Error Analysis

Preamble

Whenever a measurement is made, the result following from that measurement is always subject to uncertainty. The uncertainty can be reduced by making several measurements of the same quantity or by improving the experimental technique. But for fundamental reasons noise sources such as thermal noise or quantum fluctuations cannot be eliminated and the uncertainty cannot be reduced to zero.

Types of uncertainties

Random uncertainty and systematic error
A batch of digital watches was purchased from a factory with the expectation that they were all set to the correct time before dispatch. One might imagine that some weeks later a variety of times would be displayed by a selection of the watches.
There could be a number of reasons for the differences in the time displayed, such as
a) the crystal oscillators are running at different rates;
b) the watches were initially incorrectly set;
c) the crystal oscillators might be unstable;
d) the time standard at the factory was incorrect.

Random (statistical) uncertainties can be observed when repeated measurements of the same quantity give rise to different values. Many random factors including vibrations, variations in temperature and electronic noise, usually combine to form the 'random uncertainty' in an experiment.

Systematic error refers to a constant factor which influences all readings equally during an experiment. Such errors normally arise from the effect of a finite number of disturbances. Sources of systematic error could be anything from the contact resistance of a wire in an electrical circuit, to a stopwatch running too fast and the use of an improperly calibrated instrument.

In some experiments it might be difficult to see whether errors are due to random or systematic effects. Whenever possible, sources of systematic error should be eliminated or corrected for. Such sources of error are, however, not easily identified and the experimenter may never be aware of them.
In practice, the systematic and random uncertainties are simply added up to obtain the maximum uncertainty estimate:

$$u_{\text{max}}(X) = u_{\text{sys}}(X) + u_{\text{ran}}(X) = u(X).$$

Estimate of Uncertainties in Physical Measurements

Measured value of a physical quantity X: $x = x_{\text{best}} \pm u(X)$

x_{best} best estimate (mean) for X
$u(X)$ uncertainty or error in the measurement (systematic, random or statistical)

Relative uncertainty (error): $\delta_{\text{fract}} = \dfrac{u(X)}{x_{\text{best}}}$

Experimental uncertainties should almost always be rounded to one significant figure (digit):
measured g=(9.82$\pm$0.01) ms^{-2}

The last significant figure in any stated answer should usually be of the same order of magnitude (in the same decimal position) as the uncertainty.

Propagation of Uncertainties

If various quantities X_1, X_2, ..., X_n are measured with small uncertainties $u(X_i)$ [$u(X_i)/ X_i < 0.1$] and the measured values are used to calculate some quantity Y, then the uncertainties in the various X_i cause an uncertainty in Y. This can be estimated by a Taylor expansion to first order.

Example 1: Uncertainties in Sums and Differences

If Y is the sum or difference $Y = \sum_{i=1}^{n} (+or-) X_i$, then

$$u(Y) = \sum_{i=1}^{n} u(X_i) \qquad \text{(maximum error estimate)}$$

$$u(Y) = \sqrt{\sum_{i=1}^{n} u(X_i)^2} \qquad \text{(independent random errors)}$$

Example 2: Uncertainties in Products and Quotients

If Y is the product and quotient $Y = \frac{\prod_{i=1}^{n} X_i}{\prod_{k=1}^{m} X_k}$, then

$$\left|\frac{u(Y)}{Y}\right| = \sum_{i,k=1}^{n} \frac{u(X_{i,k})}{X_{i,k}} \qquad \text{(maximum error estimate)}$$

$$\left|\frac{u(Y)}{Y}\right| = \sqrt{\sum_{i,k=1}^{n} \left(\frac{u(X_{i,k})}{X_{i,k}}\right)^2} \qquad \text{(independent random errors)}$$

Example 3: Uncertainty in a Power

If Y is a power, $Y = X^m$, then

$$\frac{u(Y)}{|Y|} = |m| \frac{u(X)}{|X|} .$$

If $Y = Bx$, where B is known exactly, then $u(Y) = |B| \, u(X)$.

Example 4: Uncertainty in a Function of One Variable

If Y is a function of a single variable (measured quantity), $Y(X)$, then

$$u(Y) = \left|\frac{dY}{dX}\right| u(X).$$

General Case: Uncertainty in a Function of Several Variables

If Y is a function of several variables (measured quantities) X_1, X_2, ..., X_n, then

$$u(Y)_{\max} = \sum_{i=1}^{n} \left| \frac{\partial Y}{\partial X_i} \right| u(X_i) \qquad \text{(maximum error estimate)}$$

$$u(Y) = \sqrt{\sum_{i=1}^{n} \left(\frac{\partial Y}{\partial X_i} u(X_i) \right)^2} \qquad \text{(independent random errors)}$$

$$\sqrt{\sum_{i=1}^{n} \left(\frac{\partial Y}{\partial X_i} u(X_i) \right)^2} \le \sum_{i=1}^{n} \left| \frac{\partial Y}{\partial X_i} \right| u(X_i) \qquad \text{(always)}$$

Remark:

In some examples the difference

$$\Delta X = X_1 - X_2 \approx u(X)$$

between the results of experimental measurements can also be used as a measure of uncertainty (error).

Examples for the Maximum Uncertainty Estimate from Physics

1. Determination of the density ρ of a cylindrical metal rod

measured quantities mass of the rod m
length of the rod l
radius of the rod R

Equation to calculate the density $$\rho = \frac{m}{V} = \frac{m}{\pi R^2 l}$$

Relative maximum uncertainty estimate $$\frac{u_{\max}(\rho)}{\rho} = \frac{u(m)}{m} + \frac{u(l)}{l} + 2\frac{u(R)}{R}$$

2. Determination of the specific heat capacity c_f of a solid by a caloric experiment

measured quantities temperature of the hot solid ϑ_f
initial temperature of the fluid ϑ_{fl}
mixing temperature ϑ_m
mass of the fluid m_{fl}
mass of the solid m_f
heat capacity of the calorimeter C_K
specific heat capacity of the fluid c_{fl}

Equation to calculate c_f

$$c_f = \frac{(c_{fl}\, m_{fl} + C_K)\,(\vartheta_m - \vartheta_{fl})}{m_f\,(\vartheta_f - \vartheta_m)}$$

Maximum uncertainty estimate

$$u(c_f) = \frac{(\vartheta_m - \vartheta_{fl})}{m_f\,(\vartheta_f - \vartheta_m)}\left[c_{fl}\,u(m_{fl}) + u(C_K) + \frac{(c_{fl}\, m_{fl} + C_K)}{m_f}\,u(m_f) \right] +$$
$$\frac{(c_{fl}\, m_{fl} + C_K)}{m_f\,(\vartheta_f - \vartheta_m)}\left[u(\vartheta_{fl}) + \frac{(\vartheta_m - \vartheta_{fl})}{(\vartheta_f - \vartheta_m)}u(\vartheta_f) + \frac{(\vartheta_f - \vartheta_{fl})}{(\vartheta_f - \vartheta_m)}u(\vartheta_m) \right]$$

Statistical Analysis of Random Uncertainties

Let us consider *n* data points of a measured quantity *X* with measured values x_k, *k* = 1, 2, 3, ...

These values of *X* will be distributed about their mean value, which is some intermediate value not necessarily coincident with any of the data values. The mean value of the data set is written as $\overline{x}$ and is defined to be

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \ . \qquad (1)$$

Subtracting the mean value from any data value produces a residual. We can equally well view the data set as a collection of residuals (deviations from the mean),

$$\Delta x_i = x_i - \overline{x}\ , \qquad (2)$$

and we see that the mean is chosen so that the sum of the residuals is always zero

$$\sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} (x_i - \overline{x}) = 0\ . \qquad (3)$$

Since the transformation is linear the shape of the distribution of data is not altered. The residuals will have a mean value of zero.

A second piece of information from the data set, besides the mean, is related to its spread. This is given by a quantity called the standard deviation (s_x) and its square, called the variance. Data showing a large standard deviation will have the mean value poorly determined. Conversely, if the standard deviation is small, the data cluster closely, and the mean is well-determined. The standard deviation for any data point is often called its error, in the sense of 'uncertainty'. The better the data are determined the more detail they reveal.

The standard deviation (and the variance) depend on the scatter of the data about the mean and from it we find the uncertainty of the mean. The variance of the data set is defined as the mean value of the squares of the residuals :

$$\mathrm{var} = \frac{1}{n-1}\sum_{i=1}^{n} (x_i - \overline{x})^2\ . \qquad (4)$$

The standard deviation of the sample, s_x, is the square root of the variance of the sample,

$$s_x = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}\,. \qquad (5)$$

This root-mean-square result for the standard deviation is intuitive; it describes the uncertainty of every single measurement point. The mean value, however, is better defined with a smaller standard deviation. Indeed, the standard deviation of the mean value is given by

$$\bar{s}_x = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{s_x}{\sqrt{n}}\,. \qquad (6)$$

In experimental practice we adopt for the measurement uncertainty approximately twice the standard deviation of the mean:

$$u(X) \simeq 2\,\bar{s}_x\,. \qquad (7)$$

Normal or Gaussian Distribution

In addition to the measurement value and its uncertainty one should specify the probability that the true value falls into the interval $\bar{x} \pm u(X)$. In order to do so we assume that the data are distributed according to a normal distribution (Gaussian function)

$$p(x) = \frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{(x-\bar{x})^2}{2u^2}\right) \qquad (8)$$

This function is shown in the figure. Its width is given by the uncertainty $u = u(X)$. Since the Gaussian function is normalized, the probability that the true value falls in the range $\bar{x} \pm ku(X)$, where k is a numerical factor, is given by the area under the curve within this range. This leads to the confidence level of the measurement, where the confidence level simply is the probability for the true value to fall in the range $\bar{x} \pm ku(X)$. As seen from the figure the confidence level depends on k and is given by 68.3%, 95.5% and 99.7% for k = 1,2,3.