Tasks

0. During the preparation for this experiment calculate analytically the Fourier transform of a single-slit and a double-slit function and compare these with the familiar Fraunhofer diffraction-patterns for these objects. Using FFT (Fast-Fourier-Transform) in ORIGIN determine the Fourier transforms of the functions \( F(k) = \text{sinc}(2\pi k) = \sin(2\pi k)/(2\pi k) \) and \( F(k)^2 \); discuss this result in the light of the convolution theorem.

1. Measure the width of a single slit using diffraction (Fresnel, Fraunhofer) of laser light. Determine the slit width (a) from the position of the minima for various diffraction orders and (b) by FFT.

2. Measure the intensity profile of laser light diffracted by a double slit. Determine the distance between the slits and the slit width from (a) the position of the minima for various diffraction orders, (b) from the fit of the well-known Fraunhofer diffraction-pattern to the data and (c) by FFT of the intensity profile.

3. Determine the grating constant of a transmission grating from the position of the maxima.

Literature


Optics, E. Hecht, Addison Wesley, 4th ed., 10.2.1, 10.2.2., 10.2.3

Physikalisches Praktikum, 13. Auflage, Hrsg. W. Schenk, F. Kremer, Optik, 2.0, 2.2, 2.3, Fourier-Transformation und Signalanalyse, 1.0

Accessories

Diode laser (\( \lambda = 636 \text{ nm} \)), CCD-line camera, PC, printer, different slits, eyepiece-micrometer, polarizing filter, millimeter scale, transmission grating, reflection grating

Keywords for preparation

- Interference, coherence, coherence condition
- Diffraction at single slit, double slit, reflection and transmission gratings
- Fraunhofer and Fresnel diffraction
- Construction and operation principle of laser
- Operation principle of CCD sensor (CCD, Charge Coupled Devices)
- Fourier transformation
Note on Safety
The primary beam of the laser (laser class 2M) has a power of about 1 mW. Looking directly into the laser beam or its specular reflection might lead to eye injury. During the observation of interference with an eyepiece-micrometer a polarization filter placed between laser and slit has to be used to attenuate the beam intensity (Malus’ law).

Remarks
In general one distinguishes diffraction according to Fraunhofer and Fresnel. The Fraunhofer diffraction or diffraction of parallel rays describes the diffraction of plane waves. In this case of diffraction the distance between light source and slit as well as the distance between slit and screen (observation plane) are always infinite. Experimentally this kind of diffraction is realized by either placing the light source into the focal plane of a convex lens or by using a light source (e.g. laser) that emits plane waves. The diffraction pattern is observed on a screen or by a CCD-line camera placed in the principal plane of a convex lens.

Diffraction is denoted by Fresnel diffraction when the curvature of the wave fronts of the incident waves and of the diffracted waves cannot be neglected; the rays are then not parallel. Fresnel diffraction occurs when both the light source and the observation plane or only the latter have a finite distance from the diffracting object. A calculation of these diffraction patterns is usually quite intricate. In case of Fresnel diffraction at a single slit the value of the wave parameter $w$ determines the intensity profile in the observation plane

$$w = \frac{\sqrt{\lambda a}}{b}.$$ 

In the latter equation $a$ denotes the distance between the slit plane and the observation plane assumed parallel, $b$ the slit width and $\lambda$ the wavelength of the monochromatic light source. In an analogous way for the diffraction at a double slit (slit distance $g$), the Fresnel number $N_F$ with

$$N_F = \frac{g^2}{4a \lambda}$$

controls the transition for the observation of Fresnel diffraction to Fraunhofer diffraction for $N_F < 1$.

Fig. 1.1 Schematic representation of the intensity profile of a broad single slit ($w << 1$)
For \( w = 1 \) the intensity modulation contains the whole region between \( x_1 \) and \( x_2 \). Depending on the actual value of \( w \) at the center of the diffraction pattern an intensity maximum or minimum might appear. In the limit \( w \gg 1 \ (b > \lambda) \) the diffraction pattern corresponds to the Fraunhofer pattern. In this case the principal intensity maximum is located behind the slit center and is the more blurred the more narrow the slit is.

**Hints to task 0**

The single-slit function is given by \( f(x) = 1, -b/2 < x < b/2, \ f(x) = 0 \) elsewise, and the double-slit function by \( f(x) = 1, -(g+b)/2 < x < -(g-b)/2, (g-b)/2 < x < (g+b)/2, f(x) = 0 \) elsewise. \( b \) denotes the slit width and \( g \) the slit distance.

The Fourier transform of a function \( f(x) \) is defined by

\[
F(k) = \text{FT}[f(x)] = \int_{-\infty}^{\infty} f(x) \exp(-ikx)dx
\]

with the backwards transformation

\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(k) \exp(ikx)dk .
\]

In calculations the representation of the \( \delta \)-function by a Fourier transformation is often valuable:

\[
\delta(x-x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ik(x-x_0))dk .
\]

If \( x \) denotes a spatial coordinate, \( k \) has the unit \( m^{-1} \), i.e. \( k \) is a wave vector. Therefore one often refers to real space and wave-vector space (\( k \)-space); accordingly the Fourier transformation transforms functions from one into the other space.

Let \( f(x) \) and \( g(x) \) denote functions in real space, \( F(k) \) and \( G(k) \) the corresponding Fourier transforms in \( k \)-space. Then the following relations hold (\( a, b, c \) being complex numbers):

1. Parseval’s theorem

\[
\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |F(k)|^2 dk
\]

2. Linearity

\[
\text{FT}[af(x)+bg(x)]=aF(k)+bG(k)
\]

3. Scaling law

\[
\text{FT}[f(cx)]=\frac{1}{c}F\left(\frac{k}{c}\right)
\]

4. Law of displacements

\[
\text{FT}[f(x-x_0)]=\exp(ikx_0)F(k)
\]
5. Convolution theorem

\[ \text{FT} \left[ f(x) \otimes g(x) \right] = \text{FT} \left[ \int_{-\infty}^{\infty} f(x')g(x-x')dx' \right] = \text{FT} \left[ f(x) \right] \text{FT} \left[ g(x) \right] = F(k)G(k) \]  

(8)

Hints to task 1

The supervisor selects the single slit (A, B or C). Realize the observation mode according to Fraunhofer. To this end the slit (Fig. 1.2) is placed onto the optical bench such that its plane is perpendicular to the optical axis. The light is passing the slit partially undisturbed and partially it is diffracted.

![Fraunhofer diffraction](image)

All the non-diffracted rays passing the slit (width \( b \)) parallel to its axis are focused at the focal plane \( F' \) of the convex lens \( L \) at \( x = 0 \). Their path differences are zero. The path difference \( \Delta \) is defined as the difference of the optical path length \( s \) (\( \Delta = s_1 - s_2 \) for air with the refraction index \( n = 1 \)) of interfering waves. Parallel rays emitted from several points of the slit with the same angle \( \alpha \) (homologous points) are focused at \( x = f \tan \alpha \), where \( f \) is the focal length of the convex lens. Between these waves, however, path differences \( \Delta \) exist. For example, if \( \Delta = \lambda \) between the two waves emitted from the edges of the slit, all elementary waves emitted at arbitrary points on the slit separated a distance \( b/2 \) extinguish when passing \( L \). The diffraction-pattern minima of arbitrary order \( ^1 n (n = \pm 1, \pm 2, \ldots ) \) obey the relation

\[ b \sin \alpha_n = n \lambda . \]  

(9)

The minima (destructive interference, darkness) are visible in the focal plane \( F' \) at

\(^1\)In discussing minima of higher order the slit width is divided into quarter parts \((n \pm 2)\), sixth parts \((n \pm 3)\) etc.
\[ x_n = f \tan \alpha_n = f \tan \left[ \arcsin \left( \frac{n \lambda}{b} \right) \right]. \] (10)

For small angles \( \alpha \) one obtains \( x_n = f \frac{n \lambda}{b} \).

In the present experiment no collimating lens is used. Fraunhofer diffraction occurs in good approximation, if the observation plane is located sufficiently far from the slit (far field). The diffraction pattern is recorded by a CCD-line camera, the data are transferred to a computer and are imported into ORIGIN. The abscissa \( x \) is given in \( \mu m \) and refers to the sensor position on the camera.

In (a) the distances \( 2x_n \) between the minima of \( \pm nth \) order are determined; from these the slit width is obtained by linear regression.

In (b) a FFT of the intensity profile is made. To this end shift the intensity distribution along the abscissa such that the principal maximum is located by \( x = 0 \). Deduce the following relation between the wave vector \( k \) perpendicular to the optical axis and the position \( x \):

\[ k = \frac{2 \pi}{\lambda} \frac{x}{\sqrt{x^2 + d^2}} = \frac{2 \pi}{\lambda} \frac{x}{d}. \] (11)

\( d \) denotes the distance between slit and CCD-sensor. Plot the intensity as a function of \( k \), perform the FFT and determine \( b \).

In addition to Fraunhofer diffraction the Fresnel diffraction-pattern should be measured and the slit width \( b \) should be determined. To this end measure the distance \( a \) between slit plane and CCD sensor. Using the relations

\[ \tan \alpha_n = \frac{x_n}{a} \quad \text{and} \quad \sin \alpha_n = \frac{n \lambda}{b} \] (12a,b)

from the position of the minima \( b \) should be determined. For small angles the approximation \( x_n = n \frac{\lambda}{b} \frac{d}{a} \) holds.

Perform a FFT of the intensity profile measured for very small distances (near field) between slit and CCD camera. Is it possible to determine the slit width from this?

**Hints to task 2**

A double slit consists of two parallel single slits of width \( b \) and distance \( g \). In the experiment parallel, monochromatic laser-light is used. The waves coming from the points \( F_1 \) and \( F_2 \) have the path difference \( \Delta = r_1 - r_2 \) in the point \( P \) of the diffracted wave, with \( r_1 = F_1P \) and \( r_2 = F_2P \) (Fig. 2.1). All points with the same path difference \( \Delta \) (or phase difference \( \delta = \Delta 2\pi/\lambda \)) are located (as defined) on a rotational hyperboloid with \( F_1 \) and \( F_2 \) as focal points. The line \( F_1F_2 \) is the main axis (rotation axis).
The waves from F₁ and F₂ interfere constructively at phase differences of \( \delta = 2n\pi (n = \pm 1, 2, \ldots) \). In the experiment the rotational hyperboloids are intersected at the distance \( d \) from the rotation axis by a plane screen (CCD-line camera) arranged perpendicular to the optical axis (main axis of hyperbola). Maxima and minima that are at equal distances from each other can be observed in an appropriate limited area around the optical axis. The superposition of two particular waves diffracted by the double slit is shown in Fig. 2.2.

Consider waves interfering at point \( x₁ \) in the plane of the CCD-camera. The path difference \( \Delta \) between these can be calculated using the formulas:

\[
\begin{align*}
r₁² &= \left(\frac{g}{2} + x₁\right)² + d², \quad r₂² = \left(\frac{g}{2} - x₁\right)² + d².
\end{align*}
\]

With \( d \gg g \), \( d \gg x₁ \) and \( r₁ = r₂ = d \) one obtains

\[
\Delta = r₁ - r₂ = \frac{r₁² - r₂²}{r₁ + r₂} = \frac{g}{d} x₁.
\]

If the path difference is \( \Delta = n\lambda (n = \pm 1, \pm 2, \ldots) \) one observes maxima at a distance.
\[ x_n^{\text{max}} = n \lambda \frac{d}{g}, \quad (14) \]

and minima for \( \Delta = (2n-1)\lambda/2 \) \((n \pm 1, \pm 2, \ldots)\) with a distance
\[ x_n^{\text{min}} = (2n-1) \frac{\lambda}{2} \frac{d}{g}. \quad (15) \]

From the relation \( n \lambda = g \frac{x_n}{d} = g \tan \alpha_n = g \sin \alpha_n \)

one obtains for large distances \( d \left( \tan \alpha = \sin \alpha \right) \) that the formulas relating to Fraunhofer diffraction can be used. The diffraction pattern at the double slit is superimposed by the diffraction pattern of the single slit (Fig. 2.3). For the corresponding minima (dashed curve in Fig. 2.3) the same relation as for the single slit is valid: \( b \sin \alpha_n = n \lambda ; n = \pm 1, \pm 2, \ldots \).

![Fig. 2.3 Diffraction pattern of a double slit (black: double slit function, blue: envelope – single slit function)](image)

The intensity profile of light diffracted by a double slit is given by
\[ I = 4I_{\text{max}} \left( \frac{\sin p}{p} \right)^2 \cos^2 q \quad \text{with} \quad p = \frac{\pi}{\lambda} b \sin \alpha , \quad q = \frac{\pi}{\lambda} g \sin \alpha . \quad (16) \]

Whereas the factor \((\sin p/p)^2\) describes the diffraction by a single slit of width \( b \), the factor \( \cos^2 q \) characterizes the intensity profile from the interference of two point-like coherent light sources at a distance \( g \) from one another. Slit width \( b \) and ratio \((g/b)\) mainly determine the intensity ratios in the diffraction pattern. In case of an integral ratio \( \frac{g}{b} = 2m \) minima are observed in the central maximum. In the preparation to the experiment, calculate the theoretical intensity distribution of a double slit for integral ratios \( b/g = 2, 3 \) and \( 5 \) and determine the number of maxima and minima in the envelope function.

**Determination of the slit distance \( g \)**
In case of small diffraction angles $\alpha$ the position $x$ of the intensity extrema are (maxima and minima of the function $\cos^2 q$):

$$x_n = d \tan \alpha_n = d \sin \alpha_n = n \frac{\lambda}{g}.$$  \hfill (17)

Maxima: $n = 0, \pm 1, \pm 2, \ldots$ Minima: $n = \pm 1/2, \pm 3/2, \ldots$

Measure the position of the maxima and minima in the range of the central maximum of the envelope function and determine $g$ by linear regression.

**Estimate of slit width $b$**

Determine the slit width $b$ from the position of the 1st order minima of the envelope function (dashed line in Fig. 2.3).

**Determination of $g$ and $b$ using FFT**

As in the previous task perform a FFT of the intensity profile. Determine $g$ and $b$.

**Hints to task 3**

The transmission grating has to be located between the slit and the scale (Fig. 3.1) and to be adjusted (optical axis perpendicularly to the grating plane). In order to increase the measuring accuracy of the distance $l$ between the plane of the transmission grating and the glass scale one increases the distance between these until a $k$-th order maximum ($k = \pm 1, \pm 2$ or $\pm 3$) is seen at the scale end.

**Fig. 3.1** Measurement of the transmission grating constant

For the analysis equations are used which can be derived analogous to the double slit:

$$\sin \alpha_k = k \frac{\lambda}{g}, \quad \tan \alpha_k = \frac{x_k}{l},$$

where $\alpha_k$ is the angle of diffraction of the $k$-th maximum.
For the grating constant $g$ one obtains  
\[ g = k \lambda \sqrt{1 + \left( \frac{1}{x_k} \right)^2}. \]

**Additional material**


![Fig. 5 Spectral sensitivity of different optical sensors](image)

(a photodiode, b CCD sensor  
c human eye)

The manual of a Thorlabs CCD-Line-Camera is available for download in the download area of the website of the Undergraduate Physics Laboratory.