this subsection also proves a reduction from PERFECT MATCHING to our problem. (A reduction in the other direction is the object of the next subsection.)

	MAXIMUM MATCHING
Input	Non-oriented graph $\mathcal{G}$, integer k.
Output	Decide if there exists a matching of $\mathcal{G}$ with size at least k.

Consider a DL-Lite$^{\mathcal{H}}_{pos}$ TBox $\mathcal{T}$ that admits a non-trivial pairing of S and does not admit any non-trivial propagation of S or S^-. Let B and R verify the pairing conditions, that is,

$$\mathcal{T} \models \mathrm{B} \sqsubseteq \exists \mathrm{R} \quad \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S} \quad \mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^- \quad \mathcal{T} \not\models \mathrm{S} \sqsubseteq \mathrm{S}^-$$

and if $\mathrm{B} = \exists \mathrm{T}$, then either $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}$ or $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}^-$.

Consider an instance of MAXIMUM MATCHING given by the undirected graph $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ and integer k. Let $\leq_{\mathcal{V}}$ be any total order on the vertices of $\mathcal{G}$. We encode $\mathcal{G}$ using the following ABox $\mathcal{A}_{\mathcal{G}}$:

$$\begin{aligned}\mathcal{A}_{\mathcal{G}} := & \left| \begin{array}{ll} \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{V}\} & \text{if } \mathrm{B} \in \mathsf{N_C} \\ \{\mathrm{T}(\mathsf{u}, \mathsf{a}) \mid u \in \mathcal{V}\} & \text{else, with } \mathrm{B} = \exists \mathrm{T} \end{array} \right. && \text{(Representation of the vertices)} \\ & \cup \{\mathrm{S}(\mathsf{u}, \mathsf{v}) \mid \{u, v\} \in \mathcal{E}, u \leq_{\mathcal{V}} v\} && \text{(Representation of the edges)} \\ & \cup \{\mathrm{U}(\mathsf{a}, \mathsf{a}) \mid \mathrm{U} \in \mathsf{sig}(\mathcal{T})_{\mathsf{R}}\} && \text{(Saturation of the auxiliary individual)}\end{aligned}$$

Let $\mathcal{K}_{\mathcal{G}}$ be the KB $(\mathcal{T}, \mathcal{A}_{\mathcal{G}})$. Let $m_{\mathcal{A}}$ be the number of matches in the ABox. Notice each edge $\{u, v\}$ gives one match in the ABox, through the added assertion $\mathrm{S}(\mathsf{u}, \mathsf{v})$ with $u \leq_{\mathcal{V}} v$, and *exactly* one as $\mathcal{T} \not\models \mathrm{S}^- \sqsubseteq \mathrm{S}$. We claim:

$$[m_{\mathcal{A}} + |\mathcal{V}| - k + 1, +\infty] \text{ is a certain answer for } q_{\mathrm{S}} \text{ w.r.t. } \mathcal{K}_{\mathcal{G}} \;\Leftrightarrow\; (\mathcal{G}, k) \notin \text{MAXIMUM MATCHING}$$

Notice that both $\mathcal{A}_{\mathcal{G}}$ and the integer $m_{\mathcal{A}} + |\mathcal{V}| - k + 1$ are easily computable in logarithmic space from any reasonable representation of the instance $(\mathcal{G}, k)$, so we will get the desired within logspace reduction.

($\Leftarrow$). Assume $(\mathcal{G}, k) \notin$ MAXIMUM MATCHING. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$. Consider the following matching:

$$M_{\mathcal{I}} := \{\{u, v\} \in \mathcal{E} \mid f(\mathsf{u}\mathrm{R}) = \mathsf{v}, f(\mathsf{v}\mathrm{R}) = \mathsf{u}\}$$

As f is a function, it is clear that each vertex is incident to at most one of the edges from $M_{\mathcal{I}}$, ensuring $M_{\mathcal{I}}$ is a matching. In particular, it yields $|M_{\mathcal{I}}| < k$. Each edge from $M_{\mathcal{I}}$ provides exactly one additional match, since there was already exactly one match per edge, and the role R is a subrole of both S and S^-. Each vertex that is not incident to any edge in $M_{\mathcal{I}}$ provides at least one additional match: recall that since $\mathcal{T} \not\models \mathrm{T} \sqsubseteq \mathrm{S}$ or $\mathcal{T} \not\models \mathrm{T}^- \sqsubseteq \mathrm{S}$, either $(f(\mathsf{u}), f(\mathsf{u}\mathrm{R}))$ or $(f(\mathsf{u}\mathrm{R}), f(\mathsf{u}))$ is a new match.Therefore there are at least $m_{\mathcal{A}} + |M_{\mathcal{I}}| + |\mathcal{V}| - 2|M_{\mathcal{I}}| > m_{\mathcal{A}} + |\mathcal{V}| - k$ matches in $\mathcal{I}$.

($\Rightarrow$). Assume $(\mathcal{G}, k) \in$ MAXIMUM MATCHING. Consider a matching $M \subseteq \mathcal{E}$ with $|M| \geq k$. Consider the enriched ABox $\mathcal{A}_M$ such that for each $\{u, v\} \in M$ and each positive role $\mathrm{U} \in \mathsf{N}^{\pm}_{\mathsf{R}}$, we have $\mathrm{U}(\mathsf{u}, \mathsf{v}) \in \mathcal{A}_M$. This yields exactly one additional match per edge in M, again because exactly one match per edge was already present. For each $u \in \mathcal{V}$ such that u is not incident to any edge in M, also add all the assertions $\mathrm{U}(\mathsf{u}, \mathsf{u}) \in \mathcal{A}_M$. This yields exactly one new match per vertex not incident to any edge in M. Up to adding the entailed concepts wherever needed, this provides a model with at most: $m_{\mathcal{A}} + |\mathcal{E}| + |\mathcal{V}| - 2|\mathcal{E}| \leq m_{\mathcal{A}} + |\mathcal{V}| - k$ matches of q_{S}, being a counter model for $[m_{\mathcal{A}} + |\mathcal{V}| - k + 1, +\infty]$.

B.3 Proof of reduction to co-Maximum-Matching.

In this subsection, we complete the proof of the second part of Theorem 2 by showing how OMQA$(\mathcal{T}, q_{\mathrm{S}})$ can be reduced, via logspace reductions, to the complement of MAXIMUM MATCHING in the case in which $\mathcal{T}$ is a DL-Lite$^{\mathcal{H}}_{pos}$ without non-trivial propagation. Again, this yields a logspace reduction to the complement of PERFECT MATCHING due to the previously cited logspace-equivalence between these two matching problems. We start with some general remarks.

Compared with the tractable settings of Section 3, with DL-Lite$^{\mathcal{H}}_{pos}$, we no longer need to take care of negative concept inclusions, but we will now need to take into account role inclusions when handling role cardinality queries. In particular, role inclusions allow for a class $\mathcal{B}_{\mathcal{T}}$ of what we call *bipotent* roles, *i.e.,* subroles of both S and S^- (formally: positive roles U such that $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$ and $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}^-$). On the other hand, the class $\mathcal{N}_{\mathcal{T}}$ of positive roles not being a subrole of S nor a subrole of S^- are called *nilpotent* (formally: positive roles U such that $\mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S}$ and $\mathcal{T} \not\models \mathrm{U} \sqsubseteq \mathrm{S}^-$).

Recall that our previous notion of type aimed to characterize individuals based on their ability to receive some roles (is there a negative concept preventing my anonymous element to merge with this individual?) and to provide ABox matches on which to fold (is there an ABox match on which to fold matches propagated by a given anonymous element?). This typing notion needs to be modified for the setting we consider here. On the one hand, negative inclusions being disallowed, all individuals are able to receive all roles. On the other hand, we must now distinguish ABox matches on which we can fold bipotent roles from those on which we can only fold non-bipotent roles. We also extend our typing notion to nilpotent roles: their type being a characterization of the subroles they propagate.

Definition 20 (Type of an individual, of a nilpotent role)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. The* type $\theta_{\mathcal{K}}(d)$ of an element $d \in \mathsf{Ind}(\mathcal{A})$ over $\mathcal{K}$ *is the set:*

$$\theta_{\mathcal{K}}(d) \ := \ \left\{ \mathfrak{R} \ \middle| \ \begin{array}{l} \mathfrak{R} \in \{\{\mathrm{S}, \mathrm{S}^-\}, \{\mathrm{S}\}, \{\mathrm{S}^-\}\} \\ \exists e \in \Delta^{\mathcal{C}_{\mathcal{K}}} \ \forall \mathrm{R} \in \mathfrak{R}, \ \mathcal{C}_{\mathcal{K}} \models \mathrm{R}(d, e) \end{array} \right\}.$$

The type $\theta_{\mathcal{K}}(R)$ of a nilpotent role $\mathrm{R} \in \mathcal{N}_{\mathcal{T}}$ over $\mathcal{K}$ *is the set:*

$$\theta_{\mathcal{K}}(R) \ := \ \left\{ \mathfrak{U} \ \middle| \ \begin{array}{l} \mathfrak{U} \in \{\{\mathrm{S}, \mathrm{S}^-\}, \{\mathrm{S}\}, \{\mathrm{S}^-\}\} \\ \exists \mathrm{V} \in \mathsf{N}_{\mathsf{R}}^{\pm}, \forall \mathrm{U} \in \mathfrak{U}, \ \mathcal{T} \models \exists \mathrm{R}^- \sqsubseteq \exists \mathrm{V} \ \wedge \ \mathcal{T} \models \mathrm{V} \sqsubseteq \mathrm{U} \end{array} \right\}.$$

The set of possible types is $\Theta := \{\{\{\mathrm{S}, \mathrm{S}^-\}, \{\mathrm{S}\}, \{\mathrm{S}^-\}\}, \{\{\mathrm{S}\}, \{\mathrm{S}^-\}\}, \{\{\mathrm{S}^-\}\}, \{\{\mathrm{S}\}\}, \emptyset\}$.

Following the line of the TC^0 membership proofs for role cardinality queries, we are still interested in *demanding elements*. In particular, bipotent roles might create a new kind of such elements: *bidemanding elements*, which are defined as follows.

Definition 21 (Bidemanding elements)**.** *We distinguish* bidemanding individuals $\mathcal{D}^{\pm}_{\mathcal{K}}$ *and* bidemanding roles $\mathcal{D}^{\pm}_{\sigma}$*:*

$$\mathcal{D}^{\pm}_{\mathcal{K}} \ := \ \left\{ \mathsf{a} \ \middle| \ \begin{array}{l} \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \\ \{\mathrm{S}, \mathrm{S}^-\} \in \theta_{\mathcal{K}}(\mathsf{a}) \\ \forall \mathsf{b} \in \mathsf{Ind}(\mathcal{A}), \ (\mathcal{K} \not\models \mathrm{S}(\mathsf{a}, \mathsf{b})) \vee (\mathcal{K} \not\models \mathrm{S}^-(\mathsf{a}, \mathsf{b})) \end{array} \right\} \qquad \mathcal{D}^{\pm}_{\sigma} \ := \ \left\{ \mathrm{R} \ \middle| \ \begin{array}{l} \mathrm{R} \in \mathsf{gen}_{\mathcal{K}} \\ \{\mathrm{S}, \mathrm{S}^-\} \in \theta_{\mathcal{K}}(R) \end{array} \right\}$$

Notice here the assumptions that bidemanding roles should be nilpotent and not only "non-bipotent" (which should feel more natural in this case... but would provide a non-trivial propagation schema!).

We now redefine for our setting the notions of positive / negative demanding individuals.

Definition 22 (Demanding individuals in a KB)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. Define* positive demanding individuals $\mathcal{D}^{+}_{\mathcal{K}}$*, resp.* negative demanding individuals $\mathcal{D}^{-}_{\mathcal{K}}$ *as:*

$$\mathcal{D}^{+}_{\mathcal{K}} \ := \ \left\{ \mathsf{a} \ \middle| \ \begin{array}{l} \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \\ \mathsf{a} \notin \mathcal{D}^{\pm}_{\mathcal{K}} \\ \{\mathrm{S}\} \in \theta_{\mathcal{K}}(\mathsf{a}) \\ \forall \mathsf{b} \in \mathsf{Ind}(\mathcal{A}), \ \mathcal{K} \not\models \mathrm{S}(\mathsf{a}, \mathsf{b}) \end{array} \right\} \qquad \mathcal{D}^{-}_{\mathcal{K}} \ := \ \left\{ \mathsf{a} \ \middle| \ \begin{array}{l} \mathsf{a} \in \mathsf{Ind}(\mathcal{A}) \\ \mathsf{a} \notin \mathcal{D}^{\pm}_{\mathcal{K}} \\ \{\mathrm{S}^-\} \in \theta_{\mathcal{K}}(\mathsf{a}) \\ \forall \mathsf{b} \in \mathsf{Ind}(\mathcal{A}), \ \mathcal{K} \not\models \mathrm{S}^-(\mathsf{a}, \mathsf{b}) \end{array} \right\}$$

Strategies are no longer needed in our setting, as negative inclusions have been removed. Due to the adaptation of our notions of types, a choice of well-typed elements is redefined to now apply to types (of roles) instead of applying to the positive roles themselves. This is simply because the absence of negative concept inclusions allows us to apply the same choice to all nilpotent roles having the same type.

Definition 23 (Choice of well-typed element)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. A choice of well-typed elements for $\mathcal{K}$ is a function* $\mathsf{choice}_{\mathcal{K}} : \Theta \to \mathsf{Ind}(\mathcal{A})$ *such that for each type $\mathsf{t} \in \Theta$, if there exists a nilpotent generated role* $\mathrm{R} \in \mathsf{gen}_{\mathcal{K}} \cap \mathcal{N}_{\mathcal{T}}$ *such that* $\theta_{\mathcal{K}}(\mathrm{R}) = \mathsf{t}$*, then we have* $\mathsf{t} \subseteq \theta_{\mathcal{K}}(\mathsf{choice}_{\mathcal{K}}(\mathsf{t}))$.

We now state our fundamental theorem, which proves that, if a choice of well-typed elements is available and in the absence of demanding individuals, then the canonical model can fully fold on the individuals without creating any additional match. This central property crucially relies on the absence of a non-trivial propagation schema.

Theorem 11 (Fitting model)**.** *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. If there is a choice* $\mathsf{choice}_{\mathcal{K}}$ *of well-typed elements over* $\mathsf{Ind}(\mathcal{A})$ *and if $\mathcal{K}$ admits no bidemanding individuals, then there exists a mapping* $\chi : \Delta^{\mathcal{C}_{\mathcal{K}}} \to \mathsf{Ind}(\mathcal{A})$ *s.t the matches in the resulting model* $\chi(\mathcal{C}_{\mathcal{K}})$ *are exactly the ABox matches.*

Before starting with the proper proof, we need some additional definitions.

First, why do we start from a demanding individual free KB? We want to take advantage of the absence of non-trivial propagation, in particular of violation of its Condition 2 (see Definition 11), which is involving roles generated by $\exists \mathrm{S}^-$ (resp $\exists \mathrm{S}$). Therefore, we somehow need these generated roles to be here as soon as possible: we need their causes, that are S-assertions, in our initial ABox.

Speaking about *causes*, take a look at Condition 3 from Definition 11. Here is a handy definition to take advantage of the cases in which this latter condition is broken.

Definition 24 (Cause of an element)**.** *Let $w\mathrm{R}$ be an anonymous element of $\Delta^{\mathcal{C}_{\mathcal{K}}}$. A* cause *of $w\mathrm{R}$ is a positive concept such that: if $w \in \mathsf{Ind}(\mathcal{A})$, then* $\mathsf{cause}(w\mathrm{R})$ *is either an atomic concept* B *such that* $\mathcal{K} \models \mathrm{B}(w)$ *and* $\mathcal{T} \models \mathrm{B} \sqsubseteq \exists \mathrm{R}$*, or a positive concept* $\exists \mathrm{T}$ *such that there exists some* b *with* $\mathcal{K} \models \mathrm{T}(w, \mathsf{b})$ *and* $\mathcal{T} \models \exists \mathrm{T} \sqsubseteq \exists \mathrm{R}$*. Otherwise* $w = w_0\mathrm{T}$*, then* $\mathsf{cause}(w\mathrm{R}) := \exists \mathrm{T}^-$.

Following this line, here is a definition capturing the role provided by a violation of Condition 1 (again from Definition 11).

Definition 25 (Leader of an element)**.** *For an element $w \in \Delta^{\mathcal{C}_{\mathcal{K}}}$, if there exists a positive role* U *such that* $w\mathrm{U} \in \Delta^{\mathcal{C}_{\mathcal{K}}}$*,* $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$ *and* $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}^-$*, then we pick such a role* U *and say it is the* leader *of the element w, denoted* $\mathsf{leader}(w)$.

Our construction proceed by induction on $\mathcal{C}_\mathcal{K}$, so here is the order we pick.

Definition 26 (Order on $\mathcal{C}_\mathcal{K}$)**.** *We pick an order $\leq$ on $\Delta^{\mathcal{C}_\mathcal{K}}$ such that: $\leq$ is breadth-first and for all $w \in \mathcal{C}_\mathcal{K}$, if $\mathsf{leader}(w)$ is defined, then $\forall \mathrm{R}, w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}} \Rightarrow w \cdot \mathsf{leader}(w) \leq w\mathrm{R}$.*

We are now all setup for the main construction. Here is some intuition before this two-page long definition. Recall we explore the canonical model, especially anonymous elements being words ending by a particular positive role. Whenever we encounter a nilpotent role, we send it on its choice, because if ever it propagates some non-nilpotent roles, then the choice of well-typed elements ensures there are some further pre-existing matches on which to fold. Otherwise (and that is a big otherwise), if we previously encountered a bipotent role or a bidirectionnal match (that is a pair of element (a, b) such that both (a, b) and (b, a) are pre-existing matches), then it is costless to reuse it (protip: that's what the "flag" is for!). Otherwise, we look for such a bidirectionnal match around which would solve all further problems. If none, then the role you are encountering surely isn't bipotent: a nilpotent role propagating a bipotent role could not have let you end up on an element without a bidirectionnal match around (it would contradict the definition of a choice of well-typed element!), and non-nilpotent nor bipotent roles propagating a bipotent role could not have either (it would violate the absence of non-trivial propagation!). Therefore, at this point, the role you are encountering is either a subrole of S or of S^-, but not both. In both cases, you are ensured to find a pre-existing match on which to fold (otherwise it would again violate either the choice of well-typed elements of the absence of non-trivial propagation).

Here is the more formal approach. Various properties are carried along the construction. Property 1 ensures nilpotent roles behave as expected. Properties 2^+ and 2^- ensures we stay within the ABox matches. Property 3 ensures the flag is used as expected. Property 4 ensures violations of Conditions 1 and 2 are being used. Property 5^+ and 5^- ensure violations of Condition 3 are being used.

Proof. By induction on $(\Delta^{\mathcal{C}_\mathcal{K}}, \leq)$, we build two mappings $\mathsf{flag} : \Delta^{\mathcal{C}_\mathcal{K}} \to \{0, 1\}$ and $\chi : \Delta^{\mathcal{C}_\mathcal{K}} \to \mathsf{Ind}(\mathcal{A})$ and ensure alongside that any element $e \in \Delta^{\mathcal{C}_\mathcal{K}}$ satisfies the following properties:

1. If $e = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ with R nilpotent, then $\chi(w\mathrm{R}) = \mathsf{choice}_\mathcal{K}(\theta_\mathcal{K}(\mathrm{R}))$.

2^+. If $e = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ and $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}$, then $\mathcal{K} \models \mathrm{S}(\chi(w), \chi(w\mathrm{R}))$.

2^-. If $e = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ and $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^-$, then $\mathcal{K} \models \mathrm{S}(\chi(w\mathrm{R}), \chi(w))$.

3. If $e = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ and $\mathsf{flag}(w\mathrm{R})$, then $\mathcal{K} \models \mathrm{S}(\chi(w), \chi(w\mathrm{R}))$ and $\mathcal{K} \models \mathrm{S}(\chi(w\mathrm{R}), \chi(w))$.
4. If $e = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ with R non-nilpotent and $\mathsf{leader}(w)$ is defined, then $\mathsf{flag}(w\mathrm{R})$ and $\chi(w\mathrm{R}) = \chi(w \cdot \mathsf{leader}(w))$.

5^+. If $e = w\mathrm{R}_1 \in \Delta^{\mathcal{C}_\mathcal{K}}$ with $\mathcal{T} \models \mathrm{R}_1 \sqsubseteq \mathrm{S}$ and $\neg\mathsf{flag}(w\mathrm{R}_1)$ and $\mathsf{cause}(w\mathrm{R}_1) = \exists \mathrm{T}$ with $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}$ and such that there exists $w\mathrm{R}_1\mathrm{R}_2 \in \Delta^{\mathcal{C}_\mathcal{K}}$ with $\mathcal{T} \models \mathrm{R}_2 \sqsubseteq \mathrm{S}$, then $\mathcal{K} \models \mathrm{T}(\chi(w), \chi(w\mathrm{R}_1))$ or $w = w'\mathrm{T}^-$ and $\chi(w'\mathrm{T}^-\mathrm{R}_1) = \chi(w')$.

5^-. If $e = w\mathrm{R}_1 \in \Delta^{\mathcal{C}_\mathcal{K}}$ with $\mathcal{T} \models \mathrm{R}_1 \sqsubseteq \mathrm{S}^-$ and $\neg\mathsf{flag}(w\mathrm{R}_1)$ and $\mathsf{cause}(w\mathrm{R}_1) = \exists \mathrm{T}$ with $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}^-$ and such that there exists $w\mathrm{R}_1\mathrm{R}_2 \in \Delta^{\mathcal{C}_\mathcal{K}}$ with $\mathcal{T} \models \mathrm{R}_2 \sqsubseteq \mathrm{S}^-$, then $\mathcal{K} \models \mathrm{T}(\chi(w), \chi(w\mathrm{R}_1))$ or $w = w'\mathrm{T}^-$ and $\chi(w'\mathrm{T}^-\mathrm{R}_1) = \chi(w')$.

Initialization: Individuals. For all $\mathsf{a} \in \mathsf{Ind}(\mathcal{A})$, we set $\chi(\mathsf{a}) := \mathsf{a}$ and $\mathsf{flag}(\mathsf{a}) := 0$. All properties are trivially satisfied on individuals.

Induction: Anonymous elements. Let $w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$. Assume all properties hold for $e < w\mathrm{R}$.

- If R is nilpotent, then we set $\chi(w\mathrm{R}) := \mathsf{choice}_\mathcal{K}(\theta_\mathcal{K}(\mathrm{R}))$ and $\mathsf{flag}(w\mathrm{R}) := 0$. Property 1 is satisfied and all other properties trivially hold.
- Else if $\mathsf{leader}(w)$ is defined and $w \cdot \mathsf{leader}(w) < w\mathrm{R}$, then $\chi(w \cdot \mathsf{leader}(w))$ is already defined, and we set $\chi(w\mathrm{R}) := \chi(w \cdot \mathsf{leader}(w))$ and $\mathsf{flag}(w\mathrm{R}) := 1$. By induction hypothesis, properties hold for $w \cdot \mathsf{leader}(w)$ and all transfer to $w\mathrm{R}$, but Property 4. By definition however, Property 4 also holds for $w\mathrm{R}$.
- Else if $\mathsf{flag}(w)$, then w must have shape $w = w_0\mathrm{R}_0$ (recall the initialization sets the flag of all individuals to 0). We set $\chi(w\mathrm{R}) := \chi(w_0)$ and $\mathsf{flag}(w\mathrm{R}) := 1$. Property 1, 5^+ and 5^- trivially hold for $w\mathrm{R}$. By induction hypothesis on $w_0\mathrm{R}_0$, Property 3 for $w_0\mathrm{R}_0$ ensures that Properties 2^+, 2^- and 3 continue to hold for $w\mathrm{R}$. Property 4 for $w\mathrm{R}$ is only relevant if $\mathsf{leader}(w) = \mathrm{R}$, in which case it trivially holds.
- Else if there exists an individual name b such that $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b}) \wedge \mathrm{S}(\mathsf{b}, \chi(w))$, then we set $\chi(w\mathrm{R}) := \mathsf{b}$ and $\mathsf{flag}(w\mathrm{R}) := 1$. Therefore Property 2^+, 2^-, 3 hold for $w\mathrm{R}$. Property 1, 5^+ and 5^- trivially hold for $w\mathrm{R}$. Again, Property 4 for $w\mathrm{R}$ is only relevant if $\mathsf{leader}(w) = \mathrm{R}$, in which case it trivially holds.
- Else if R is bipotent, that is $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}$ and $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^-$. We distinguish several subcases, each leading to a contradiction.
  - If $w \in \mathsf{Ind}(\mathcal{A})$, then $w \in \mathcal{D}^\pm_\mathcal{K}$. Contradicts the absence of bidemanding individuals.

– If $w = w_0\mathrm{R}_0$ with R_0 nilpotent. By induction assumption and Property 1 and the definition of $\mathsf{choice}_\mathcal{K}$, $\theta_\mathcal{K}(\chi(w)) = \{\{\mathrm{S},\mathrm{S}'\},\{\mathrm{S}\},\{\mathrm{S}'\}\}$. This is a contradiction with not being in the previous case.
– Otherwise $w = w_0\mathrm{R}_0$, then $\mathsf{leader}(w_0)$ is not defined as Property 4 from induction hypothesis on $w_0\mathrm{R}_0$ would then contradicts $\mathsf{flag}(w)$ being false. In particular R_0 cannot be bipotent. Hence either $\mathcal{T} \models \mathrm{R}_0 \sqsubseteq \mathrm{S}$ or $\mathcal{T} \models \mathrm{R}_0 \sqsubseteq \mathrm{S}^-$, but not both. Both cases being symmetrical, we now focus on $\mathcal{T} \models \mathrm{R}_0 \sqsubseteq \mathrm{S}$. For the triple $(\mathsf{cause}(w_0), \mathrm{R}_0, \mathrm{R})$, we have a propagation of S. As there are no non-trivial propagation of S, there must be an interference (Definition 11). Note that there cannot be an interference of the first type. Indeed, if U were such an interference, then $\mathsf{flag}(w)$ would be set, which we excluded in a previous case. Hence an interference should be of one of the other types:
  * If it is of type 2, then we have a bipotent U generated by $\exists\mathrm{S}^-$. Property 2^+ from induction hypothesis gives $\mathcal{K} \models \mathrm{S}(\chi(w_0), \chi(w))$. Hence $\mathcal{K} \models \exists z\ \mathrm{U}(\chi(w), z)$. As U is bipotent and $\chi(w)$ cannot be a bidemanding element, there exists an individual b such that $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b})$ and $\mathcal{K} \models \mathrm{S}^-(\chi(w), \mathsf{b})$, which we excluded in a previous case.
  * If it is of type 3, then $\mathsf{cause}(w_0\mathrm{R}_0) = \exists\mathrm{T}$ with $\exists\mathrm{T}^-$ generating a bipotent role. Property 5^+ by induction hypothesis on $w_0\mathrm{R}_0$ provides either $\mathcal{K} \models \mathrm{T}(\chi(w_0), \chi(w_0\mathrm{R}_0))$ or $w_0 = w_0'\mathrm{T}^-$ and $\chi(w) = \chi(w_0'\mathrm{T}^-)$. If $\mathcal{K} \models \mathrm{T}(\chi(w_0), \chi(w_0\mathrm{R}_0))$, since $\chi(w)$ cannot be a bidemanding element, there exists an individual b such that $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b})$ and $\mathcal{K} \models \mathrm{S}^-(\chi(w), \mathsf{b})$, which we excluded in a previous case. Otherwise $w_0 = w_0'\mathrm{T}^-$ and $\chi(w) = \chi(w_0'\mathrm{T}^-)$. Properties 2^+ and 2^- from induction hypothesis on w ensure $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b})$ and $\mathrm{S}^-(\chi(w_0), \chi(w))$, which leads to the same excluded case.

• Otherwise either $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}$ or $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^-$ (but not both, as we already dealt with bipotent R). These two cases are symmetrical, we focus on $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}$. We investigate the various possibilities for w and $\mathsf{cause}(w\mathrm{R})$:
– If $\mathsf{cause}(w\mathrm{R}) = \exists\mathrm{T}$ with $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}$ and $w = w_0\mathrm{T}^-$, then we set $\chi(w\mathrm{R}) := \chi(w_0)$ and $\mathsf{flag}(w\mathrm{R}) := 0$. In particular, Property 5^+ is satisfied. Property 2^+ from induction hypothesis on w gives Property 2^+ for $w\mathrm{R}$. Other properties trivially hold.
– Else if $\mathsf{cause}(w\mathrm{R}) = \exists\mathrm{T}$ with $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}$ and $w \in \mathsf{Ind}(\mathcal{A})$, then there exists $\mathsf{b} \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{K} \models \mathrm{T}(w, \mathsf{b})$, and we set $\chi(w\mathrm{R}) := \mathsf{b}$ and $\mathsf{flag}(w\mathrm{R}) := 0$. In particular, Property 2^+ and Property 5^+ are satisfied. Other properties trivially hold.
– Else if $\mathsf{cause}(w\mathrm{R}) = \exists\mathrm{T}$ with $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}^-$ and $w = w_0\mathrm{T}^-$, from all the preceding tests, $(\mathsf{cause}(w_0), \mathrm{T}^-, \mathrm{R})$ provides a propagation of S. As there are no non-trivial propagation, there must be an interference. It cannot be of the first type (otherwise $\mathsf{flag}(w)$ would be set), hence it must be of type 2 or 3:
  * If it is of type 2, then we have a role U generated by $\exists\mathrm{S}^-$ and with $\mathcal{T} \models \mathrm{U} \sqsubseteq \mathrm{S}$. Property 2^+ from induction hypothesis gives $\mathcal{K} \models \mathrm{S}(\chi(w_0), \chi(w))$. Since $\mathcal{K}$ does not contain any positive demanding individuals, there exists an individual b such that $\mathcal{K} \models \mathrm{U}(\chi(w), \mathsf{b})$, and we set $\chi(w\mathrm{R}) := \mathsf{b}$ and $\mathsf{flag}(w\mathrm{R}) := 0$.
  * If it is of type 3, then we set $\chi(w\mathrm{R}) := \chi(w_0)$ and $\mathsf{flag}(w\mathrm{R}) = 0$. Applying Property 5^+ by induction hypothesis on $w_0\mathrm{R}_0$ provides the desired properties.
– Else if $w \in \mathsf{Ind}(\mathcal{A})$, then, since there are no demanding individuals, there exists $\mathsf{b} \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b})$, then set $\chi(w\mathrm{R}) = \mathsf{b}$ and $\mathsf{flag}(\chi(w\mathrm{R}) = 0$. In particular, Properties 2^+ and 5^+ hold.
– Otherwise $\mathsf{cause}(w\mathrm{R}) = \exists\mathrm{T}$ with T nilpotent, then by Property 1 of induction hypothesis applied on w we have $\chi(w) = \mathsf{choice}_\mathcal{K}(\theta_\mathcal{K}(T))$. By definition of the choice of well-typed elements, there exists $\mathsf{b} \in \mathsf{Ind}(\mathcal{A})$ such that $\mathcal{K} \models \mathrm{S}(\chi(w), \mathsf{b})$, and we set $\chi(w\mathrm{R}) = \mathsf{b}$ and $\mathsf{flag}(\chi(w\mathrm{R}) = 0$.

□

With this key-result in hand, we can now notice that, in the absence of bidemanding individuals, our problem is easy to decide: within TC^0. Indeed, without bidemanding individuals, the best way to combine positive and negative demanding individual is still to pair them 1-to-1. Therefore, the optimal amount of matches can easily be decided by counting such elements.

Lemma 12. *Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$. If $\mathcal{K}$ admits no bidemanding individuals, then the minimal amount of matches can be decided within TC^0.*

Proof. Assume $\mathcal{K}$ does not admit any bidemanding individuals. Set a classic pairing $\mathsf{p} := (\mathsf{p}^+, \mathsf{p}^-)$ for positive and negative demanding individuals. We distinguish several cases, but the proof idea is always the same: in each case we exhibit the optimal amount of matches that can be easily computed from the types of individuals. We then prove it is minimal and exhibit a model with this precise amount of matches using Theorem 11 on ABox $\mathcal{A}^*$ and some $\mathsf{choice}_{\mathcal{K}^*}$ that will be specified in each case:

$$\mathcal{A}^* := \mathcal{A} \cup \{\mathrm{S}(\mathsf{x},\mathsf{y}) \mid \mathsf{p}^+(x) = y\} \cup \left\{\mathrm{S}(\mathsf{x},\mathsf{x}) \,\middle|\, \begin{array}{l} x \in \mathcal{D}^+_\mathcal{K} \\ x \notin \mathsf{dom}(\mathsf{p}^+) \end{array}\right\} \cup \left\{\mathrm{S}(\mathsf{x},\mathsf{x}) \,\middle|\, \begin{array}{l} x \in \mathcal{D}^-_\mathcal{K} \\ x \notin \mathsf{dom}(\mathsf{p}^-) \end{array}\right\}$$

The redundant arguments to prove minimality are the following mappings, always defined and injective, $\mathcal{I}$ being a model of $\mathcal{K}$:

$$\begin{array}{rcl} \rho^+ : \mathcal{D}^+_\mathcal{K} & \to & \Delta^\mathcal{I} \times \Delta^\mathcal{I} \\ x & \mapsto & (x, \mathsf{succ}^\mathcal{I}_\mathrm{S}(x)) \end{array} \qquad \begin{array}{rcl} \rho^- : \mathcal{D}^-_\mathcal{K} & \to & \Delta^\mathcal{I} \times \Delta^\mathcal{I} \\ x & \mapsto & (\mathsf{succ}^\mathcal{I}_{\mathrm{S}^-}(x), x) \end{array}$$

1. If there exists an individual a such that $\{\mathrm{S}, \mathrm{S}^-\} \in \theta_{\mathcal{K}}(\mathsf{a})$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$. $\mathcal{A}^*$ does not admit demanding elements, and we choose, for all $\mathsf{t} \in \Theta$, and $\mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \mathsf{a}$.
2. Else if $|\mathcal{D}^+_{\mathcal{K}}| > |\mathcal{D}^-_{\mathcal{K}}|$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) \in \mathcal{D}^+_{\mathcal{K}} \setminus \mathsf{dom}(\mathsf{p}^+)$.
3. Else if $|\mathcal{D}^+_{\mathcal{K}}| < |\mathcal{D}^-_{\mathcal{K}}|$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) \in \mathcal{D}^-_{\mathcal{K}} \setminus \mathsf{dom}(\mathsf{p}^-)$.
4. Else if there exists an individual $\mathsf{a} \in \mathcal{D}^+_{\mathcal{K}} \cap \mathcal{D}^-_{\mathcal{K}}$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements, assuming w.l.o.g $\mathsf{p}^+(\mathsf{a}) = \mathsf{a}$, and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \mathsf{a}$.
5. Else if there exists $(\mathsf{a}, \mathsf{b}) \in \mathcal{D}^+_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}}$ such that $\mathcal{K} \models \mathrm{S}(\mathsf{b}, \mathsf{a})$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements, assuming w.l.o.g $\mathsf{p}^+(\mathsf{a}) = \mathsf{b}$, and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \mathsf{a}$.
6. Else if there exists a role $\mathrm{R} \in \mathcal{N}_{\mathcal{T}} \cap \mathsf{gen}_{\mathcal{K}}$ such that $\{\mathrm{S}, \mathrm{S}^-\} \in \theta_{\mathcal{K}}(R)$. Let V be a bipotent role generated by R^-. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|) + 1$ reached with $\mathcal{A}^* \cup \{\mathrm{S}(\bot, \bot)\}$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \bot$. To ensure this amount of matches is still a lower bound for the amount of matches in any model $\mathcal{I}$, we need to specify where the extra match can be found in any model $\mathcal{I}$ of $\mathcal{K}$. Consider $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$ a homomorphism. Because of all the excluded previous cases, it can be verified that either $(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{V}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))))$ or $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{V}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))), f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})))$ is an additional match in $\mathcal{I}$ (in particular, not already counted by one's favorite mapping ρ^+ or ρ^-).
7. Else if there exists an individual a such that $\{\mathrm{S}\}, \{\mathrm{S}^-\} \in \theta_{\mathcal{K}}(\mathsf{a})$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \mathsf{a}$.
8. Else if there exists a role $\mathrm{R} \in \mathcal{N}_{\mathcal{T}} \cap \mathsf{gen}_{\mathcal{K}}$ and $\{\mathrm{S}\}, \{\mathrm{S}^-\} \in \theta_{\mathcal{K}}(\mathrm{R})$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|) + 1$ reached with $\mathcal{A}^* \cup \{\mathrm{S}(\bot, \bot)\}$ not admitting demanding elements and $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \bot$. Again, we need to specify where the extra match can be found in any model $\mathcal{I}$ of $\mathcal{K}$. Consider $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$ a homomorphism. It can be verified that either $(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))))$ or $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{S}^-}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))), f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})))$ is an additional match in $\mathcal{I}$ (in particular, not already counted by one's favorite mapping ρ^+ or ρ^-).
9. Else if there exists an individual a such that $\{\mathrm{S}\} \in \theta_{\mathcal{K}}(\mathsf{a})$. Optimum is $m_{\mathcal{A}} + \max(|\mathcal{D}^+_{\mathcal{K}}|, |\mathcal{D}^-_{\mathcal{K}}|)$ reached with $\mathcal{A}^*$ not admitting demanding elements and $\mathsf{choice}_{\mathcal{K}}(\{\mathrm{S}\}) := \mathsf{a}$ and $\mathsf{choice}_{\mathcal{K}}(\{\mathrm{S}^-\}) := \mathsf{b}$ with b the other endpoint (either certain or from pairing).
10. Else if there exists a role $\mathrm{R} \in \mathcal{N}_{\mathcal{T}} \cap \mathsf{gen}_{\mathcal{K}}$ and either $\{\mathrm{S}\} \in \theta_{\mathcal{K}}(\mathrm{R})$ or $\{\mathrm{S}^-\} \in \theta_{\mathcal{K}}(R)$. Optimum is $m_{\mathcal{A}} + 1$ reached with $\mathcal{A}^* \cup \{\mathrm{S}(\bot, \bot)\}$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \bot$. Again, we need to specify where the extra match can be found in any model $\mathcal{I}$ of $\mathcal{K}$. Consider $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$ a homomorphism. It can be verified that either $(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})), \mathsf{succ}^{\mathcal{I}}_{\mathrm{S}}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))))$ or $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{S}^-}(f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R}))), f(\mathsf{repr}_{\mathcal{K}}(\mathrm{R})))$ is an additional match in $\mathcal{I}$ (in particular, not already counted by one's favorite mapping ρ^+ or ρ^-).
11. Otherwise. Optimum is $m_{\mathcal{A}}$ with $\mathcal{A}$ not admitting demanding elements and setting $\forall \mathsf{t} \in \Theta, \mathsf{choice}_{\mathcal{K}^*}(\mathsf{t}) := \bot$.

To conclude the TC^0 membership proof, we describe the slight changes required to adapt the circuits already provided for the role cardinality queries:

- In the block "A closer look at roles and concepts over the input", one should extend the inputs to all subroles of R.
- The typing block should be adapted to fit the new typing notion (see Definition 20).
- All the blocks, each dedicated to a single strategy, can now be united as a single block computing $|\mathcal{D}^+_{\mathcal{K}}|$ and $|\mathcal{D}^-_{\mathcal{K}}|$.
- From this previous step, the typing block and the generated roles block, deciding if Situation 6, 8 or 10 occurs is easy, in which case one should add 1 to the final amount of matches.

□

We can now prove the desired reduction from our problem to MAXIMUM MATCHING.

Proof of the reduction. Let $\mathcal{A}$ be an ABox and $\mathcal{K} := (\mathcal{T}, \mathcal{A})$.

If $\mathcal{K}$ does not admit bidemanding individuals, then Lemma 12 ensures we can actually compute the answer within TC^0, in particular within L, and create a trivial instance of MAXIMUM MATCHING co-equivalent to it.

Otherwise, there are some bidemanding individuals. Consider then the following graph $\mathcal{G}_{\mathcal{K}}$:

$$\mathcal{V} := (\mathcal{D}^+_{\mathcal{K}} \times \{1\}) \cup (\mathcal{D}^-_{\mathcal{K}} \times \{-1\}) \cup \mathcal{D}^\pm_{\mathcal{K}}$$

$$
\begin{aligned}
\mathcal{E} \quad := \quad & \{\{(x,1),(y,-1)\} \mid (x,y) \in \mathcal{D}^+_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}}\} \\
& \cup \left\{ \{x,(y,1)\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^+_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(x,y)} \end{array} \right\} \\
& \cup \left\{ \{x,(y,-1)\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S^-(x,y)} \end{array} \right\} \\
& \cup \left\{ \{x,y\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^\pm_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(x,y)} \end{array} \right\}
\end{aligned}
$$

Claim:

$$[k,+\infty] \text{ is a certain answer for } q_{\mathrm{S}} \text{ w.r.t. } \mathcal{K} \Leftrightarrow (\mathcal{G}, m_{\mathcal{A}} + |\mathcal{V}| - k + 1) \notin \textsc{Maximum Matching}$$

Notice the graph $\mathcal{G}$ and the integer $m_{\mathcal{A}} + |\mathcal{V}| - k + 1$ are easily computable within L.

($\Leftarrow$). Assume $(\mathcal{G}, m_{\mathcal{A}} + |\mathcal{V}| - k + 1) \notin$ MAXIMUM MATCHING. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$. For each demanding individual x, we denote R_x a role causing this element to be demanding (that is, a bipotent role for bidemanding elements, a subrole of S for positive demanding elements, a subrole of S^- for negative demanding elements such that $x\mathrm{R}_x \in \Delta^{\mathcal{C}_{\mathcal{K}}}$). Consider the following matching, induced by $\mathcal{I}$:

$$
\begin{aligned}
M_{\mathcal{I}} \quad := \quad & \{\{(x,1),(y,-1)\} \mid (x,y) \in \mathcal{D}^+_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}}, f(x\mathrm{R}_x) = y \wedge x = f(y\mathrm{R}_y)\} \\
& \cup \left\{ \{x,(y,1)\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^+_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(x,y)} \end{array}, f(x\mathrm{R}_x) = y \wedge x = f(y\mathrm{R}_y) \right\} \\
& \cup \left\{ \{x,(y,-1)\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S^-(x,y)} \end{array}, f(x\mathrm{R}_x) = y \wedge x = f(y\mathrm{R}_y) \right\} \\
& \cup \left\{ \{x,y\} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^\pm_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(x,y)} \end{array}, f(x\mathrm{R}_x) = y \wedge x = f(y\mathrm{R}_y) \right\}
\end{aligned}
$$

Being a matching, $|M_{\mathcal{I}}| < m_{\mathcal{A}} + |\mathcal{V}| - k + 1$. Each edge from $M_{\mathcal{I}}$ provides exactly one additional match: either through the pairing of a positive with a negative, or through the pairing of a bidemanding with another demanding given that one match was already present in between. Each non-covered vertex provides one additional match, being $(x, \mathsf{succ}^{\mathcal{I}}_{\mathrm{R_x}}(x))$ for positive demanding uncovered elements, $(\mathsf{succ}^{\mathcal{I}}_{\mathrm{R_y}}(y), y)$ for negative demanding uncovered elements, and at least one of the latter two shapes for bidemanding elements. In addition with ABox matches, all these matches are distinct, hence there are at least $m_{\mathcal{A}} + |M_{\mathcal{I}}| + |\mathcal{V}| - 2|M_{\mathcal{I}}| = m_{\mathcal{A}} + |\mathcal{V}| - M_{\mathcal{I}} > k - 1$ matches in $\mathcal{I}$. That is at least k matches.

($\Rightarrow$). Assume $(\mathcal{G}, m_{\mathcal{A}} + |\mathcal{V}| - k + 1) \in$ MAXIMUM MATCHING. Consider a matching $M \subseteq \mathcal{E}$ with $|M| \geq m_{\mathcal{A}} + |\mathcal{V}| - k + 1$. Consider the enriched ABox $\mathcal{A}_M$:

$$
\begin{aligned}
\mathcal{A}_M \quad := \quad & \mathcal{A} && \text{(“Shape” 0: ABox matches)} \\
& \cup \{\mathsf{S(x,y)} \mid (x,y) \in \mathcal{D}^+_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}}, \{(x,1),(y,-1)\} \in M\} && \text{(Shape 2: Pairing pos-neg)} \\
& \cup \left\{ \mathsf{S(x,y)} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^+_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(x,y)} \end{array} \{x,(y,1)\} \in M \right\} && \text{(Shape } 3^+\text{: Pairing bi-pos)} \\
& \cup \left\{ \mathsf{S(x,y)} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^-_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(y,x)} \end{array}, \{x,(y,-1)\} \in M \right\} && \text{(Shape } 3^-\text{: Pairing bi-neg)} \\
& \cup \left\{ \mathsf{S(x,y)} \,\middle|\, \begin{array}{l} (x,y) \in \mathcal{D}^\pm_{\mathcal{K}} \times \mathcal{D}^\pm_{\mathcal{K}} \\ \mathcal{K} \models \mathsf{S(y,x)} \end{array}, \{x,y\} \in M \right\} && \text{(Shape 4: Pairing bi-bi)} \\
& \cup \{\mathsf{S(x,x)} \mid x \in \mathcal{V}, x \text{ uncovered by } M\} && \text{(Shape 5: Unpaired)}
\end{aligned}
$$

Notice $\mathcal{K}_M := (\mathcal{T}, \mathcal{A}_M)$ does not admit any demanding individuals. Since there exists at least one bidemanding individual a for $\mathcal{K}$, setting $\mathsf{choice}_{\mathcal{K}}(*) := \mathsf{a}$ provides a well-typed choice of elements for both $\mathcal{K}$ and $\mathcal{K}_M$. Applying Theorem 11 provides a model of $\mathcal{K}_M$, hence of $\mathcal{K}$, in which the matches are exactly: $m_{\mathcal{A}}$ ABox matches (Shape 0), $|M|$ matches from shapes 2, 3^+, 3^-, and 4, and $|\mathcal{V}| - 2|M|$ for uncovered by M elements of $\mathcal{V}$ (Shape 5). Hence a total of exactly: $m_{\mathcal{A}} + |\mathcal{V}| - |M|$ matches. Recall $|M| \geq m_{\mathcal{A}} + |\mathcal{V}| - k + 1$, hence that is at most $k - 1$ matches, that is less than k, hence this model is a countermodel for k.

□

B.4 Proof of TC^0 membership.

We now prove that if a TBox $\mathcal{T}$ does not admit a non-trivial propagation of S or S^-, and does not admit a non-trivial pairing, then $\mathrm{OMQA}(q_\mathrm{S}, \mathcal{T})$ is in TC^0.

Notice that if $\mathcal{T}$ satisfies $\mathcal{T} \not\models \mathrm{S} \sqsubseteq \mathrm{S}^-$, then for any ABox $\mathcal{A}$, $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ does not admit bidemanding individuals. Indeed, the existence of a bidemanding individual a implies the existence of B and R such that $\mathrm{B}(\mathsf{a}) \in \mathcal{A}$, $\mathcal{T} \models \mathrm{B} \sqsubseteq \exists\mathrm{R}$, $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}$ and $\mathcal{T} \models \mathrm{R} \sqsubseteq \mathrm{S}^-$. If B is a concept name, this is non-trivial pairing. If $\mathrm{B} = \exists\mathrm{T}$, then to prevent a non-trivial pair, $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}$ and $\mathcal{T} \models \mathrm{T} \sqsubseteq \mathrm{S}^-$, which would prevent a from being bidemanding. In that case, Lemma 12 holds and solves the problem.

Otherwise $\mathcal{T} \models \mathrm{S} \sqsubseteq \mathrm{S}^-$, in which case the only possible demanding individuals are bidemanding individuals (which disallows Shapes 2, 3^+ and 3^- from the proof just above) not touching any pre-existing match as $\mathcal{T} \models \mathrm{S} \sqsubseteq \mathrm{S}^-$ (which also disallows Shape 4). In particular the easiest way to minimize the amount of matches is simply by introducing a self-S-loop on each bidemanding individual, and the optimal amount of matches is therefore $m_\mathcal{A} + \left|\mathcal{D}^\pm_\mathcal{K}\right|$ in general except if $m_\mathcal{A} = \left|\mathcal{D}^\pm_\mathcal{K}\right| = 0$ and there exists a generated bipotent role R, in which case it is exactly 1. This is easily shown through the following injective mapping, providing at least $\left|\mathcal{D}^\pm_\mathcal{K}\right|$ non-ABox matches in any model $\mathcal{I}$:

$$\begin{array}{rcl} \rho^\pm : \mathcal{D}^\pm_\mathcal{K} & \rightarrow & \Delta^\mathcal{I} \times \Delta^\mathcal{I} \\ x & \mapsto & (x, \mathsf{succ}^\mathcal{I}_\mathrm{S}(x)) \end{array}$$

Furthermore, in the exception stated above the single match is found in any model $\mathcal{I}$ by considering where the representative $\mathsf{repr}_\mathcal{K}(\mathrm{R}) = w\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$ maps in $\mathcal{I}$ through a homomorphism $f : \mathcal{C}_\mathcal{K} \rightarrow \mathcal{I}$. It gives a match $(f(w), f(w\mathrm{R}))$, or alternatively $(f(w\mathrm{R}), f(w))$ as R is bipotent (but if the model $\mathcal{I}$ is optimal enough, these two are the same match!). Notice that again, with slight adaptations of the circuits, this is still easily computable within TC^0, the threshold gates being here essential to count the amount of bidemanding individuals.

C Proofs for Section 5.

C.1 Proof of Theorem 3.

Proof. Consider the ABox:

$$\mathcal{A} \quad = \quad \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{U}\} \,\cup\, \{\mathrm{S}(\mathsf{u},\mathsf{s}^*) \mid u \in s \in \mathcal{S}\} \,\cup\, \{\mathrm{C}(\mathsf{s}) \mid s \in \mathcal{S}\} \,\cup\, \{\mathrm{S}(\mathsf{s},\mathsf{s}^*) \mid s \in \mathcal{S}\}$$

and set $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. Notice there are $|\mathcal{S}| + \sum_{s\in\mathcal{S}} |s|$ ABox matches. We claim:

$$[|\mathcal{S}| + \sum_{s\in\mathcal{S}} |s| + k + 1, +\infty] \text{ is a certain answer of } q_{\mathrm{S}} \text{ w.r.t. } \mathcal{K} \iff (\mathcal{U}, \mathcal{S}, k) \notin \textsc{Set Cover}$$

($\Rightarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \in$ SET COVER. Consider a covering $F \subseteq \mathcal{S}$ of $\mathcal{U}$ with $|F| \leq k$. Consider the interpretation obtained from $\mathcal{K}$ in which we add, for each $u \in s \in F$ the fact $\mathrm{U}(\mathsf{u},\mathsf{s}^*)$ and $\mathrm{V}(\mathsf{s},\mathsf{s})$, which provide k additional matches from $\mathrm{S}(\mathsf{s},\mathsf{s})$. For the remaining $s \in \mathcal{S}$, we can add the fact $\mathrm{V}(\mathsf{s},\mathsf{s}^*)$, which does not provide an additional match. We obtain a model $\mathcal{I}_F$, with exactly $|\mathcal{S}| + \sum_{s\in\mathcal{S}} |s| + k$ matches, being a countermodel.

($\Leftarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \notin$ SET COVER. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$. For each $u \in \mathcal{U}$, we associate a subset $\rho(u) = s$ if $f(\mathsf{u}\mathrm{U}) = \mathsf{s}^*$ and $u \in s \in \mathcal{S}$, otherwise set $\rho(u) = s_u$, where s_u is an arbitrary set containing u. The image $\rho(\mathcal{U})$ is a covering of $\mathcal{U}$, hence $|\rho(\mathcal{U})| \geq k+1$. By definition, for each $s \in \rho(\mathcal{U})$ there exists $u \in \mathcal{S}$ such that: either $f(\mathsf{u}\mathrm{U}) = \mathsf{s}^*$, or $f(\mathsf{u}\mathrm{U}) \neq \hat{\mathsf{s}}^*$ for all $\hat{s}$ such that $u \in \hat{s} \in \mathcal{S}$. In the first case, $(\mathsf{s}, f(\mathsf{s}\mathrm{V}))$ must be a new match as $f(\mathsf{s}\mathrm{V})$ cannot be s^*. In the second case $(\mathsf{u}, f(\mathsf{u}\mathrm{U}))$ is a new match. Therefore there are at least $|\mathcal{S}| + \sum_{s\in\mathcal{S}} |s| + k + 1$ matches in $\mathcal{I}$. □

C.2 Proof of Theorem 4.

Proof. Consider the ABox:

$$\mathcal{A} \quad = \quad \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{U}\} \,\cup\, \{\mathrm{S}(\mathsf{u},\mathsf{s}) \mid u \in s \in \mathcal{S}\} \,\cup\, \{\mathrm{W}(\mathsf{s},\mathsf{u}) \mid u \in s \in \mathcal{S}\}$$

and set $\mathcal{K} = (\mathcal{T}, \mathcal{A})$. Notice there are $\sum_{s\in\mathcal{S}} |s|$ ABox matches. We claim:

$$[\sum_{s\in\mathcal{S}} |s| + k + 1, +\infty] \text{ is a certain answer of } q_{\mathrm{S}} \text{ w.r.t. } \mathcal{K} \iff (\mathcal{U}, \mathcal{S}, k) \notin \textsc{Set Cover}$$

($\Rightarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \in$ SET COVER. Consider a covering $F \subseteq \mathcal{S}$ of $\mathcal{U}$ with $|F| \leq k$. Consider the interpretation obtained from $\mathcal{K}$ in which we add, for each $u \in s \in F$ the fact $\mathrm{U}(\mathsf{u},\mathsf{s})$ and $\mathrm{V}(\mathsf{s},\mathsf{s})$, which provide k additional matches from $\mathrm{S}(\mathsf{s},\mathsf{s})$. We obtain a model $\mathcal{I}_F$, with exactly $\sum_{s\in\mathcal{S}} |s| + k$ matches, being a countermodel.

($\Leftarrow$). Assume $(\mathcal{U}, \mathcal{S}, k) \notin$ SET COVER. Consider a model $\mathcal{I}$ of $\mathcal{K}$ and a homomorphism $f : \mathcal{C}_{\mathcal{K}} \to \mathcal{I}$. For each $u \in \mathcal{U}$, we associate a subset $\rho(u) = s$ if $f(\mathsf{u}\mathrm{U}) = \mathsf{s}$ and $u \in s \in \mathcal{S}$, otherwise set $\rho(u) = s_u$, where s_u is an arbitrary set containing u. The image $\rho(\mathcal{U})$ is a covering of $\mathcal{U}$, hence $|\rho(\mathcal{U})| \geq k+1$. By definition, for each $s \in \rho(\mathcal{U})$ there exists $u \in \mathcal{S}$ such that: either $f(\mathsf{u}\mathrm{U}) = \mathsf{s}$, or $f(\mathsf{u}\mathrm{U}) \neq \hat{\mathsf{s}}^*$ for all $\hat{s}$ such that $u \in \hat{s} \in \mathcal{S}$. In the first case, $(f(\mathsf{u}\mathrm{UV}), f(\mathsf{u}\mathrm{U}))$ must be a new match as $f(\mathsf{u}\mathrm{UV})$ cannot be any v with $v \in \mathcal{U}$ (roles W prevent it!). In the second case $(\mathsf{u}, f(\mathsf{u}\mathrm{U}))$ is a new match. Therefore there are at least $\sum_{s\in\mathcal{S}} |s| + k + 1$ matches in $\mathcal{I}$. □

C.3 Proof of Theorem 5.

Proof. Given a 3DNF formula $\phi(x_1, \ldots x_m) = \bigvee_{i=1}^{n} l_i$, with $l_i = \bigwedge_{j=1}^{3} (\neg)^{p_{i,j}} v_{i,j}$. Introduce the following individual names:

$$\mathsf{Ind}_\phi = \{\mathsf{x_1}, \ldots \mathsf{x_m}, \mathsf{l_1}, \ldots \mathsf{l_n}, \mathsf{t_1}, \ldots \mathsf{t_m}, \mathsf{f_1}, \ldots \mathsf{f_m}\}$$

Consider now the ABox given by:

$$\begin{aligned} \mathcal{A}_\phi \quad = \quad & \{\mathrm{A}(\mathsf{x_1}), \ldots \mathrm{A}(\mathsf{x_m}), \mathrm{B}(\mathsf{l_1}), \ldots \mathrm{B}(\mathsf{l_n}), \mathrm{C}(\mathsf{t_1}), \ldots \mathrm{C}(\mathsf{t_m}), \mathrm{C}(\mathsf{f_1}), \ldots \mathrm{C}(\mathsf{f_m})\} \\ & \cup \{\mathrm{U}'(\mathsf{x_k}, \mathsf{a}) \mid k \in \{1, \ldots m\},\ \mathsf{a} \in \mathsf{Ind}_\phi \setminus \{\mathsf{t_k}, \mathsf{f_k}\}\} \\ & \cup \left\{ \mathrm{V}'(\mathsf{l_i}, \mathsf{a}) \,\middle|\, i \in \{1, \ldots n\},\ \mathsf{a} \in \mathsf{Ind}_\phi \setminus \left\{ \mathsf{t_i} \,\middle|\, \begin{array}{l} v_{i,j} = x_i \\ p_{i,j} = 0 \end{array} \right\} \cup \left\{ \mathsf{f_i} \,\middle|\, \begin{array}{l} v_{i,j} = x_i \\ p_{i,j} = 1 \end{array} \right\} \right\} \end{aligned}$$

Set $\mathcal{K}_\phi = (\mathcal{T}, \mathcal{A}_\phi)$. Notice there are $2m$ ABoxes matches: $\mathsf{t_1} \ldots \mathsf{t_m}$ and $\mathsf{f_1}, \ldots \mathsf{f_m}$. We claim:

$$[2m+1, +\infty] \text{ is a certain answer of } q_{\mathrm{C}} \text{ w.r.t. } \mathcal{K}_\phi \quad \Leftrightarrow \quad \forall \mathbf{x}\, \phi(\mathbf{x})$$

($\Leftarrow$). Assume $\forall \mathbf{x}\ \phi(\mathbf{x})$. Let $\mathcal{I}$ be a model of $\mathcal{K}_\phi$ and $f : \mathcal{C}_{\mathcal{K}_\phi} \to \mathcal{I}$ a homomorphism. If there exists a $k \in \{1, \dots m\}$ such that $f(\mathsf{x_k}\mathrm{U}) \notin \{\mathsf{t_k}, \mathsf{f_k}\}$, then $f(\mathsf{x_k}\mathrm{U})$ is an anonymous element, since U' prevents $f(\mathsf{x_k}\mathrm{U})$ to be equal to other individuals. As $f(\mathsf{x_k}\mathrm{U}) \in \mathrm{C}^{\mathcal{I}}$, it provides a new match. Otherwise, define the assignment induced by $\mathcal{I}$ as $\rho_{\mathcal{I}}(x) = 1$ if $f(\mathsf{x_k}\mathrm{U}) = \mathsf{t_k}$, and $\rho_{\mathcal{I}}(x) = 0$ if $f(\mathsf{x_k}\mathrm{U}) = \mathsf{f_k}$. Since $\forall \mathbf{x}\ \phi(\mathbf{x})$, there exists a satisfied clause l_i. For this i, the element $f(\mathsf{l_i}\mathrm{V})$ cannot be equal to any individual (as V' and $\exists \mathrm{U}^-$ prevent it), and therefore provides a new match for q_{C}. In all cases $[2m+1, +\infty]$ is a certain answer of q_{C} w.r.t. $\mathcal{K}_\phi$.

($\Rightarrow$). Assume $\exists \mathbf{x}\ \neg\phi(\mathbf{x})$. Consider such a valuation $\rho : \mathbf{x} \to \{0, 1\}$ such that $\neg\phi(\rho(\mathbf{x}))$. For each clause l_i, there exists (at least) a variable x_{k_i} which invalidates l_i. Consider the interpretation $\mathcal{I}_\rho$ obtained from $\mathcal{K}_\phi$ in which we add facts $\mathrm{U}(\mathsf{x_k}, \mathsf{t_k})$ iff $\rho(x_k) = 1$, resp $\mathrm{U}(\mathsf{x_k}, \mathsf{f_k})$ iff $\rho(x_k) = 0$, and $\mathrm{V}(\mathsf{l_i}, \mathsf{t_{k_i}})$ if $\rho(x_{k_i}) = 0$, resp $\mathrm{V}(\mathsf{l_i}, \mathsf{f_{k_i}})$ if $\rho(x_{k_i}) = 1$. By definition of variables x_{k_i}, we are ensured this interpretation $\mathcal{I}_\rho$ is a model. It only has $2m$ matches, hence $[2m+1, +\infty]$ is not a certain of q_{C} w.r.t. $\mathcal{K}_\phi$. ☐

C.4 Proof of Theorem 6.

For the two L lower bounds, we proceed by reduction from the Undirected Forest Accessibility problem, known to be L-complete [Cook and McKenzie, 1987].

	Undirected Forest Accessibility (UFA)
Input	Undirected acyclic graph $(\mathcal{V}, \mathcal{E})$ with two components, vertices $s, t \in \mathcal{V}$
Output	Is t reachable from s ?

Proof. We start with L membership. Let us first describe how to compute, given an ABox $\mathcal{A}$, the minimal number of matches of q_{S}. Intuitively, whenever an outgoing $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ is required (by the presence of $\mathrm{B}(\mathsf{v})$) but not already provided in the ABox, one aims at adding $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ in such a way that $\mathrm{S}(\mathsf{v}, \mathsf{v}')$ is already present in the ABox. This is always possible, except for two cases: *(i)* there are no outgoing S from v, or *(ii)* all the $\mathrm{S}(\mathsf{v}, \mathsf{v}')$ are such that $\mathrm{B}(\mathsf{v}')$ holds and $\mathrm{S}(\mathsf{v}', \mathsf{v})$ holds as well.

In case *(i)*, a new atom of the shape $\mathrm{S}(\mathsf{v}, \mathsf{v}')$ has to be added, creating a new match. In the second case, since $\mathrm{R} \sqsubseteq \neg\mathrm{R}^-$, one could create an inconsistency if the choice were to be done in a local fashion. Let us study how to perform optimally these choices.

We call *exit point* an individual v such that one of the three following conditions holds:

- $\mathrm{B}(\mathsf{v}) \notin \mathcal{A}$;
- $\exists \mathsf{v}'\ \mathrm{R}(\mathsf{v}, \mathsf{v}') \in \mathcal{A}$;
- $\exists \mathsf{v}'\ \mathrm{S}(\mathsf{v}, \mathsf{v}') \in \mathcal{A}$ and either $\mathrm{B}(\mathsf{v}') \notin \mathcal{A}$ or $\mathcal{K} \not\models \mathrm{S}(\mathsf{v}', \mathsf{v})$.

Intuitively, an exit point either already satisfies the concept inclusion $\mathrm{B} \sqsubseteq \exists\mathrm{R}$ (the first two conditions) or can satisfy it in a globally optimal way by adding $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ (in the third case, if a model minimizing the number of matches contains $\mathrm{R}(\mathsf{v}', \mathsf{v})$ and $\mathrm{S}(\mathsf{v}', \mathsf{v})$, one can get another minimal mode by adding $\mathrm{S}(\mathsf{v}', \mathsf{v}^*)$ and $\mathrm{S}(\mathsf{v}', \mathsf{v}^*)$), where v^* is a fresh element).

Let us thus consider the tradeoff graph of $\mathcal{A}$ having as vertices the individuals of $\mathcal{A}$ and an edge between u and v if it holds that $\mathrm{S}(\mathsf{u}, \mathsf{v}), \mathrm{S}(\mathsf{v}, \mathsf{u}), \mathrm{B}(\mathsf{u}), \mathrm{B}(\mathsf{v}) \in \mathcal{A}$, and $\mathrm{R}(\mathsf{u}, \mathsf{v}), \mathrm{R}(\mathsf{v}, \mathsf{u}) \notin \mathcal{A}$. This graph may contain several connected components, which can be of several types:

a. the connected component contains a cycle: there exists a consistent way to add R atoms wherever necessary in such a way that all the new R atoms fold on S atoms present in $\mathcal{A}$;

b. the connected component contains an exit point: similarly, add R atoms wherever necessary in such a way that all the new R atoms fold on S atoms present in $\mathcal{A}$;

c. the connected component is a tree and does not contain an exit point: an atom $\mathrm{R}(\mathsf{v}, x)$ for which $\mathrm{S}(\mathsf{v}, x) \notin \mathcal{A}$ has to be added. v can be chose arbitrarily among the vertices of the connected component, and x can be chosen to be a fresh element.

Thus, the minimal number of matches is the number of pairs $(\mathsf{v}, \mathsf{v}')$ such that either $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ or $\mathrm{S}(\mathsf{v}, \mathsf{v}')$ holds, plus the number of connected components of type c. in the previous case distinction. Algorithm 1 computes this minimum number of matches, and compare it to the number provided in input. Let us notice that checking for the existence of a cycle in a connected component can be done by making calls to an oracle for reachability in undirected graphs.

Algorithm 1 runs in logarithmic space, as undirected reachability is decidable in L, and L is low for itself. This proves membership to L.

For the lower bound, let us reduce UFA to our problem. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected acyclic graph with two components and let $s, t \in \mathcal{V}$ be two vertices. Consider the following ABox:

$$\mathcal{A} = \{\mathrm{B}(\mathsf{u}) \mid u \in \mathcal{V}\} \cup \{\mathrm{S}(\mathsf{u}, \mathsf{v}) \mid \{u, v\} \in \mathcal{E}\} \cup \{\mathrm{S}(\mathsf{s}, \mathsf{v}^*), \mathrm{S}(\mathsf{t}, \mathsf{v}^*)\} \cup \{\mathrm{R}(\mathsf{s}, \mathsf{v}^*), \mathrm{R}(\mathsf{t}, \mathsf{v}^*)\},$$

Data: An ABox $\mathcal{A}$, an integer n
Result: Yes if and only if $[n, +\infty]$ is a certain answer for q_{S} w.r.t. $(\mathcal{T}, \mathcal{A})$
$m \leftarrow |\{(\mathsf{v}, \mathsf{v}') \mid \mathrm{S}(\mathsf{v}, \mathsf{v}') \in \mathcal{A} \vee \mathrm{R}(\mathsf{v}, \mathsf{v}') \in \mathcal{A}\}|$;
$\mathcal{G} \leftarrow$ tradeoff graph of $\mathcal{A}$
for $i \leftarrow 1$ **to** n **do**
  if *no* $\mathsf{v_j}$ *with* $j < i$ *is reachable from* $\mathsf{v_i}$ *in* $\mathcal{G}$ **then**
    if *no exit point is reachable from* $\mathsf{v_i}$ *in* $\mathcal{G}$ **then**
      if *the connected component of* $\mathsf{v_i}$ *in* $\mathcal{G}$ *does not contain a cycle* **then**
        $m \leftarrow m + 1$;
      end
    end
  end
  return *Yes if* $n \leq m$*, no otherwise*
end

Algorithm 1: An algorithm for checking whether $[n, +\infty]$ is a certain answer for q_{S} w.r.t. $(\mathcal{T}, \mathcal{A})$

where v^* is a fresh individual. Note that we have thus made both s and t exit points, and they are the only such individuals. Let us notice that $\mathcal{A}$ is first-order definable from $\mathcal{G}$. We thus focus on the following claim:

$$((\mathcal{V}, \mathcal{E}), s, t) \in \text{UFA} \Leftrightarrow [2|\mathcal{E}| + 3, +\infty] \text{ is a certain answer for } q_{\mathrm{S}} \text{ w.r.t. } (\mathcal{T}, \mathcal{A})$$

Let us first notice that in any model, there are $2|\mathcal{E}| + 2$ matches of q_{S}, as there are that many matches from q_{S} in $\mathcal{A}$.

Let us consider the case where s is not reachable from t. As $\mathcal{G}$ has exactly two connected components, for any vertex v (distinct from both s and t), there exists a unique vertex among $\{s, t\}$ that is reachable from v and a unique S-edge $\mathrm{S}(\mathsf{v}, \mathsf{f}(\mathsf{v}))$ outgoing from v on the shortest path to s or t (depending on which connected component v belongs). Let us consider the interpretation $\mathcal{I} = \mathcal{A} \cup \{\mathrm{R}(\mathsf{v}, \mathsf{f}(\mathsf{v}))| \, v \in \mathcal{V} \setminus \{s, t\}\}$. $\mathcal{I}$ is a model of $\mathcal{T}$: for any v such that $\mathrm{B}(\mathsf{v})$ holds, there is an atom $\mathrm{R}(\mathsf{v}, \mathsf{v}')$. Moreover, if v is on the shortest path from v' to s (resp. to t), then v' cannot be on the shortest path from v to s (resp. to t), hence $\mathrm{R}^{\mathcal{I}} \cap (\mathrm{R}^{-})^{\mathcal{I}} = \emptyset$. $\mathcal{I}$ is thus a model of $\mathcal{A}$ and $\mathcal{T}$ in which there are exactly $2|\mathcal{E}| + 2$ matches of q_{S}, proving that if $((\mathcal{V}, \mathcal{E}), s, t) \notin$ UFA, then $[2|\mathcal{E}| + 3, +\infty]$ is not a certain answer of q_{S} w.r.t. $(\mathcal{T}, \mathcal{A})$.

Let us now consider the case where s is reachable from t. We already know that in any model of $\mathcal{A}$ and $\mathcal{T}$, there are $2|\mathcal{E}| + 2$ matches of q_{S}. We prove there must be another match of q_{S}. We show that there must be some $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ in any model such that $\mathrm{S}(\mathsf{v}, \mathsf{v}') \notin \mathcal{A}$. Let v be in the connected component that contains neither s nor t. Let us consider a maximal (possibly infinite) sequence $\mathsf{v}_1, \mathsf{v}_2, \ldots, \mathsf{v}_n$ with $\mathsf{v}_1 = \mathsf{v}$ and such that for any i, $\mathrm{R}(\mathsf{v}_i, \mathsf{v}_{i+1})$ belongs to $\mathcal{I}$. As there are no cycle in $\mathcal{G}$ and that $\mathrm{R} \sqsubseteq \neg\mathrm{R}^{-}$, there exists i such that $\mathrm{S}(\mathsf{v}_i, \mathsf{v}_{i+1}) \notin \mathcal{A}$, which provides a new match for q_{S}, which concludes the proof. ☐

C.5 Proof of Theorem 6 for Concept Cardinality Queries.

Let us first state our theorem in the case of concept cardinality queries.

Theorem 12 (L-complete situation for concept cardinality queries)**.** *For the following TBox:*

$$\mathcal{T} = \{\, \mathrm{B} \sqsubseteq \exists\mathrm{R},\ \exists\mathrm{R}^{-} \sqsubseteq \mathrm{C},\ \mathrm{R} \sqsubseteq \neg\mathrm{R}^{-},\ \mathrm{R} \sqsubseteq \neg\mathrm{T} \,\}$$

the problem $\mathrm{OMQA}(q_{\mathrm{C}}, \mathcal{T})$ *is* L*-complete.*

Proof. We start by proving L membership. Let us first notice that the minimum number of matches can only be one of the two following values:

- $n = |\{\mathsf{v} \mid C(\mathsf{v}) \in \mathcal{A} \vee \exists \mathsf{v}' \; R(\mathsf{v}', \mathsf{v}) \in \mathcal{A}\}|$, which is the number of matches in the ABox on which concept inclusions have been applied;
- $n + 1$, which can be obtained by introducing a fresh element α, and adding $\mathrm{R}(\mathsf{v}, \alpha)$ for any v in $\mathsf{Ind}(\mathcal{A})$, as well $\mathrm{C}(\alpha)$.

Let us consider a model $\mathcal{I}$ having n matches. Let f be a homomorphism from $\mathcal{C}_{\mathcal{K}}$ to $\mathcal{I}$. Let $\mathsf{v} \in \mathsf{Ind}(\mathcal{A})$ such that $\mathsf{v}\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}}}$. Then:

- $f(\mathsf{v}\mathrm{R}) \in \mathsf{Ind}(\mathcal{A})$ (otherwise, a new match would be created);
- $\mathrm{C}(f(\mathsf{v}\mathrm{R})) \in \mathcal{A}$ or there is v' s.t. $\mathrm{R}(\mathsf{v}', \mathsf{v}) \in \mathcal{A}$ (otherwise, a new match would be created);
- $\mathrm{T}(\mathsf{v}, f(\mathsf{v}\mathrm{R})) \notin \mathcal{A}$ (otherwise $\mathcal{I}$ would not be a model)
- $\mathrm{R}(f(\mathsf{v}\mathrm{R}), \mathsf{v}) \notin \mathcal{A}$ (otherwise $\mathcal{I}$ would not be a model)
- if $f(\mathsf{v}\mathrm{R})\mathrm{R} \in \Delta^{\mathcal{C}_{\mathcal{K}}}$, then $f(f(\mathsf{v}\mathrm{R})\mathrm{R}) \neq \mathsf{v}$ (otherwise $\mathcal{I}$ would not be a model).

All the conditions except the last one can be checked for each individual separately. We call *exit point* an individual v for which either $\mathsf{v}R \notin \Delta^{\mathcal{C}_\mathcal{K}}$ or there exists v' such that by setting $f(\mathsf{v}R) = \mathsf{v}'$, the first four conditions are satisfied, and the fifth one is satisfied by vacuity, *i.e.*, $\mathsf{v}'\mathrm{R} \notin \Delta^{\mathcal{C}_\mathcal{K}}$.

Let us define the tradeoff graph $\mathcal{G}$ of $\mathcal{A}$ having as vertices the individuals of $\mathcal{A}$ and an edge $\{\mathsf{v}, \mathsf{v}'\}$ if and only if $\{\mathrm{B}(\mathsf{v}), \mathrm{B}(\mathsf{v}'), \mathrm{C}(\mathsf{v}), \mathrm{C}(\mathsf{v}')\} \in \mathcal{A}$ and $\{\mathrm{T}(\mathsf{v}, \mathsf{v}'), \mathrm{T}(\mathsf{v}', \mathsf{v}), \mathrm{R}(\mathsf{v}, \mathsf{v}'), \mathrm{R}(\mathsf{v}', \mathsf{v})\} \cap \mathcal{A} = \emptyset$. This is called a tradeoff graph because if $\{\mathsf{v}, \mathsf{v}'\}$ is an edge, then we could either set $f(\mathsf{v}\mathrm{R}) = \mathsf{v}'$ or $f(\mathsf{v}'\mathrm{R}) = \mathsf{v}$ without creating new matches, but not both, as this would violate the negative role inclusion $\mathrm{R} \sqsubseteq \neg\mathrm{R}^-$.

We claim that there exists a model with exactly n matches if and only if in every connected component of G there is either an exit point or a cycle. Indeed, notice that if $\{\mathsf{v}, \mathsf{v}'\}$ is an edge of the tradeoff graph, then adding an atom $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ does not increase the number of matches of q_C. If there is an exit point v^* in a connected component, there is a way to add an atom $\mathrm{R}(\mathsf{v}^*, \hat{\mathsf{v}})$ without adding a match and with $\hat{\mathsf{v}}$ not being in the same connected component as v^* (by definition of the tradeoff graph). Then, by a breadth first traversal of the connected component, one can add R atoms as required. Similarly, when there is a cycle, one starts by such a cycle, and add other atoms in a breadth first fashion.

Conversely, if there exists a model with n matches, then $f(\mathsf{v}\mathrm{R}) \in \mathsf{Ind}(\mathcal{A})$ for any v such that $\mathsf{a}\mathrm{R}$ is defined. Let $\mathsf{v}_1, \ldots, \mathsf{v}_\mathsf{n}, \ldots$ be a sequence such that $f(\mathsf{v}_\mathsf{i}\mathrm{R}) = \mathsf{v}_{\mathsf{i}+1}$ whenever $\mathsf{v}_\mathsf{i}\mathrm{R} \in \Delta^{\mathcal{C}_\mathcal{K}}$, and such that v_i is the last element of the sequence otherwise. If $f(\mathsf{v}_\mathsf{i}\mathrm{R})$ is not an exit point, then there is an edge $\{\mathsf{v}_\mathsf{i}, \mathsf{v}_{\mathsf{i}+1}\}$ in the tradeoff graph. If the sequence if finite, then the one before the last is an exit point. Otherwise, there must be a cycle in the connected component containing v_1.

Algorithm 2 checks this condition. As it amounts to several reachability checks in an undirected graph, this algorithm can be made to run in L.

Data: An ABox $\mathcal{A}$, an integer n
Result: Yes if and only if $[n, +\infty]$ is a certain answer of q_C w.r.t. $(\mathcal{T}, \mathcal{A})$
$m \leftarrow |\{(\mathsf{v}) \mid \mathrm{R}(\mathsf{v}', \mathsf{v}) \in \mathcal{A} \vee \mathrm{C}(\mathsf{v}) \in \mathcal{A}\}|$;
$r \leftarrow m$;
$\mathcal{G} \leftarrow$ tradeoff graph of $\mathcal{A}$ **for** $i \leftarrow 1$ **to** n **do**
    if *no* v_j *with* $j < i$ *is reachable from* v_i *in* $\mathcal{G}$ **then**
        if *no exit point is reachable from* v_i *in* $\mathcal{G}$ **then**
            if *the connected component of* v_i *in* $\mathcal{G}$ *does not contain a cycle* **then**
                $r \leftarrow m + 1$; `// r can take only two values`
            end
        end
    end
    return *Yes if* $n \leq r$*, no otherwise*
end

Algorithm 2: An algorithm for checking whether $[n, +\infty]$ is a certain answer of q_C w.r.t $(\mathcal{T}, \mathcal{A})$

To prove L-hardness, we again proceed by reduction from UFA. Consider the following ABox:

$$\mathcal{A} = \{\mathrm{B}(\mathsf{u}), \mathrm{C}(\mathsf{u}) \mid u \in \mathcal{V}\} \cup \{\mathrm{T}(\mathsf{u}, \mathsf{v}) \mid \{u, v\} \notin \mathcal{E}\} \cup \{\mathrm{T}(\mathsf{u}, \mathsf{v}^*), \mid u \in \mathcal{V} \setminus \{s, t\}\} \cup \{\mathrm{R}(\mathsf{s}, \mathsf{v}^*), \mathrm{R}(\mathsf{t}, \mathsf{v}^*)\},$$

There are $|\mathcal{V}| + 1$ matches of q_C in $\mathcal{A}$. We prove that:

$$((\mathcal{V}, \mathcal{E}), s, t) \in \mathrm{UFA} \Longleftrightarrow [|\mathcal{V}| + 2, +\infty] \text{ is a certain answer of } q_\mathrm{C} \text{ w.r.t. } (\mathcal{T}, \mathcal{A}).$$

Let us consider the case where s is not reachable from t. As $(\mathcal{V}, \mathcal{E})$ has exactly two connected components, for any vertex v (distinct from both s and t), there exists a unique vertex among $\{s, t\}$ that is reachable from v and a unique vertex $f(v)$ that is on the shortest path from v to s (or t). Let us consider the interpretation $\mathcal{I} = \mathcal{A} \cup \{\mathrm{R}(\mathsf{v}, \mathsf{f}(\mathsf{v})) \mid v \in \mathcal{V} \setminus \{s, t\}\}$. $\mathcal{I}$ is a model of $\mathcal{T}$: for any v such that $\mathrm{B}(\mathsf{v})$ holds, there is an atom $\mathrm{R}(\mathsf{v}, \mathsf{v}')$. Moreover, if v is on the shortest path from v' to s, then v' cannot be on the shortest path from v to s, hence $\mathrm{R}^\mathcal{I} \cap (\mathrm{R}^-)^\mathcal{I} = \emptyset$. Moreover, $\{v, f(v)\} \in \mathcal{E}$, hence $(v, f(v)) \notin \mathrm{T}^\mathcal{I}$. $\mathcal{I}$ is thus a model of $\mathcal{A}$ and $\mathcal{T}$ in which there are exactly $|\mathcal{V}| + 1$ matches of q_C, proving that if $((\mathcal{V}, \mathcal{E}), s, t) \notin \mathrm{UFA}$, then $[|\mathcal{V}| + 2, +\infty] \notin q_\mathrm{C}^{(\mathcal{T}, \mathcal{A})}$.

Let us now consider the case where s is reachable from t. We already know that in any model of $\mathcal{A}$ and $\mathcal{T}$, there are at least $|\mathcal{V}| + 1$ matches of q_C. As there are no cycle in the connected component not containing s and t, in any model of $(\mathcal{A}, \mathcal{T})$ there must be an individual v having an outgoing edge $\mathrm{R}(\mathsf{v}, \mathsf{v}')$ with $\{\mathsf{v}, \mathsf{v}'\} \notin \mathcal{E}$. As $\mathrm{T}(\mathsf{v}, \mathsf{u})$ holds for any u such that $\{\mathsf{v}, \mathsf{u}\} \notin \mathcal{E}$, as well as for $\mathsf{u} = \mathsf{v}^*$, v' provides a novel match for q_C, concluding the proof. □