E17e “Fourier analysis of coupled electric oscillations”

Tasks

0. During the preparation of this experiment calculate the Fourier coefficients of a square and a triangular oscillation analytically. Using appropriate software (e.g. ORIGIN) calculate the Fast-Fourier-Transform (FFT) of a sine-, a triangle- and a square-signal. Compare the results for the triangular and square signal with the analytical results.

1. Measure the time dependence of a sine-, a triangular- and a square-signal with an USB oscilloscope using the internal voltage generator. Analyze these time traces using Fourier transformation. Compare the experimentally obtained with the theoretically calculated results. Study the aliasing effect (see Nyquist-Shannon sampling theorem) by measuring the FFT of a sine-signal for a fixed FFT cutoff frequency $f_G$ when tuning the frequency $f$ of the sine-signal from $f < f_G$ to $f > f_G$. Follow the evolution of the Fourier spectra and explain your observations.

2. For two capacitively coupled resonant circuits, in each case for ten different coupling capacitance values, measure the time traces and frequency spectra of the free beat oscillations in two configurations (low-point and high-point circuit). For each circuit, plot the ten frequency spectra into one graph. Determine the frequencies of the in-phase and out-of-phase oscillation modes. Calculate the coupling factors from the measured frequencies. Fit the theoretical expressions to the data to determine the value of the capacitance $C$.

3. Measure the beat period for one selected coupling factor. Compare the values to those obtained from the frequencies of the in-phase and out-of-phase oscillation modes.

4. Measure the coupling factor for two inductively coupled resonant circuits as a function of the distance between the inductance coils. Plot the coupling factor as a function of distance and analyze the distance dependence. Plot the frequency spectra in one graph.

Literature

- M. Alonso, E.J. Finn, Physics, 1992, section 27.12

Instruments and Accessories

USB oscilloscope with FFT function, switch, circuit board, resistances, capacitors, coils, notebook
Basic principles

The mathematical treatment of coupled resonant circuits is presented here under the assumption that the losses in the coils and capacitors are small (negligible damping) and that the values of the capacitors and inductances in both resonant circuits are the same.

Capacitive coupling, low-point circuit

In case of capacitively coupled resonant circuits one distinguishes between two measuring circuits, the so-called low-point and high-point circuits. In the first case, see Fig. 1, one obtains for the voltages and currents from Kirchhoff’s laws

\begin{equation}
U_{L,A} + U_{C,A} + U_K = L \frac{d^2 I_A}{dt^2} + \frac{1}{C} I_A + \frac{1}{C_K} (I_A - I_B) = 0,
\end{equation}

with the coupling current \( I_K = I_A - I_B \).

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{capacitive_coupling.png}
\caption{Capacitively coupled resonant circuits, low-point circuit}
\end{figure}

Using the relations for the inductive and capacitive voltages in terms of current and charge one obtains

\begin{equation}
L \frac{d^2 I_A}{dt^2} + \frac{1}{C} I_A + \frac{1}{C_K} (I_A - I_B) = 0,
\end{equation}

(2)

\begin{equation}
L \frac{d^2 I_B}{dt^2} + \frac{1}{C} I_B - \frac{1}{C_K} (I_A - I_B) = 0.
\end{equation}

(3)

\( C_K \) denotes the coupling capacitor. Adding Eqs. (2) and (3) yields

\begin{equation}
L \frac{d^2 (I_A + I_B)}{dt^2} + \frac{1}{C} (I_A + I_B) = 0,
\end{equation}

(4)

and as a solution of this differential equation one obtains

\begin{equation}
(I_A + I_B) = (I_{A,0} + I_{B,0}) \cos(\omega_1 t).
\end{equation}

(5)

\( \omega_1 \) denotes the angular eigenfrequency of the oscillation with

\begin{equation}
\omega_1 = \omega_0 = \frac{1}{\sqrt{LC}}.
\end{equation}

(6)

When Eqs. (2) and (3) are subtracted, one obtains

\begin{equation}
L \frac{d^2 (I_A - I_B)}{dt^2} + \left( \frac{1}{C} + \frac{2}{C_K} \right) (I_A - I_B) = 0,
\end{equation}

(7)

and as a solution:

\begin{equation}
(I_A - I_B) = (I_{A,0} - I_{B,0}) \cos(\omega_2 t).
\end{equation}

(8)
with the angular eigenfrequency

$$\omega_2 = \frac{1}{\sqrt{L \left( \frac{1}{\tau} + \frac{2}{C_K} \right)^{-1}}}.$$  \hfill (9)

Rearranging Eqs. (5) and (9) yields

$$I_A = \frac{1}{2} (I_{A,0} + I_{B,0}) \cos(\omega_1 t) + \frac{1}{2} (I_{A,0} - I_{B,0}) \cos(\omega_2 t), \hfill (10)$$

$$I_B = \frac{1}{2} (I_{A,0} + I_{B,0}) \cos(\omega_1 t) - \frac{1}{2} (I_{A,0} - I_{B,0}) \cos(\omega_2 t). \hfill (11)$$

If the resonant circuits are excited with in-phase voltages of the same amplitude, both resonant circuits oscillate with the same frequency $f_1 = \omega_1 / (2\pi)$ (in-phase oscillation mode). In case the excitation is made with voltages of the same amplitude, but with a phase difference of $180^\circ$ ($I_{A,0} = -I_{B,0}$), the frequency is given by $f_2 = \omega_2 / (2\pi)$ (out-of-phase oscillation mode). If only one circuit is excited at time $t = 0$ ($I_A \neq 0$, $I_B = 0$), beat oscillations are obtained for small coupling factors (Fig. 2) that contain both frequencies $f_1$ and $f_2$.

![Figure 2: Left: free oscillation with period $T$, beat period $T_S$. Right: Frequency spectra of capacitively coupled resonant circuits: in-phase (black curve), out-of-phase (grey curve) and (3) beat oscillation (blue curve).](image)

From the definition of the coupling factor $k$ with

$$k = \frac{\omega_2^2 - \omega_1^2}{\omega_1^2 + \omega_2^2}; \hfill (12)$$

one obtains with Eqs. (6) and (10) the coupling factor for the low-point circuit:

$$k_{C,T} = \frac{C}{C + C_K}. \hfill (13)$$

The angular frequency of the beat is given by

$$\omega_5 = \frac{2\pi}{T_S} = \omega_2 - \omega_1. \hfill (14)$$
Capacitive coupling, high-point circuit  The basic design of the high-point circuit for the investigation of capacitively coupled oscillations is shown in Fig. 3.

Applying Kirchhoff’s rules to mesh C as well as nodes D and E one obtains the equations

\[ 0 = I_K - \frac{I_{C,A}}{C} - \frac{I_{C,B}}{C} \]  (mesh C). \hspace{1cm} (15)

\[ I_{L,A} = I_K + I_{C,A} \]  (node D). \hspace{1cm} (16)

\[ I_{L,B} = I_K + I_{C,B} \]  (node E). \hspace{1cm} (17)

Using

\[ 0 = U_{L,A} + U_{C,A} = \frac{1}{C} I_{C,A} + L \frac{d^2 I_{L,A}}{dt^2} \]  (18)

yields for resonant circuit A with the Eqs. (15) and (16) the differential equation

\[ 0 = \frac{I_{C,A}}{C} + L \left( \frac{d^2 I_{C,A}}{dt^2} + \frac{C_K}{C} \frac{d^2 I_{L}}{dt^2} \right) \]  (19)

with \( I_+ = I_{C,A} + I_{C,B} \). With Eqs. (15) und (17) one obtains for circuit B:

\[ 0 = \frac{I_{C,B}}{C} + L \left( \frac{d^2 I_{C,B}}{dt^2} + \frac{C_K}{C} \frac{d^2 I_{L}}{dt^2} \right) \]  (20)

Equations (19) und (20) are again a coupled system of differential equations. Adding and subtracting these equations and using the currents \( I_+ \) and \( I_- = I_{C,A} - I_{C,B} \) one obtains the decoupled differential equations:

\[ 0 = \frac{I_+}{C} + L \left( 1 + 2 \frac{C_K}{C} \right) \frac{d^2 I_+}{dt^2} \]  (21)

\[ 0 = \frac{I_-}{C} + L \frac{d^2 I_-}{dt^2} \]  (22)

Mathematically the time dependence of the current is expressed as a linear combination

\[ I = A_+ \cos(\omega_+ t) + A_- \cos(\omega_- t) \]  (23)

with the angular frequencies

\[ \omega_+ = \sqrt{\frac{1}{L(C + 2C_K)}} \]  (24)

\[ \omega_- = \frac{1}{\sqrt{LC}} \]  (25)
\((C_k > 0, \omega_- > \omega_+)\). The amplitudes of both oscillation modes as well as their contribution to the total oscillation depend on the initial conditions. The coupling factor is obtained in analogy to the low-point circuit as

\[
k_{C,H} = \frac{\omega_-^2 - \omega_+^2}{\omega_-^2 + \omega_+^2} = \frac{C_K}{C + C_K}.
\] (26)

**Inductive coupling** The differential equations for the case of two inductively coupled resonant circuits are again obtained using Kirchhoff’s rules.

\[
I_A \frac{d^2 I_A}{dt^2} + L d^2 I_B \frac{d I_B}{dt^2} + M d^2 I_A \frac{d I_B}{dt^2} = 0, \tag{27}
\]

\[
I_B \frac{d^2 I_B}{dt^2} + L d^2 I_A \frac{d I_A}{dt^2} + M d^2 I_A \frac{d I_B}{dt^2} = 0. \tag{28}
\]

The strength of the coupling is parameterized by the mutual inductance \(M\). A decoupling of this system of differential equations is done in analogy to the capacitively coupled case. Adding and subtracting Eqs. (27) and (28) and introducing the currents \(I_+ = I_A + I_B\) and \(I_- = I_A - I_B\) yields:

\[
I_+ \frac{d^2 I_+}{dt^2} + (L + M) d^2 I_+ \frac{d I_+}{dt^2} = 0, \tag{29}
\]

\[
I_- \frac{d^2 I_-}{dt^2} + (L - M) d^2 I_- \frac{d I_-}{dt^2} = 0. \tag{30}
\]

The corresponding angular frequencies are

\[
\omega_+ = \frac{1}{\sqrt{C(L + M)}} \tag{31}
\]

\[
\omega_- = \frac{1}{\sqrt{C(L - M)}} \tag{32}
\]

With the angular frequency \(\omega_0 = 1/\sqrt{LC}\) of the uncoupled oscillation one obtains

\[
\omega_{\pm} = \frac{\omega_0}{\sqrt{1 \pm \frac{M}{L}}} = \frac{\omega_0}{\sqrt{1 \pm k_L}}, \tag{33}
\]
where \( k_L \) denotes the coupling factor of the inductively coupled circuits with

\[
k_L = \frac{M}{L}.
\]  

(34)

In the case of weak coupling one has

\[
\Delta \omega = \omega_+ - \omega_- = k_L \omega_0.
\]  

(35)

**Performing the experiment**

During the preparation of this laboratory session, for **task 0**, check the dependence of the FFT spectra on the number of periods contained in the data set used for FFT analysis as well as the dependence on the windowing-method (e.g. rectangle, Hanning). What is the difference of the FFT spectra of a sine- and a cosine-function? What is the meaning of the real/imaginary and amplitude/phase representations of the FFT spectra? Determine the FFT spectra of the function \( \sin(2\pi f_1 t) \cos(2\pi f_2 t) \) for two appropriately chosen frequencies \( f_1 \) and \( f_2 \) and the two cases \( f_1 \approx f_2 \) (beat) and \( f_1 = 10f_2 \) and interpret these. How do the FFT spectra of the functions \( \sin(2\pi f_1 t) \cos(2\pi f_2 t) \) and \( \cos(2\pi f_1 t) \sin(2\pi f_2 t) \) differ? Why?

In **task 1** that serves to familiarize yourself with the experimental setup and the practical application of the Fast-Fourier transformation (FFT), the internal voltage source of the oscilloscope is connected to input A of the oscilloscope. Set the oscilloscope views in such a way as to show the time representation and the frequency spectrum at the same time. Set the sampling rate to a fixed value and observe the frequency spectrum of a sine voltage when the frequency of this signal is varied from \( f < f_G \) to \( f > f_G \). Further, observe the frequency spectra of a triangular and a square voltage.

In **task 2**, in a preparatory experiment an uncoupled resonant circuit, see Fig. 5, should be set up with the electronic parts that will be further used in the experiment.

![Figure 5: Circuit for the measurement of the eigenfrequencies of a single uncoupled resonant circuit](image)

The capacitor (capacitance \( C \)) is charged by the dc voltage \( U_a \). After the circuit is closed by the switch \( (S) \) exponentially damped free oscillations are observed. Record time traces and frequency spectra of these oscillations. Measure and compare the eigenfrequencies of both uncoupled circuits. If the eigenfrequencies are different, a frequency tune can be made using additional electronic spares. To start with the main part of **task 2**, set up the circuit in Fig. 6. The capacitors are charged with the dc voltage \( U_a \), the charge states determine the initial conditions.

As can be seen in the sketch of Fig. 6, in the low-point circuit one has two series circuits of the left and right capacitor and coil each in parallel with the variable coupling capacitor \( C_k \). Accordingly, the left and right capacitor are differently charged (but with charges of the same sign). Therefore one has mixed initial conditions, and oscillations with frequencies \( f_1 \) and \( f_2 \) occur as indicated in Eqs. (6) and (9). To start the measurement the switch \( (S) \) is closed. For ten different values of the coupling capacitance measure the free oscillation and frequency spectrum with the digital oscilloscope, and transfer the latter data to the computer. Plot all FFT spectra in a single graph. Determine the frequencies \( f_1 \) and \( f_2 \) and calculate the coupling factor \( k_{C,T} \).
The so-called high-point circuit is set up following Fig. 7. This circuit allows for a separate measurement of the fundamental oscillation modes of this capacitively coupled system, since the initial charging states can be uniquely defined.

The initial condition \( U_a = U_b \) realizes the case of in-phase oscillations (frequency \( f_1 \)), whereas for \( U_a = -U_b \) out-of-phase oscillations (frequency \( f_2 \)) are obtained. Measure the frequency spectra for ten different values of the coupling capacitance, transfer the data to the computer and plot all spectra in one graph. Determine the frequencies \( f_1 \) and \( f_2 \) and calculate the corresponding coupling factors \( k_{CT} \) and \( k_{CH} \).

Plot the coupling factors \( k_{CT} \) and \( k_{CH} \) in one diagram as a function of the coupling capacitance. Fit Eqs. (13) and (26) to the data and determine the value of \( C \); compare with the nominal value.

In choosing a selection of values for the coupling capacitor \( C_k \) for the measurements in the low- and high-point circuits, take the different functional dependence of the coupling factors \( k_{CT} \) and \( k_{CH} \) as a function of \( C_k \) into account, to obtain an even distribution of the measurement values in the graph.

In task 3 beat oscillations can be measured after short circuiting the right switch (Fig. 7), charging the left capacitor and starting the measurement by closing the left switch. Determine the oscillation period as well as the beat period for one selected coupling capacitance from task 2. Compare the values \( T \) and \( T_S \) calculated from the frequencies \( f_1 \) and \( f_2 \) with the directly measured values.

For task 4 set up the circuit in Fig. 8. As coupling coils two flat identical circular coils are used that face each other coaxially and parallel.

After charging the capacitor \( C_A \) the switch is closed, the free oscillation and the frequency spectrum are measured. Determine for about eight distances between the coupling coils the frequencies \( f_+ \) and \( f_- \); determine the frequency \( f_0 \) of the uncoupled resonant circuit. Calculate the coupling factor \( k_L \) as well as the mutual inductance \( M \). Plot \( M \) as a function of distance \( d \) between the coils and discuss the result. To this end fit a general power law \( M \propto d^{-n} \) to the data. Is the exponent \( n \) compatible with the axial field distribution of a circular coil (experiment E7e)?

\( \text{Figure 6: Low-point circuit of capacitively coupled resonant circuits} \)

\( \text{Figure 7: High-point circuit of capacitively coupled resonant circuits} \)
The measurements are made with a USB-oscilloscope (Picoscope 5000 Series). The graphical user interface is shown in Fig. 9. A brief introduction into the use of the software is given at the beginning of the experiment.