UNIVERSITÄT LEIPZIG

Fakultät für Physik und Geowissenschaften
Physikalisches Grundpraktikum

M12e "Coupled pendulums and degree of coupling"

Tasks

1. Measure for three different positions of the coupling spring:

a) the oscillation period T_1 of in-phase oscillations,
b) the oscillation period T_2 of out-of-phase oscillations,
c) the oscillation period T of the 'beat' mode oscillation and
d) the beat period T_S.

2. Calculate the beat oscillation period T and the beat period T_{SI} from the measured values T_1 and T_2; compare calculated and measured results.

3. Calculate the degree of coupling using T_1 and T_2 as well as the measured values of T and T_S for three positions of the coupling spring. Compare and discuss the uncertainty of the results.

4. Study the influence of the coupling spring position on the ratio of the oscillation periods T_1 and T_2 for several different positions of the spring analogously to *task 1*. Determine the spring constant of the coupling spring.

Literature

Physics, M. Alonso & J. F. Finn, Chap. 10.10
The Physics of Vibration and Waves, H. J. Pain, Wiley 1968, Chap. 3
Physikalisches Praktikum, 13. Auflage, Hrsg. W. Schenk, F. Kremer, Mechanik, 2.0, 2.4
http://www.cmt.phys.kyushu-u.ac.jp/~M.Sakurai/phys/physmath/union-e.html (Java Applet)

Accessories

Coupled pendulums, PC-workstation with PACs-Interface

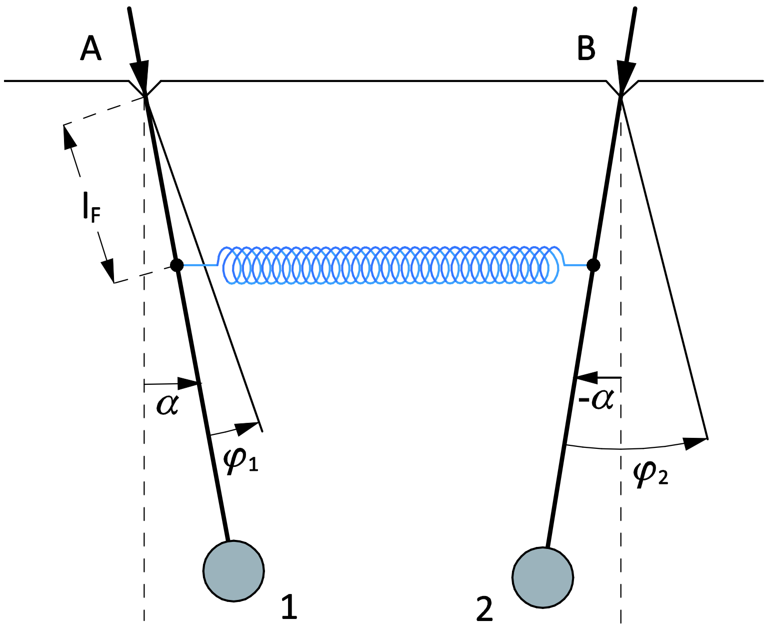
Keywords for preparation

- Physical pendulum, equation of motion, oscillation period
- Coupled pendulums, equation of motion, spring constant, restoring moment, differential equations
- Fundamental (in-phase and out-of-phase) oscillation modes, degree of coupling
- Beat oscillation, law of energy conservation, phase jump

Remarks

At the beginning of the experiment the demonstrator gives a brief introduction to the PACs-interface system.
For task 4 derive the relation $T_1^2/T_2^2 = f\,(l_F^2)$. For graphical representation use the values measured in *tasks 1 and 4*. Calculate the spring constant from the slope.

Fig. 1 Two coupled pendulums



<u>Torque M_0 of each pendulum:</u>
In the initial condition α=|-α| the pendulums are at rest and

$$M_0 = D\,\alpha\,, \qquad (1)$$

where *D* describes the restoring moment of the two identical stiff pendulums.

After deflections by φ_1 (pendulum 1) and φ_2 (pendulum 2) the restoring torques are

pendulum 1: $-D(\varphi_1 + \alpha)$
spring: $M_0 + D^*(\varphi_1 - \varphi_2)$ (2)
pendulum 2: $-D(\varphi_2 - \alpha)$

*D** is the restoring moment of the spring.

For the net value of the torque acting on pendulum 1 one obtains with Eq. (1)

$$-D(\varphi_1 + \alpha) + M_0 + D^*(\varphi_2 - \varphi_1) = -D\,\varphi_1 - D^*(\varphi_1 - \varphi_2)\,. \qquad (3)$$

Analogously one obtains for pendulum 2:

$$-D(\varphi_2-\alpha)-M_0-D^*(\varphi_2-\varphi_1)=-D\,\varphi_2+D^*(\varphi_1-\varphi_2)\,. \tag{4}$$

The symbols *A* and *B* in Fig. 1 denote the axis of rotation. The following equations describe the motion of the coupled pendulums:

$$I_1\frac{\mathrm{d}^2\varphi_1}{\mathrm{d}t^2}=-D\varphi_1-D^*\left(\varphi_1-\varphi_2\right) \tag{5-1}$$

$$I_2\frac{\mathrm{d}^2\varphi_2}{\mathrm{d}t^2}=-D\varphi_2+D^*\left(\varphi_1-\varphi_2\right) \tag{5-2}$$

With substitutions $\psi_1=\varphi_1+\varphi_2, \psi_2=\varphi_1-\varphi_2$ (normal coordinates) and assuming that the pendulums are identical (the same restoring moments $D = D_1 = D_2$ and moments of inertia $I=I_1=I_2$) one obtains the simplified equations

$$I\frac{\mathrm{d}^2\psi_1}{\mathrm{d}t^2}=-D\psi_1\,, \tag{6-1}$$

$$I\frac{\mathrm{d}^2\psi_2}{\mathrm{d}t^2}=-(D+2D^*)\psi_2 \tag{6-2}$$

with the solutions

$$\psi_1=a_1\cos\omega_1\,t+b_1\sin\omega_1\,t\,,\psi_2=a_2\cos\omega_2\,t+b_2\sin\omega_2\,t\;\;. \tag{7-1}$$

With the original angles φ_1 and φ_2 one obtains

$$\begin{aligned}\varphi_1&=\frac{1}{2}(a_1\cos\omega_1 t+b_1\sin\omega_1 t+a_2\cos\omega_2 t+b_2\sin\omega_2 t)\,,\\ \varphi_2&=\frac{1}{2}(a_1\cos\omega_1 t+b_1\sin\omega_1 t-a_2\cos\omega_2 t-b_2\sin\omega_2 t)\,.\end{aligned} \tag{7-2}$$

The circular frequencies of the normal modes in Eqs. (7-1) and (7-2) are

$$\omega_1=\frac{2\pi}{T_1}=\sqrt{\frac{D}{I}}\quad\text{and}\quad\omega_2=\frac{2\pi}{T_2}=\sqrt{\frac{D+2D^*}{I}}=\omega_1\sqrt{1+2\frac{D^*}{D}}\,. \tag{8}$$

Depending on the initial conditions one might distinguish three different solutions:

(i) '*In phase mode*'

$$\varphi_1(0)=\varphi_2(0)=\varphi_0\,,\ \frac{d\varphi_1}{dt}(0)=\frac{d\varphi_2}{dt}(0)=0\,,\ a_1=2\varphi_0,\ b_1=a_2=b_2=0,\ \varphi_1=\varphi_2=\varphi_0\cos\omega_1 t\,. \tag{9}$$

The two pendulums oscillate with the same period T_1. (Check this experimentally.)

(ii) '*Out of phase mode*':

$$\varphi_1(0)=-\varphi_2(0)=\varphi_0\,,\ \frac{d\varphi_1}{dt}(0)=\frac{d\varphi_2}{dt}(0)=0\,,\ a_2=2\varphi_0,\ a_1=b_1=b_2=0,\ \varphi_1=-\varphi_2=\varphi_0\cos\omega_2 t\,. \tag{10}$$

The two pendulums oscillate with the same period $T_2 < T_1$, but in opposite directions (phase shift π).

(iii) '*Beat mode*':

$$\varphi_1(0)=0,\ \varphi_2(0)=\varphi_0\,,\ \frac{d\varphi_1}{dt}(0)=\frac{d\varphi_2}{dt}(0)=0\,,\ a_1=-a_2=\varphi_0,\ b_1=b_2=0\,. \tag{11}$$

$$\varphi_1=\frac{\varphi_0}{2}(\cos\omega_1 t-\cos\omega_2 t)=\varphi_0\sin\left[\frac{1}{2}(\omega_2-\omega_1)t\right]\sin\left[\frac{1}{2}(\omega_2+\omega_1)t\right] \tag{12}$$

$$\varphi_2=\frac{\varphi_0}{2}(\cos\omega_1 t+\cos\omega_2 t)=\varphi_0\cos\left[\frac{1}{2}(\omega_2-\omega_1)t\right]\cos\left[\frac{1}{2}(\omega_2+\omega_1)t\right] \tag{13}$$

In general the coupled oscillations in the 'beat mode' show a complex behavior, but for weak coupling (D>>D^*; how you can check that experimentally?) you will observe an oscillation with the circular frequency $\omega = (\omega_1+\omega_2)/2$ modulated by the 'beat frequency' $\omega_S = (\omega_2-\omega_1)$. The corresponding periods are then

$$\frac{1}{T}=\frac{1}{2}\left(\frac{1}{T_2}+\frac{1}{T_1}\right)\ \text{and}\qquad \frac{1}{T_S}=\frac{1}{T_2}-\frac{1}{T_1}\qquad . \tag{14}$$

Using the definition of the 'degree of coupling'

$$k=\frac{D^*}{D+D^*} \tag{15}$$

and Eqs. (8) one can express the latter equation by the measured periods T_1 and T_2:

$$k=\frac{\omega_2^2-\omega_1^2}{\omega_2^2+\omega_1^2}=\frac{T_1^2-T_2^2}{T_1^2+T_2^2}\ . \tag{16}$$

This can be easily rewritten to express the degree of coupling by T and T_S; derive the corresponding equation.

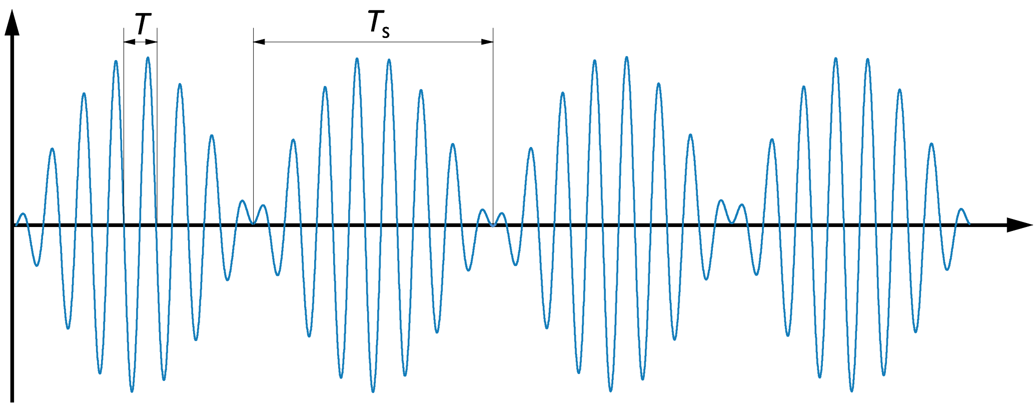


Fig 2. Beat oscillation of one pendulum in the case of weak coupling. The vertical axis gives the amplitude in arbitrary units while the horizontal axis measures time.

In the 4th task study the influence of the coupling spring position on the ratio of the oscillation periods of in-phase and out-of-phase oscillations using the equation

$$\frac{T_1^2}{T_2^2} = \frac{2D^* + D}{D} = 1 + 2\frac{D^*}{D} \qquad \text{(derive it)}, \tag{17}$$

In Eq. (17) D is the restoring moment of the two identical pendulums and D^* the restoring moment of the spring. For the restoring moment of the spring derive the relationship

$$D^* = l_F^2\, c\,. \tag{18}$$

l_F is the distance between the rotation axis and the coupling position of the spring.

The coupling or force constant of the spring can be determined by a simple experiment using Hooke's law. The directional moment of the pendulums is provided at the workplace.

Data for pendulums:

Place Nr.	number of pendulum	serial number	*m* (kg)	s_A (m)
1	I	034	1,318	0,725
1	II	041	1,325	0,724
2	III	040	1,308	0,582
2	IV	036	1,304	0,582
3	V	K05	1,338	0,845
3	VI	K12	1,339	0,845
4	VII	-	1,329	0,747
4	VIII	-	1,330	0,747

m ... mass of pendulum
s_A ... distance center of gravity to rotation axis

FS	pendulum I,II l_F (m)	pendulum III,IV l_F (m)	pendulum V,VI l_F (m)	pendulum VII,VIII l_F (m)
1	0,282	0,282	0,282	0,282
2	0,382	0,382	0,382	0,382
3	0,482	0,482	0,482	0,482
4	0,531	0,531	0,582	0,582
5	0,582	0,582	0,682	0,682
6	0,682	0,682	0,782	0,782
7	0,882	0,782	0,882	0,882
8	---	0,882	1,022	1,022
9	1,022	1,022		

FS ... number of spring position
l_F ... distance spring position to rotation axis