Semantics

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Sets

Sets

- Formalizes the idea of a *collection of elements*

Group membership

Given:

- the members of *Westgohliser Gartenkolonie*
- the members of *Michaeliskirchgemeinde Posaunenchor*

we can ask:

- are all *Posaunenchor* members also *Gartenkolonie* members?
- which members of the *Gartenkolonie* are also in the *Posaunenchor*?
- which members of the *Gartenkolonie* are not in the *Posaunenchor*?

Venn Diagrams

Relations between sets

- subset
- superset
- disjoint

Making new sets from old

- intersection
- union
- relative complement

Defining sets

Define a set
by describing its members

- the set whose members are: John, Bob, Sally, Alice
  $\{\text{John}, \text{Bob}, \text{Sally}, \text{Alice}\}$

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comprehension

- Φ is some condition ('driving a Honda')
- the set of things that satisfy Φ is written as

$$\{x : \Phi\}$$

Equality of sets

Define a set
by describing its members

Sets are equal
iff they have the same members

- $\{1, 2, 3\} = \{2, 1, 1, 3\}$
- $\{x : x^2 + 2x + 1 = 0\} = \{x : (x + 1)^2 = 0\}$

Membership

$x \in A$

x is one of the members of A

Recasting Relations

subset $A \subseteq B$ iff for all x, if $x \in A$ then $x \in B$

superset $A \supseteq B$ iff for all x, $x \in A$ only if $x \in B$

Membership

$x \in A$

x is one of the members of A

Recasting Operations

union $A \cup B := \{x : x \in A \text{ or } x \in B\}$

intersection $A \cap B := \{x : x \in A \text{ and } x \in B\}$

complement $A - B := \{x : x \in A \text{ and } x \notin B\}$

Empty sets

Stipulation:
There is a set with no members

Theorem (Uniqueness)
There is exactly one empty set

Notation $\emptyset$

Cardinality

$|A|$
the number of elements in A

- $|\emptyset| = 0$
- $|\{\emptyset\}| = 1$

Powerset

2^A (sometimes written $\wp(A)$)
all possible sets made from elements of A

$2^{\{0,1\}}$

- $\emptyset$
- $\{0\}$
- $\{1\}$
- $\{0, 1\}$

Theorem (Powerset cardinality)
$\left|2^A\right| = 2^{|A|}$

Sets of sets (of sets ...)

If $\mathcal{A}$ is a set *of sets*

- $\bigcup \mathcal{A}$
  - is the set which contains an element x just in case $x \in A$ for some $A \in \bigcup \mathcal{A}$
  - $\{x : \exists A \in \mathcal{A}. x \in A\}$
- $\bigcap \mathcal{A}$
  - is the set which contains an element x just in case $x \in A$ for every $A \in \bigcup \mathcal{A}$
  - $\{x : \forall A \in \mathcal{A}. x \in A\}$

Model Theory (I)

Models

> *A model is an abstract mathematical structure that we construct for describing hypothetical situations. Models are used for analyzing natural language expressions (words, phrases and sentences) by associating them with abstract objects.* *Winter*

need

1. the individuals in the world
2. an *interpretation* of the words of our language

$\mathcal{M} = (E, I)$

- E is a *set* of individuals
- I *interprets* the words of our language in the model

Interpretations

I interprets words in a model

What should $I(Bob)$ be?

- Bob is a person
- so: $I(Bob) \in E$

What should $I(kind)$ be?

- kind is not a person, but rather a property of a person
- to be able to decide whether a sentence like *Bob is kind* is true in our model, we need to know
  - who the kind people are
  - whether Bob is one of them
- so: $I(kind) \subseteq E$

From words to sentences

what we want is
modeling our semantic judgements of

- truth in a situation
- entailment

i.e. 'meanings' of sentences

- since we have given ourselves meanings for words
- we must find a way to build meanings for sentences out of these

we write $[\![\phi]\!]^{\mathcal{M}}$ for the meaning of arbitrary expressions in model $\mathcal{M}$

Bob is kind

Bob is kind

$$[\![\textit{Bob is kind}]\!] = \begin{cases} 1 & \text{if } [\![\textit{Bob}]\!] \in [\![\textit{kind}]\!] \\ 0 & \text{if } [\![\textit{Bob}]\!] \notin [\![\textit{kind}]\!] \end{cases}$$

in general

$$[\![\textit{name is Adj}]\!] = \begin{cases} 1 & \text{if } [\![\textit{name}]\!] \in [\![\textit{Adj}]\!] \\ 0 & \text{if } [\![\textit{name}]\!] \notin [\![\textit{Adj}]\!] \end{cases}$$

Bob is kind and chubby

Bob is kind and chubby

$$[\![\textit{Bob is kind and chubby}]\!] = \begin{cases} 1 & \text{if } [\![\textit{Bob}]\!] \in [\![\textit{kind and chubby}]\!] \\ 0 & \text{if } [\![\textit{Bob}]\!] \notin [\![\textit{kind and chubby}]\!] \end{cases}$$

in general

$$[\![\textit{Adj}_1 \textit{ and } \textit{Adj}_2]\!] = ???$$

Bob is kind and chubby

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in general

$$[\![\textit{Adj}_1 \textit{ and } \textit{Adj}_2]\!] = [\![\textit{Adj}_1]\!] \cap [\![\textit{Adj}_2]\!]$$

Models and Entailment

Consider
Bob is kind and chubby $\Longrightarrow$ *Bob is kind*

Assume that $[\![$*Bob is kind and chubby*$]\!] = 1$
what do we know about $[\![$*Bob is kind*$]\!]$?

The question
Does A entail B?

Its analysis
setting $1 =$ **true** and $0 =$ **false**:

$$\text{Is every model } \mathcal{M} \text{ such that } [\![A]\!]^{\mathcal{M}} \leq [\![B]\!]^{\mathcal{M}}?$$

Negation

Bob is not kind and chubby
is Bob chubby?

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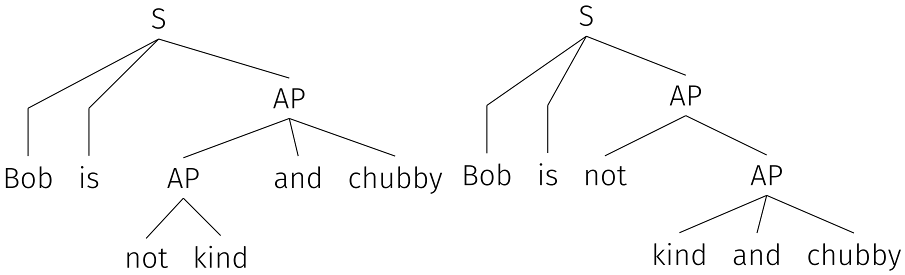

Analysing negation

in general

$$[\![\textit{not Adj}]\!] = ???$$

Analysing negation

in general

$$[\![\textit{not Adj}]\!] = E - [\![\textit{Adj}]\!]$$

Bob is not kind and chubby
is Bob chubby?

Sequences

Sequences

- Formalizes the idea of a *list of elements*

Defining a sequence

1. how long is it?
2. what is at each position?

- $|s|$ is the *length* of s
- s_i is the object in the i^{th} position of s

An example sequence

$\langle a, b, b, a \rangle$
is the sequence where:

1. it's length is 4
2. it has
   - a in the first position
   - b in the second position
   - b in the third position
   - a in the fourth position

Remember!

- sequences are 'pointy' ($\langle a, b \rangle$)
- sets are 'curly' ($\{a, b\}$)

Sequence Equality

Sequences are equal
iff they have

1. the same length
2. the same elements at each position

Examples

- $\{a, b, c\} = \{b, c, a\}$
- $\langle a, b, c \rangle \neq \langle b, c, a \rangle$

Empty sequence

A sequence is empty

- if its length is 0
- (and has, therefore, nothing at any position)

Theorem (Uniqueness)
There is exactly one empty sequence

Notation ϵ

Relations

a set of sequences of some fixed length

unary relations
a set of sequences of length 1

binary relations
a set of sequences of length 2

ternary relations
a set of sequences of length 3

Being in front of
is a relation between two individuals

- can model this as a binary relation *IFO*:
- a pair $\langle a, b \rangle \in IFO$ iff a is in front of b

Cross-Product

$A \times B$
$\{\langle a, b\rangle : a \in A \textit{ and } b \in B\}$

$A_1 \times \cdots \times A_n$
$\{\langle a_1, \ldots, a_n\rangle : a_1 \in A_1 \textit{ and } \ldots \textit{ and } a_n \in A_n\}$

A^n
$\underbrace{A \times \cdots \times A}_{n \text{ times}}$

Functions

A special kind of binary relation - $f \subseteq A \times B$

- each left-hand-side is paired with exactly one right-hand-side
- A is the domain
- B is the codomain

Properties of functions

injective each lhs is paired with a *unique* rhs (no two lhs' have the same rhs)

surjective each element of codomain is paired with some lhs

Counting

If there is an *injection* between two sets, A and B

- then B must be *at least as* large as A

Theorem (Cantor's lemma)
$A < 2^A$