Black holes

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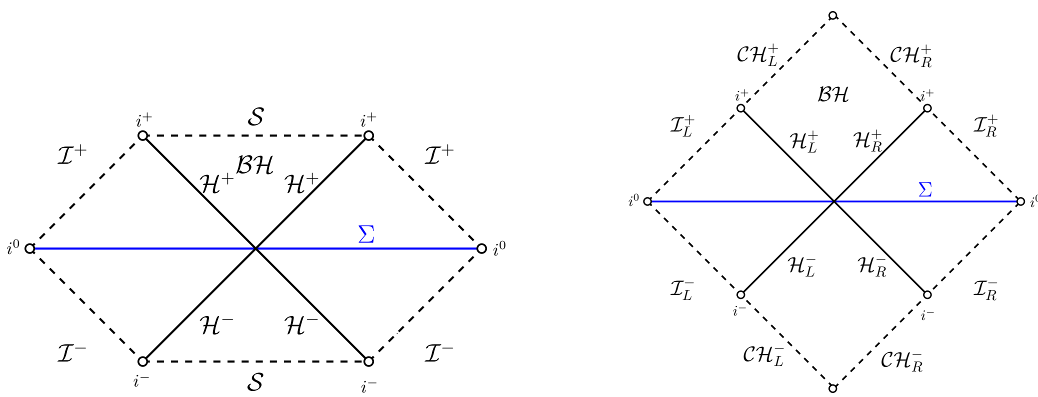
1 Introduction

Black holes stand as objects *par excellence*, showcasing the radical divergence between Einstein's and Newton's theories of gravity. They are spacetime solutions to Einstein's equations of general relativity that are fascinating from a geometric, analytic and astrophysical perspective and form the stage for an interplay between mathematics, theoretical physics and astrophysics.

In these lectures, we will cover the following topics on black holes:

- The geometric properties of **Schwarzschild black hole spacetimes** and the dynamics of geodesics in these spacetimes. We will see how to make sense of the following pictures, representing the maximally-extended Schwarzschild spacetime and the (sub-extremal) electromagnetically charged **Reissner–Nordström spacetimes**, respectively:



(a) maximally-extended Schwarzschild

(b) Sub-extremal Reissner–Nordström with asymptotically flat two-ended data

Figure 1: Penrose diagrams of Schwarzschild and Reissner–Nordström spacetimes

- The **Einstein equations coupled to matter in spherical symmetry**. We will prove that in the absence of matter, all spherically symmetric solutions to the Einstein equations are locally isometric to a Schwarzschild or the Minkowski spacetime (*Birkhoff's theorem*). We will discuss how to represent general, dynamical spherically symmetric black hole spacetimes pictorially and understand the following picture representing the formation of a dynamical black hole:

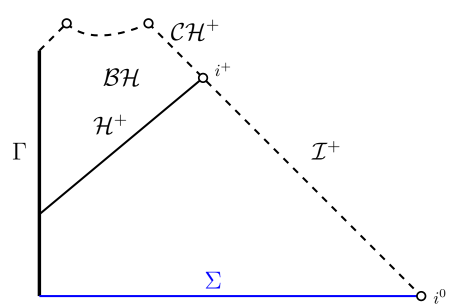


Figure 2: A possible Penrose diagram of a spacetime describing gravitational collapse to a black hole in spherical symmetry.

In this part of the course, we will encounter "baby versions" of some of the big theorems in black hole dynamics.

- The geometric properties of rotating **Kerr black hole spacetimes** and the dynamics of geodesics in these spacetimes;
- The **initial value** or **Cauchy problem**, which provides a systematic way to study dynamics of spacetimes;
- **Asymptotically flat initial data** and notion of *mass/energy*, *momentum* and *angular momentum* in general relativity;
- Trapped surfaces, causal geodesic incompleteness of spacetimes and the stability of the property of "black hole-ness" via **Penrose's incompleteness theorem**.
- Formulations of the **weak and strong cosmic censorship conjectures**.
- (*if time permits*) **linear waves** on Schwarzschild as a first tool towards understanding the dynamics of spacetimes arising from the evolution of small perturbations of Schwarzschild initial data.

We will present the above topics in a mathematically precise style. This means that we will organize the material with definitions, propositions, theorems and lemmas and that we will try to keep precise track of the various domains and codomains of the functions and maps that we encounter. This will serve as a way to eliminate any ambiguities and confusions that have historically appeared in the study of black holes.

Throughout the lecture notes, there will be EXERCISES that are meant to encourage an active reading of the text.

These lecture notes are still work in progress! Further chapters will be added as the course progresses. For this reason, there will inevitably be typos throughout the text. If you spot any typos or other errors, please do let me know.

1.1 Some useful literature

These lectures will not follow any existing textbooks too closely. Nevertheless, the list below *complements* the lectures and also provides different styles of presentation on some of the topics that we will cover.

- Robert M. Wald, *General relativity.* University of Chicago press, 1984.
  A classic text on general relativity. Note the use of "abstract index notation" for tensor fields that we will not follow in this course.
- Stephen W. Hawking and George F.R. Ellis, *The large scale structure of space-time.* Cambridge University Press, 1973.
  A complete discussion on many topics in general relativity. Includes Penrose diagrams of several solutions to the Einstein equations that we will also discuss. Contains a discussion on causality and Lorentzian geometry that goes beyond this course.
- Harvey S. Reall, *Mathematical Tripos Part III: General relativity.* `https://www.damtp.cam.ac.uk/user/hsr1000/part3_gr_lectures.pdf`, 2022
  Lecture notes for a course on general relativity taught at the University of Cambridge.
- Harvey S. Reall, *Mathematical Tripos Part III: Black holes.* `http://www.damtp.cam.ac.uk/user/hsr1000/black_holes_lectures_2020.pdf`, 2020
  Lecture notes for a course on black holes taught at the University of Cambridge. Contains a discussion on quantum field theory on curved spacetimes that will not be covered in this course.

- John M. Lee, *Introduction to Riemannian manifolds.* Springer, 1997.
  In contrast with the above references, this is a mathematics textbook. Introduces concepts of Lorentzian Riemannian geometry in a mathematically precise manner.

- Stefanos Aretakis, *General Relativity.* https://www.math.toronto.edu/aretakis/General%20Relativity-Aretakis.pdf, 2018
  Lecture notes on general relativity with a more mathematical focus and an accessible discussion of Penrose's incompleteness theorem.

- Demetrios Christodoulou, *Mathematical problems of general relativity I.* European Mathematical Society, 2008.
  An introduction to general relativity from a mathematical and PDE (partial differential equations) angle. Contains a discussion of conserved quantities on asymptotically flat spacetimes.

- Piotr T. Chruściel, *Geometry of black holes.* Oxford University Press, 2020
  Extensive, mathematically rigorous, discussion on geometric properties of several families of black hole spacetimes.

- Barrett O'Neill, *Semi-Riemannian geometry with applications to relativity.* Academic press, 1983.
  Exhaustive, mathematically precise treatment covering various aspects of Lorentzian geometry.

2 Preliminaries: basics of spacetime geometry

Before we start discussing black hole spacetimes, we will briefly review the main mathematical machinery that is required to introduce black hole spacetimes. The material in this section would typically be covered in a general relativity course or differential geometry course. This section is by no means a complete discussion and is meant to serve as a review as well as an opportunity to set the geometric notation used throughout the course.

Let main mathematical objects in the field of general relativity are so-called "spacetimes", which are pairs consisting of a 4-dimensional *smooth manifold* $\mathcal{M}$ and a Lorentzian metric g:

$$(\mathcal{M}, g),$$

together with a "time-orientation", an unambiguous notion of past and future. We will define each of these objects.

2.1 Manifolds

An n-dimensional smooth manifold is a mathematical object that locally "looks like" $\mathbb{R}^n$. This roughly means the following: $\mathcal{M}$ is a topological space[1] that can be covered by open sets $U \subset \mathcal{M}$ which come equipped with coordinate charts $\phi = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$. Then we demand that the coordinate transformations $\phi' \circ \phi^{-1}$, with $\phi' : U' \to \mathbb{R}^n$ another coordinate chart, are smooth on the intersection $U \cap U'$ and have smooth inverses.

Rather than making the above precise, we will mostly restrict to concrete special cases of manifolds. In particular, the following two special cases are sufficient to capture all the intricate properties of single black holes and their dynamics in four dimensions:[2]

$$\mathcal{M} \cong \mathbb{R} \times \mathbb{R}^3 =: \mathbb{R}^{3+1},$$
$$\mathcal{M} \cong \mathbb{R}^2 \times \mathbb{S}^2.$$

Here $\mathbb{S}^n$ denotes the unit round n-sphere: $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1}, \sum_{i=1}^n x_i^2 = 1\}$, which is an example of an n-dimensional manifold. The symbol $\cong$, in the context of manifolds, indicates equality *up to diffeomorphism*. For example, $\mathcal{M} \cong \mathbb{R} \times \mathbb{R}^3$ means that there exists a map $\psi : \mathcal{M} \to \mathbb{R} \times \mathbb{R}^3$ that is differentiable, bijective and has a differentiable inverse ψ^{-1}. A C^k-diffeomorphism is k times continuously differentiable and a smooth or C^∞-diffeomorphism is arbitrarily many times continuously differentiable.

We will also use the notation $\mathcal{M} = \mathbb{R} \times \Sigma$, with $\Sigma = \mathbb{R}^3$ or $\Sigma = \mathbb{R} \times \mathbb{S}^2$, to treat these two cases simultaneously. Later in the course, we will see that $(n+1)$-dimensional *globally hyperbolic spacetimes*, the class of spacetimes relevant for studying dynamical aspects of general relativity, always take the form $\mathcal{M} = \mathbb{R} \times \Sigma$, where Σ is an n-dimensional manifold.

In the case $\Sigma \cong \mathbb{R} \times \mathbb{S}^2$, we already notice an important property: *there exists no global coordinate chart on* Σ. Instead we often consider *standard spherical coordinates* (θ, φ), which are defined as follows:

$$x = \cos\varphi \sin\theta,$$
$$y = \sin\varphi \sin\theta,$$

[1] It comes with an notion of "open subset" and, related to this, continuity of maps on $\mathcal{M}$.

[2] In the case of spacetimes describing gravitational collapse to a black hole, $\mathcal{M} \cong \mathbb{R}^4$. It can also be shown that under very general assumptions (Hawking 1972), the exterior region of a stationary (time-independent) 4-dimensional black hole spacetime with reasonable matter is diffeomorphic to $\mathbb{R}^2 \times \mathbb{S}^2$. The event horizon (the boundary of the black hole region) at any fixed time, is diffeomorphic to $\mathbb{S}^2$. For dynamical spacetimes containing black hole regions, or in the case of higher spacetimes dimensions, the topology of the horizon at an instant in time may be more complicated.

$$z = \cos\theta.$$

Since we cannot always work in a single coordinate chart it will be convenient to introduce a few geometric objects that make sense even if we do not specify exactly what coordinate chart we are working in.

2.2 Tensors and tensor fields

In the physics literature, you may have encountered tensor fields as objects defined in the following way:

A tensor field is a list of functions that transforms like a tensor field.

Though this "definition" might seem rather circular, it can be made mathematically precise. We will, however, define tensors and tensor fields in a different, less utilitarian manner.

Before discussing tensor *fields*, we first need to introduce *tensors*. Before we introduce tensors, we recall the definition of *vectors* and *covectors*.

Definition 2.1. *Let V be a finite-dimensional vector space over $\mathbb{R}$. Then elements $v \in V$ are referred to as* vectors. *The* dual space *V^* is defined as follows:*

$$V^* := \{f : V \to \mathbb{R}\,,\ f \text{ is linear}\}.$$

Elements $f \in V^*$ are called covectors. *Sometimes elements of V are referred to as* contravariant vectors *and elements of V^* as* covariant vectors.

We can also consider the *double dual space* of V: $V^{**} := (V^*)^*$. It turns out that $(V^*)^*$ is *isomorphic* to V if V is finite dimensional. This means that there exist a bijective (injective and surjective, or "onto" and "one-to one") linear map:

$$\Phi : V \to V^{**}.$$

Indeed, we define Φ as follows: let $f \in V^*$, then

$$\Phi(v)(f) := f(v)$$

is clearly linear. We will also use "$\cong$" to denote to indicate an isomorphism between vector spaces, so $V \cong V^{**}$.

[Exercise: Show that Φ is injective, i.e. $\ker\Phi = \{0\}$. Then use the *rank-nullity theorem*, which says that general linear maps $L : V \to W$, with V, W vector spaces satisfy $\dim(\ker L) + \dim(\operatorname{ran} L) = \dim V$, to conclude that Φ is bijective.]

We can therefore also identify vectors $v \in V$ with linear maps of the form $v : V^* \to \mathbb{R}$.

Definition 2.2. *An (r, s)-tensor in V is a* multilinear *(over $\mathbb{R}$) map of the form:*

$$T : \underbrace{V^* \times \ldots \times V^*}_{r \text{ times}} \times \underbrace{V \times \ldots \times V}_{s \text{ times}} \to \mathbb{R}.$$

The set of (r, s)-tensors in V forms a n^{r+s}-dimensional vector space, which we denote as follows:

$$T^{(r,s)}(V).$$

We also use the following alternative notation for $T^{(r,s)}(V)$:

$$\underbrace{V \otimes \ldots \otimes V}_{r \text{ times}} \otimes \underbrace{V^* \otimes \ldots \otimes V^*}_{s \text{ times}}.$$

Let $\dim V = n$ and let $\{e_\mu\}_{1\le\mu\le n}$ be a basis of V. Let $f^\mu \in V^*$ be the unique linear maps satisfying: $f^\mu(e_\nu) = \delta^\mu_\nu$, with δ^μ_ν the Kronecker delta. Then $\{f^\mu\}_{1\le\mu\le n}$ form a basis of V^*, which we refer to as the dual basis associated to $\{e_\mu\}$. The (r,s)-tensors of the form

$$\underbrace{e_{\mu_1}\otimes\ldots\otimes e_{\mu_r}}_{r\text{ times}}\otimes\underbrace{f^{\nu_1}\otimes\ldots\otimes f^{\nu_s}}_{s\text{ times}}$$

form a basis of $T^{(r,s)}(V)$. The above notation is defined is defined as follows: let $\omega_i \in V^*$, $i \in \{1,\ldots,r\}$, and $v_j \in V$, $j \in \{1,\ldots,s\}$, then:

$$(e_{\mu_1}\otimes\ldots\otimes e_{\mu_r}\otimes f^{\nu_1}\otimes\ldots\otimes f^{\nu_s})(\omega_1,\ldots,\omega_r,v_1,\ldots,v_s) := e_{\mu_1}(\omega_1)\ldots e_{\mu_r}(\omega_r)f^{\nu_1}(v_1)\ldots f^{\nu_s}(v_s).$$

In other words, we can expand $T \in T^{(r,s)}(V)$ as follows:

$$T = \sum_{\mu_1=1}^{n}\ldots\sum_{\mu_r=1}^{n}\sum_{\nu_1=1}^{n}\ldots\sum_{\nu_s=1}^{n} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} e_{\mu_1}\otimes\ldots\otimes e_{\mu_r}\otimes f^{\nu_1}\otimes\ldots\otimes f^{\nu_s},$$

where $T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \in \mathbb{R}$. To shorten notation, we will usually omit the summation symbols. This is called the *Einstein summation convention*:

$$T = T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} e_{\mu_1}\otimes\ldots\otimes e_{\mu_r}\otimes f^{\nu_1}\otimes\ldots\otimes f^{\nu_s}.$$

Example 2.1. *Consider the tensor $T : V^* \times V \to \mathbb{R}$. Then $A : V \to V$, defined as*

$$A(v) := T(\cdot, v) \in V^{**} \cong V$$

is a linear map, which can be represented by a matrix after choosing a basis. The matrix coefficients are given by $T^\mu{}_\nu$.

To be able to define tensor *fields* on $\mathcal{M}$, we first need to introduce the relevant vector spaces that will play the role of V and V^*: the *tangent spaces* and *cotangent spaces* of the manifold $\mathcal{M}$.

Definition 2.3. *Let $\mathcal{M}$ be an $n+1$-dimensional manifold and let $x \in \mathcal{M}$. Suppose that $x \in U \subset \mathcal{M}$ is covered by the coordinate chart $\phi := (x^0,\ldots,x^n) : U \to \mathbb{R}^{n+1}$. Let $\gamma_1,\gamma_2 : \mathbb{R} \to \mathcal{M}$ be differentiable curves with $\gamma_1(0) = \gamma_2(0) = x$. Then we write $\gamma_1 \sim \gamma_2$ if $(\phi\circ\gamma_1)'(0) = (\phi\circ\gamma_2)'(0)$.*[3]

Let $\gamma : \mathbb{R} \to \mathcal{M}$ be differentiable with $\gamma(0) = x$. Denote

$$[\gamma] := \{\widetilde{\gamma} : \mathbb{R} \to \mathcal{M} \text{ differentiable, with } \widetilde{\gamma}(0) = x \text{ and } \tilde{\gamma} \sim \gamma\}$$

Then the tangent space at x, denoted $T_x\mathcal{M}$ is defined as follows:

$$T_x\mathcal{M} = \{[\gamma],\ \gamma : \mathbb{R} \to \mathcal{M} \text{ differentiable, with } \gamma(0) = x\}.$$

The set $T_x\mathcal{M}$ can be equipped with a vector space structure, by introducing "addition" and "scalar multiplication". This is not immediate, since we cannot simply "add" curves in $\mathcal{M}$ (unless $\mathcal{M} = \mathbb{R}^{n+1}$). Consider the following map:

$$d\phi_x : T_x\mathcal{M} \to \mathbb{R}^{n+1}$$
$$d\phi_x([\gamma]) := (\phi\circ\gamma)'(0).$$

[Exercise: Show that $d\phi_x$ is a bijection.]

Then we define for $\lambda \in \mathbb{R}$:

$$[\gamma_1] + \lambda\cdot[\gamma_2] := (d\phi_x)^{-1}(d\phi_x([\gamma_1]) + \lambda\cdot d\phi_x([\gamma_2])).$$

[3]The relation $\sim$ is an equivalence relation.

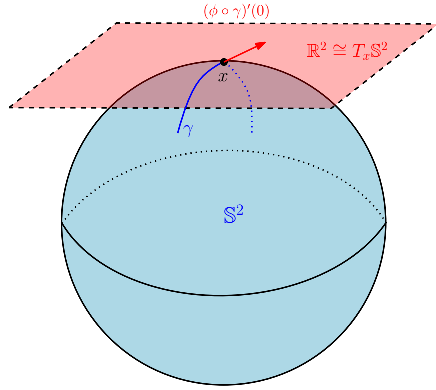


Figure 3: The tangent space $T_x\mathbb{S}^2$.

Definition 2.4. *We define the* coordinate basis vectors at $x \in \mathcal{M}$ as follows*:*

$$\frac{\partial}{\partial x^\mu}\Big|_x := d\phi_x^{-1}((\overbrace{0,\ldots,0,1}^{\mu},0,\ldots))$$

[Exercise: Convince yourself that the vectors $\frac{\partial}{\partial x^\mu}|_x$ form a basis of $T_x\mathcal{M}$, so $T_x\mathcal{M}$ is an $(n+1)$-dimensional vector space.]

For example, $T_x\mathbb{R}^4 \cong \mathbb{R}^4$ and $T_x(\mathbb{R}^2 \times \mathbb{S}^2) \cong \mathbb{R}^4$.

If $\psi = (y^0 \ldots, y^n) : U \to \mathbb{R}^{n+1}$ is another choice of coordinates and $\gamma : \mathbb{R} \to \mathcal{M}$ satisfies $(\psi\circ\gamma)'(0) = (\overbrace{0,\ldots,0,1}^{\mu},0,\ldots)$. Then

$$(x^\nu \circ \gamma)'(0) = ((x^\nu \circ \psi^{-1}) \circ (\psi \circ \gamma))'(0) = \sum_{\alpha=0}^{n} \frac{\partial(x^\nu \circ \psi^{-1})}{\partial y^\alpha}(y^\alpha \circ \gamma)'(0) = \frac{\partial(x^\nu \circ \psi^{-1})}{\partial y^\mu}.$$

Hence, we can write

$$\frac{\partial}{\partial y^\mu}|_x = \frac{\partial x^\nu}{\partial y^\mu}\frac{\partial}{\partial x^\nu}|_x,$$

where we shortened the notation by writing $\frac{\partial x^\nu}{\partial y^\mu}$ instead of $\frac{\partial(x^\nu\circ\psi^{-1})}{\partial y^\mu}$.

[Exercise: Show that $T_x\mathcal{M}$ is independent of the choice of coordinate chart $\phi : U \to \mathbb{R}^{n+1}$.]

We can think of $\frac{\partial}{\partial x^\mu}|_x$ as acting on functions $f \in C^\infty(\mathcal{M})$ in the following way: $\frac{\partial}{\partial x^\mu}|_x(f) = \frac{\partial(f\circ\phi^{-1})}{\partial x^\mu}|_x$. This leads to an alternative, but equivalent, characterization of $T_x\mathcal{M}$:

$$T_x\mathcal{M} \cong \{v : C^\infty(\mathcal{M}) \to \mathbb{R} \text{ linear, such that } v(fg) = v(f)g + fv(g)\ \forall f, g \in C^\infty(\mathcal{M})\}.$$

We denote $T_x^*\mathcal{M} = (T_x\mathcal{M})^*$. Then $dx^\mu|_x \in T_x^*\mathcal{M}$ and

$$dx^\mu|_x\left(\frac{\partial}{\partial x^\nu}\Big|_x\right) = \delta^\mu_\nu,$$

so $\{dx^\mu|_x\}$ is the dual basis associated to the basis $\{\frac{\partial}{\partial x^\mu}|_x\}$.

Definition 2.5. *A vector field X is a continuous map $X : \mathcal{M} \to T\mathcal{M}$, with $X(x) = (x, X_x)$, where $X_x \in T_x\mathcal{M}$ and $T\mathcal{M}$ is the* tangent bundle of $\mathcal{M}$*, which is defined as follows:*[4]

$$T\mathcal{M} := \bigcup_{x\in\mathcal{M}} \{x\} \times T_x\mathcal{M}.$$

[4]Note that the notion of continuity (and smoothness) of maps to $T\mathcal{M}$ makes sense since $T\mathcal{M}$ can be given the structure of a smooth manifold (it "locally looks like $\mathbb{R}^{2n}$).

We denote the space of all vector fields by $\mathcal{T}(\mathcal{M})$. A vector field X is C^k or smooth if the map X is C^k or smooth, respectively.

We can interpret $X \in \mathcal{T}(\mathcal{M})$ as a map $X : C^1(\mathcal{M}) \to C^0(\mathcal{M})$, where

$$X(f)(x) := X_x(f).$$

Note that on the right-hand side, we are interpreting vectors in $T_x\mathcal{M}$ as maps acting on functions.

Consider the coordinate chart $(U, \{x^\mu\})$. By construction of the differentiable structure on $T\mathcal{M}$, it follows that the following maps are smooth vector fields on U:

$$\frac{\partial}{\partial x^\mu} : U \to T\mathcal{M},$$
$$\frac{\partial}{\partial x^\mu}(x) := \left(x, \frac{\partial}{\partial x^\mu}\Big|_x \right).$$

We can express a general vector field $X \in \mathcal{T}(\mathcal{M})$ as follows when restricted to the domain U of the coordinate chart:

$$X|_U = X^\mu \frac{\partial}{\partial x^\mu}|_U,$$

with $X^\mu : U \to \mathbb{R}$ continuous functions (or C^k/smooth functions if the vector field X is C^k/smooth).

To shorten notation, we will write $\partial_\mu = \frac{\partial}{\partial x^\mu}$ when there is no ambiguity about the coordinate chart under consideration.

Definition 2.6. *The* cotangent bundle of $\mathcal{M}$ *is the set:*

$$T^*\mathcal{M} := \bigcup_{x \in \mathcal{M}} \{x\} \times T_x^*\mathcal{M}.$$

Covector fields *or* 1-forms *are continuous maps $\omega : \mathcal{M} \to T^*\mathcal{M}$, with $\omega(x) = (x, \omega_x)$. We denote the space of all covector fields/1-forms by $\Omega^1(\mathcal{M})$.*

We define $dx^\mu \in \Omega^1(\mathcal{M})$ as the 1-forms saitsfying $dx^\mu(x) = (x, dx^\mu|_x)$, with $\{dx^\mu|_x\}$ denoting the dual basis in $T_x^*\mathcal{M}$corresponding to the coordinate basis $\{\frac{\partial}{\partial x^\mu}|_x\}$.

Now we are ready to define *tensor fields*. Note that are several different but equivalent approaches to defining tensor fields and we will only present the one that is most convenient for the purposes of this course.

Definition 2.7. *An* (r, s)-tensor field T *is a map*

$$T : \underbrace{\Omega^1(\mathcal{M}) \times \ldots \Omega^1(\mathcal{M})}_{r \text{ times}} \times \underbrace{\mathcal{T}(\mathcal{M}) \times \ldots \times \mathcal{T}(\mathcal{M})}_{s \text{ times}} \to C^0(\mathcal{M})$$

that is multilinear over $C^0(\mathcal{M})$. This means that for all $f \in C^0(\mathcal{M})$, $\omega^i, \theta \in \Omega^1(\mathcal{M})$ with $i \in \{1, \ldots, r\}$, $X_j, Y \in \mathcal{T}(\mathcal{M})$ with $j = \{1, \ldots, s\}$:

$$T(\omega^1, \ldots, (f\omega^i + \theta), \ldots, \omega^r, X_1, \ldots, X_s) = fT(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) + T(\omega^1, \ldots, \theta, \ldots, \omega^r, X_1, \ldots, X_s),$$
$$T(\omega^1, \ldots, \omega^r, X_1, \ldots, (fX_j + Y), \ldots, X_s) = fT(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) + T(\omega^1, \ldots, \omega^r, X_1, \ldots, Y, \ldots, X_s).$$

We denote the space of (r, s)-tensor fields by $\mathcal{T}^{(r,s)}(\mathcal{M})$.

If $C^0(\mathcal{M})$ above is replaced with $C^k(\mathcal{M})$, with $k \leq \infty$, we say T is a C^k tensor field.

Example 2.2. *Note that $X \in \mathcal{T}(\mathcal{M})$ can be interpreted as a $(1,0)$-tensor field in the following way: for all $\omega \in \Omega^1(\mathcal{M})$*

$$(X(\omega))(x) := \omega|_x(X|_x).$$

Similarly, $\omega \in \Omega^1(\mathcal{M})$ can be interpreted as a $(0,1)$-tensor field in the following way: for all $X \in \mathcal{T}(\mathcal{M})$

$$(\omega(X))(x) := \omega_x(X_x).$$

Example 2.3. *The following maps are special cases of (r,s) tensor fields: let $X_i \in \mathcal{T}(\mathcal{M})$, $1 \leq i \leq r$, and $\omega^i \in \Omega^1(\mathcal{M})$, $1 \leq j \leq s$, then for all $Y_j \in \mathcal{T}(\mathcal{M})$ and $\theta^i \in \mathcal{T}(\mathcal{M})$*

$$(X_1 \otimes \ldots \otimes X_r \otimes \omega^1 \otimes \ldots \omega^s)(\theta^1, \ldots, \theta^r, Y_1, \ldots, Y_s) = X_1(\theta^1)X_2(\theta^2)\ldots X_r(\theta^r)\omega^1(Y_1)\omega^2(Y_2)\ldots \omega_s(Y_s).$$

In the domain of a coordinate chart $(U, \{x^\mu\})$, we can express a general tensor field $T \in \mathcal{T}(\mathcal{M})$ as follows: let $0 \leq \mu_i \leq n$ and $0 \leq \nu_j \leq n$, then

$$T|_U = T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s} \frac{\partial}{\partial x^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_s}|_U.$$

where $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s} \in C^0(U)$ for all $0 \leq \mu_i \leq n$ and $0 \leq \nu_1 \leq n$.

[Exercise: Let $\psi = (y^0, \ldots, y^n) : U \to \mathbb{R}^{n+1}$ be a different coordinate chart on U. Determine the relation between $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s}$ and $\widetilde{T}^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s}$, where

$$T|_U = \widetilde{T}^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s} \frac{\partial}{\partial y^{\mu_1}} \otimes \ldots \otimes \frac{\partial}{\partial y^{\mu_r}} \otimes dy^{\nu_1} \otimes \ldots \otimes dy^{\nu_s}|_U.]$$

A special class of tensor fields are the *alternating tensor fields* or *differential forms*.

Definition 2.8. *The space of s-forms on $\mathcal{M}$ (or differential forms of degree s, or alternating $(0,s)$-tensor fields) is denoted by $\Omega^s(\mathcal{M})$ and is defined as follows:*

$$\Omega^s(\mathcal{M}) := \left\{ T \in \mathcal{T}^{(0,s)}(\mathcal{M}) \,|\, T(X_{\sigma(1)}, X_{\sigma(2)}, \ldots, X_{\sigma(s)}) = \mathrm{sign}(\sigma) T(X_1, \ldots, X_s) \quad \forall \sigma \in S_s \right\},$$

with S_s the set of all permutations of the set $\{1, \ldots, s\}$ and $\mathrm{sign}(\sigma)$ the sign of the permutation σ which is $+1$ if σ can be written as an even number of transpositions (exchanges between two elements), for example, a cyclic permutation, and -1 if it can be written as an odd number of transpositions.

Example 2.4.

$$\Omega^2(\mathcal{M}) := \left\{ T \in \mathcal{T}^{(0,2)}(\mathcal{M}) \,|\, T(X,Y) = -T(Y,X) \right\}.$$

In particular, suppose $\omega, \theta \in \Omega^1(\mathcal{M})$. Then $\omega \otimes \theta$ or $\omega \otimes \theta + \theta \otimes \omega$ are not 2-forms, but $\omega \otimes \theta - \theta \otimes \omega$ is a 2-form.

With respect to a coordinate chart $(U, \{x^\mu\})$, $T \in \Omega^s(\mathcal{M})$ if and only if $T_{\nu_1 \ldots \nu_s}$ is fully anti-symmetric in its indices.

Definition 2.9. *The* wedge product *between an r-form ω and an s-form θ is defined as the following $(r+s)$-form:*

$$(\omega \wedge \theta)(X_1, \ldots, X_{r+s}) := \sum_{\sigma \in S_{r+s}} \mathrm{sign}(\sigma)(\omega \otimes \theta)(X_{\sigma(1)}, \ldots, X_{\sigma(r+s)})$$

By introducing the notation

$$T_{[\nu_1 \ldots \nu_s]} := \frac{1}{s!} \sum_{\sigma \in S_s} \mathrm{sign}(\sigma) T_{\nu_{\sigma(1)} \ldots \nu_{\sigma(s)}},$$

we can write

$$(\omega \wedge \theta)_{\nu_1 \ldots \nu_{r+s}} = (r+s)! \omega_{[\nu_1 \ldots \nu_r} \theta_{\nu_{r+1} \ldots \nu_{r+s}]}.$$

2.3 The exterior derivative

We now provide a definition of the exterior derivative d which can appear in front of a function $f : \mathcal{M} \to \mathbb{R}$ to obtain a 1-form df, but also in front of a general s-form ω to obtain an $s+1$-form $d\omega$.

Definition 2.10. *The exterior derivative $d : C^1(\mathcal{M}) \to \Omega^1(\mathcal{M})$ is defined as follows: for any $X \in \mathcal{T}(\mathcal{M})$*

$$(df)(X) := X(f).$$

With respect to a coordinate chart $(U, \{x^\mu\})$, we therefore have that

$$df|_U = \partial_\mu f dx^\mu|_U = \partial_{x^\mu} f dx^\mu|_U.$$

We can extend d as a map from the space of C^1 elements of $\Omega^s(\mathcal{M})$ to $\Omega^{s+1}(\mathcal{M})$ with $s \geq 0$ by imposing the following additional requirements:

$$\begin{aligned} d(\omega \wedge \eta) &= d\omega \wedge \eta + (-1)^r \omega \wedge d\eta \quad \text{for all } \omega \in \Omega^r(\mathcal{M}) \text{ and } \eta \in \Omega^s(\mathcal{M}), \\ d \circ d &= 0. \end{aligned}$$

For example, if $\omega = \omega_\mu dx^\mu$ with respect to a coordinate chart, then

$$d\omega|_U = d(\omega_\nu) \wedge dx^\nu|_U + 0 = \partial_\mu \omega_\nu dx^\mu \wedge dx^\nu|_U = (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu) dx^\mu \otimes dx^\nu|_U = 2\partial_{[\mu}\omega_{\nu]} dx^\mu \otimes dx^\nu|_U.$$

[Exercise: Show that, for a C^1 r-form $\omega \in \Omega^r(\mathcal{M})$, the following identity holds in the domain of a coordinate chart $(U, \{x^\mu\})$:

$$d\omega|_U = (r+1)\partial_{[\nu_1}\omega_{\nu_2 \ldots \nu_{r+1}]} dx^{\nu_1} \otimes \ldots \otimes dx^{\nu_{r+1}}|_U.]$$

Let $\omega \in \Omega^s(\mathcal{M})$. If $d\omega = 0$, we say ω is *closed*. If $s \geq 1$ and $\omega = d\eta$ for some $\eta \in \Omega^{s-1}(\mathcal{M})$, then we say ω is *exact*. Note that in the latter case, $d\omega = d^2\eta = 0$, so ω must be closed. The converse need not be true: closed forms need not be exact.[5]

2.4 Pullbacks and pushforwards

Let $\psi : \mathcal{M} \to \mathcal{N}$ be a smooth diffeomorphism between two manifolds $\mathcal{M}$ and $\mathcal{N}$.

Example 2.5. *Let $\phi : U \to \mathbb{R}^n$ be a coordinate chart on an n-manifold $\mathcal{M}$. Then we can use diffeomorphisms $\psi : U \to U$ to obtain a different coordinate chart $\widetilde{\phi} = \phi \circ \psi$.*

Conversely, given two coordinate charts $\phi, \widetilde{\phi} : U \to \mathbb{R}^n$, the map $\psi = \phi^{-1} \circ \widetilde{\phi}$ defines a diffeomorphism on U. The study of diffeomorphism on a single manifold $\mathcal{M}$ is therefore closely related to coordinate transformations. For example, let $\phi = \mathrm{id}$ be a Cartesian coordinate chart on $\mathbb{R}^3$ and $\widetilde{\phi}$ a spherical coordinate chart on s subset of $\mathbb{R}^3$. Then the change of coordinates from Cartesian to spherical corresponds to a diffeomorphism: $\psi = \widetilde{\phi}$.

There is a natural way to associate to diffeomorphism ψ maps that act between spaces of tensor fields on $\mathcal{N}$ and $\mathcal{M}$. From the perspective of coordinate charts, this is closely related to determining how components of a tensor field with respect to one coordinate chart are related to its components with respect to another coordinate chart.

[5] When restricted to a suitably small neighbourhood of any point $x \in \mathcal{M}$, all closed forms are exact. Under a topological condition on the manifold $\mathcal{M}$, namely *simply connectedness*, all closed forms are globally exact. Simply connectedness roughly means that any loop in $\mathcal{M}$ cannot be continuously contracted to a single point.

Definition 2.11. *The* pullback map $\psi^* : C^0(\mathcal{N}) \to C^0(\mathcal{M})$ *is defined as follows:*

$$\psi^* f = f \circ \psi,$$

for all $f \in C^0(\mathcal{N})$.

The pushforward map $\psi_* : \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{N})$ *is defined as follows:*

$$(\psi_* X)(f) := X(f \circ \psi) \circ \psi^{-1},$$

for all $X \in \mathcal{T}(\mathcal{M})$ *and* $f \in C^1(\mathcal{N})$.

The pullback map $\psi^* : \Omega^1(\mathcal{N}) \to \Omega^1(\mathcal{M})$ *is defined as follows:*

$$(\psi^* \omega)(X) := \omega(\psi_* X) \circ \psi,$$

for all $X \in \mathcal{T}(\mathcal{M})$.

The pullback map $\psi^* : \mathcal{T}^{(r,s)}(\mathcal{N}) \to \mathcal{T}^{(r,s)}(\mathcal{M})$ *is defined as follows:*

$$(\psi^* T)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_s) = T((\psi^{-1})^* \omega^1, \ldots, (\psi^{-1})^* \omega^r, \psi_* X_1, \ldots, \psi_* X_s) \circ \psi,$$

for all $\omega^i \in \Omega^1(\mathcal{M})$, *with* $i \in \{1, \ldots, r\}$ *and* $X_j \in \mathcal{T}(\mathcal{M})$, *with* $j \in \{1, \ldots, s\}$.

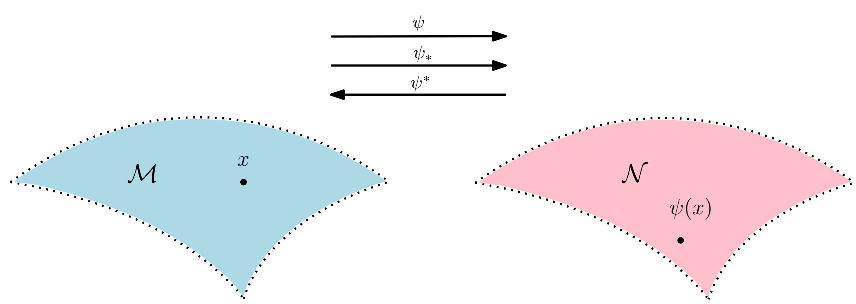


Figure 4: The pushforward and pullback maps.

The behaviour of the pullback map to linear order is determined by the *Lie derivative.*

Definition 2.12.

- *An* integral curve *of a vector field* $X \in \mathcal{T}(\mathcal{M})$ *is a map* $\gamma : \mathbb{R} \supset I \to \mathcal{M}$ *satisfying* $\gamma'(s) = X_{\gamma(s)} \in T_{\gamma(s)}\mathcal{M}$. *Here,* $\gamma'(s) = [\delta] \in T_{\gamma(s)}\mathcal{M}$, *with* $\delta : \mathbb{R} \to \mathcal{M}$ *satisfying* $\delta(0) = \gamma(s)$ *and* $(\phi \circ \delta)'(0) = (\phi \circ \gamma)'(s)$, *where* $\phi : U \to \mathcal{M}$ *is a coordinate chart and* $\gamma(s) \in U$.
- *A* global flow *is a map* $\psi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ *satisfying* $\psi(s, p) = \gamma(s)$, *with* $\gamma : \mathbb{R} \to \mathcal{M}$ *an integral curve of* X *such that* $\gamma(0) = p$, *assuming that the domain of all integral curves* γ *is* $\mathbb{R}$. *Denote* $\psi_s = \psi(s, \cdot)$. *Then the following properties hold (*Exercise*)*

$$\psi_0 = \mathrm{id}, \tag{2.1}$$
$$\psi_{s+\tau} = \psi_s \circ \psi_\tau \quad \text{for all } s, \tau \in \mathbb{R}. \tag{2.2}$$

  For general $X \in \mathcal{T}(\mathcal{M})$, ψ *need not be defined for all* $s \in \mathbb{R}$. *In this case, we simply refer to* ψ *as a* flow.
- *Conversely, given a* C^1 *map* $\psi : \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ *satisfying* (2.1) *and* (2.2), *the map* $X(f) = \frac{d}{ds}|_{s=0}(f \circ \psi_s)$, *with* $f \in C^1(\mathcal{M})$ *defines a vector field* $X \in \mathcal{T}(\mathcal{M})$, *such that* ψ *is the global flow corresponding to* X *(*Exercise*).*

- *The* Lie derivative in the direction of X *of a tensor field* $T \in \mathcal{T}^{(r,s)}(\mathcal{M})$ *is defined as follows:*

$$\mathcal{L}_X T = \frac{d}{ds}\Big|_{s=0} \psi_s^* T,$$

with ψ_s *corresponding to the flow of* X.

For $X, Y \in \mathcal{T}(\mathcal{M})$, we can alternatively express:

$$\mathcal{L}_X Y = \frac{d}{ds}\Big|_{s=0} (\psi_{-s})_* Y = [X, Y],$$

with $[X,Y](f) = X(Y(f)) - Y(X(f))$ the commutator.

It can be shown that the Lie derivative satisfies the following additional properties: for all $X, X_i \in \mathcal{T}(\mathcal{M})$, with $i \in \{i, \dots, s\}$, $\omega^j \in \Omega^1(\mathcal{M})$, with $j \in \{1, \dots, r\}$, $f \in C^\infty(\mathcal{M})$, $T \in \mathcal{T}^{(r,s)}(\mathcal{M})$, $S \in \mathcal{T}^{(r',s')}(\mathcal{M})$

$$\begin{aligned}
\mathcal{L}_X(f) &= X(f), \\
\mathcal{L}_X(S \otimes T) &= (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_S T), \\
\mathcal{L}_X(T(\omega^1, \dots, \omega^r, X_1, \dots, X_s)) &= (\mathcal{L}_X T)(\omega^1, \dots, \omega^r, X_1, \dots, X_s) \\
&\quad + T(\mathcal{L}_X \omega^1, \dots, \omega^r, X_1, \dots, X_s) + \dots + T(\omega^1, \dots, \mathcal{L}_X \omega^r, X_1, \dots, X_s) \\
&\quad + T(\omega^1, \dots, \omega^r, \mathcal{L}_X X_1, \dots, X_s) + \dots + T(\omega^1, \dots, \omega^r, X_1, \dots, \mathcal{L}_X X_s) \\
[\mathcal{L}_X, d] f &= 0.
\end{aligned}$$

2.5 Metrics

Definition 2.13. *A* metric *(tensor field) is a* $(0,2)$*-tensor field* $g \in \mathcal{T}^{(0,2)}(\mathcal{M})$ *that satisfies the additional two properties:*

- *(symmetry)* $g(X,Y) = g(Y,X)$,
- *(non-degeneracy) if* $g(X,Y) = 0$ *for all* $Y \in \mathcal{T}(\mathcal{M})$, *then* $X = 0$.

We say g *is a* Riemannian metric *(tensor field) if moreover*

- *(positive-definiteness)* $g(X,X) \geq 0$ *with equality if and only if* $X = 0$.

To define a Lorentzian metric (tensor field) we first make the following observation. The following map is bilinear and non-degenerate:[6]

$$\begin{gathered}
g_x : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}, \\
g_x(v,w) = g(V,W)(x), \quad V|_x = v, W|_x = w.
\end{gathered}$$

Pick a basis $\{e_\mu\}$ of $T_x\mathcal{M}$. Let G be the matrix with coefficients $G_{\mu\nu} = g_x(e_\mu, e_\nu)$. Then, by the symmetry and non-degeneracy of g, G is invertible and symmetric. This means we can diagonalize G and that its eigenvalues are non-zero.

We define the *signature* (n_+, n_-) of g_x as follows: n_+ is the total number of positive eigenvalues and n_- is the total number of negative eigenvalues of G. This definition only makes sense if the signature is a quantity that is invariant under a choice of basis. That is to say, if we define $\tilde{G}$ as the matrix with coefficients $\tilde{G}_{\mu\nu} = g_x(\tilde{e}_\mu, \tilde{e}_\nu)$, then we need to show that n_+ and n_- of $\tilde{G}$ are the same as n_+ and n_- of G. This follows from *Sylvester's Law of Inertia* (Exercise).[7]

For a Riemannian metric g, we have that $n_- = 0$ for all $x \in \mathcal{M}$.

[6] Exercise: Convince yourself of the existence of V, W and the fact that $g(V,W)$ only depends on $V|_x$ and $W|_x = v$.

[7] Sylvester's Law of Inertia: if A is an $n \times n$ symmetric matrix with n_+ positive eigenvalues and n_- negative eigenvalues, then, for any $n \times n$ matrix B, the symmetric matrix BAB^T will also have n_+ positive eigenvalues and n_- negative eigenvalues.

Definition 2.14. *A metric g is* Lorentzian *if $n_- = 1$ for all $x \in \mathcal{M}$.*

Not every manifold $\mathcal{M}$ admits a Lorentzian metric. For example, the sphere $\mathbb{S}^4$ does not admit a Lorentzian metric.[8]

[Exercise: Manifolds of the form $\mathcal{M} = \mathbb{R} \times \Sigma$ always admit a Lorentzian metric. *Hint*: You may use that Σ always admits a Riemannian metric.]

2.5.1 Basic concepts of causality

The structure of a Lorentzian metric allows us to talk about *causality* and make sense of whether spacetime points are in the future or past of other spacetime points.

Definition 2.15. *A vector $v \in T_x\mathcal{M}$ is:*

- timelike *if $g_x(v, v) < 0$,*
- null *if $g_x(v, v) = 0$ and $v \neq 0$,*
- spacelike *if $g_x(v, v) > 0$.*

Via the isometric identification $T_x\mathcal{M} \cong \mathbb{R}^{n+1}$, we can identify the union of all null vectors at $x \in \mathcal{M}$ with a double cone in $\mathbb{R}^{n+1}$ minus the vertex:

$$C_x = \left\{ v \in \mathbb{R}^{n+1} \setminus \{0\} \,|\, v_0^2 = \sum_{i=1}^{n} (v^i)^2 \right\}.$$

The cone C_x is called the lightcone*.*

The union of all timelike vectors at $x \in \mathcal{M}$ can be identified with the interior of a double cone in $\mathbb{R}^{n+1}$:

$$I_x = \left\{ v \in \mathbb{R}^{n+1} \,|\, v_0^2 > \sum_{i=1}^{n} (v^i)^2 \right\}.$$

The union of all spacelike vectors is the interior of the following double cone:

$$S_x = \left\{ v \in \mathbb{R}^{n+1} \,|\, v_0^2 < \sum_{i=1}^{n} (v^i)^2 \right\}.$$

A vector field $X \in \mathcal{T}(\mathcal{M})$ is timelike/spacelike/null if X_x satisfies exactly one of the three above three conditions for all $x \in \mathcal{M}$.

We similarly say that a curve $\gamma : \mathbb{R} \supset I \to \mathcal{M}$ is timelike/spacelike/null if its tangent vector $\gamma'(s)$ is everywhere timelike/spacelike/null, respectively. We say a curve is causal *if it is timelike or null and* achronal *if it is spacelike or null.*

We will frequently refer to timelike curves of timelike vector fields as (idealized) *observers*.

Definition 2.16. *A (smooth)* hypersurface *is a subset $\Sigma \subset \mathcal{M}$, which has the structure of an n-dimensional manifold, such that the inclusion map $\iota : \Sigma \to \mathcal{M}$, $\iota(x) = x$, is a (smooth) diffeomorphism onto its image (i.e. $\iota : \Sigma \to \iota(\Sigma)$ is a diffeomorphism).*[9]

[8] By the so-called Hairy Ball Theorem, the sphere $\mathbb{S}^4$ does not admit a vector field that is non-vanishing everywhere. If we could equip $\mathbb{S}^4$ with a Lorentzian metric, then once can show that there must exist a vector field that does not vanish everywhere, which is a contradiction. Conversely, if a manifold admits a non-vanishing vector field X, we can construct the following Lorentzian metric: $g = -2\frac{X^\flat \otimes X^\flat}{g(X,X)} + h$, with h a Riemannian metric on $\mathcal{M}$ and where $X^\flat$ is a 1-form dual to X that we define below.

[9] More generally, a subset $\Sigma \subseteq \mathcal{M}$ is called a (smooth) *embedded k-dimensional submanifold* if it is a k-dimensional manifold and the inclusion map $\iota : \Sigma \to \mathcal{M}$ is a (smooth) diffeomorphism onto its image.

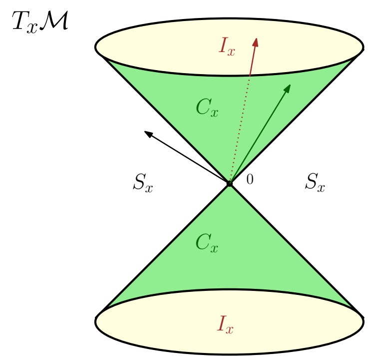


Figure 5: The double lightcone C_x (drawn as a subset of $\mathbb{R}^{n+1}$) and its interior I_x and exterior S_x.

- *If the induced metric*[10] *on Σ is Riemannian, we say Σ is a* spacelike hypersurface*. Equivalently, there exists a timelike vector field $N \in \mathcal{T}(\mathcal{M})$ such that $g(N, X) = 0$ for all $X \in \mathcal{T}(\mathcal{M})$ with $X_x \in T_x\Sigma$ (i.e. N is normal to Σ).*
- *If the induced metric is Lorentzian, we say Σ is a* timelike hypersurface*. Equivalently, there exists a spacelike vector field $N \in \mathcal{T}(\mathcal{M})$ such that $g(N, X) = 0$ for all $X \in \mathcal{T}(\mathcal{M})$ with $X_x \in T_x\Sigma$.*
- *If there exists a vector field $L \in \mathcal{T}(\mathcal{M})$ that is null along Σ, such that $g(L, X)|_\Sigma \equiv 0$ for all $X \in \mathcal{T}(\mathcal{M})$ with $X_x \in T_x\Sigma$, we say Σ is a* null hypersurface*.*

Definition 2.17. *The* arc length *of a spacelike curve γ between $s = 0$ and $s = s_0$ is defined as follows:*

$$\ell[\gamma](s_0) := \int_0^{s_0} \sqrt{g_{\gamma(s)}(\gamma'(s), \gamma'(s))}\, ds.$$

In the case of a timelike curve γ, we define the proper time *of γ between $s = 0$ and $s = s_0$ as follows:*

$$\tau[\gamma](s_0) := \int_0^{s_0} \sqrt{-g_{\gamma(s)}(\gamma'(s), \gamma'(s))}\, ds.$$

We need to equip our manifold with an additional structure to be able to define the notion of a spacetime.

Definition 2.18. *A Lorentzian manifold $(\mathcal{M}, g)$ is said to be* time-orientable *if there exists a global timelike vector field T. We define a* spacetime *to be the triple*

$$(\mathcal{M}, g, [T]),$$

with $[T] := \{X \in \mathcal{T}(\mathcal{M}) \,|\, g(X, T) < 0\}$ the time orientation*. We will refer to a spacetime as $(\mathcal{M}, g)$ and omit $[T]$ from the notation.*

Remark 2.1. *In our definition of a "spacetime", we will also include "manifolds-with-boundary" $\mathcal{M}$. These, strictly speaking, are not "manifolds" because in a neighbourhood of points on their boundary, they are look likes a half-space $\{(x^1, \ldots, x^n) \mid x^n \geq 0\} \subset \mathbb{R}^{n+1}$ rather than $\mathbb{R}^n$. For example, the set $\{x^n \geq 0\} \subset \mathbb{R}^{n+1}$ itself is a manifold-with-boundary, or the set $[0, 1]$.*

[10]If $\iota : \Sigma \to \mathcal{M}$ is the inclusion map, which just maps $x \in \Sigma$ to $x \in \Sigma \subset \mathcal{M}$, then the induced metric g_Σ is defined as $g_\Sigma = \iota^* g$ (the pullback makes sense, even though ι is not a diffeomorphism!), i.e. for $X, Y \in \mathcal{T}(\Sigma)$ and $x \in \Sigma$, $g_\Sigma(X, Y)(x) := g(\iota_* X, \iota_* Y)(x)$ for all $x \in \Sigma$, with $(\iota_* X)(f) = X(f \circ \iota)$. In practice, you may use that Σ can locally be expressed as the level set of a function with respect to a coordinate chart. For example $x^0 = h(x^1, \ldots, x^n)$ along Σ, so you can use that $dx^0 = \partial_i h dx^i$, $i = 1, \ldots, n$ on Σ to write $g_\Sigma = g_{00}(\partial_i h dx^i) \otimes (\partial_i h dx^i) + g_{j0} dx^j \otimes (\partial_i h dx^i) + g_{ij} dx^i \otimes dx^j$, with $j = 1, \ldots, n$.

A time-orientation allows us to define the future and past lightcones C_x^+ and C_x^-:

$$C_x^+ = \{v \in C_x \,|\, g(v, T_x) < 0\},$$
$$C_x^- = \{v \in C_x \,|\, g(v, T_x) > 0\},$$

with $C = C_x^+ \cup C_x^-$.

We similarly define:

$$I_x^+ = \{v \in I_x \,|\, g(v, T_x) < 0\},$$
$$I_x^- = \{v \in I_x \,|\, g(v, T_x) > 0\},$$

with $I = I_x^+ \cup I_x^-$.

The structures of a Lorentzian metric and time orientation allow us to determine the regions of $\mathcal{M}$ that can be influenced by a given subset S of $\mathcal{M}$. This is a notion that is fundamentally Lorentzian; it does not have a Riemannian analogue.

Definition 2.19. *Let $S \subset \mathcal{M}$. The* causal future/past $J^{\pm}(S)$ *is the following subset of $\mathcal{M}$:*

$$J^{\pm}(S) = \{p \in \mathcal{M}\,,\ \exists\ \textbf{causal}\ \text{future-/past-directed curve}\ \gamma : [0,1] \to \mathcal{M},\ \text{with}\ \gamma(0) \in S\ \text{and}\ \gamma(1) = p\}\,.$$

The chronological future/past $I^{\pm}(S)$ *of S is the following subset of $\mathcal{M}$:*

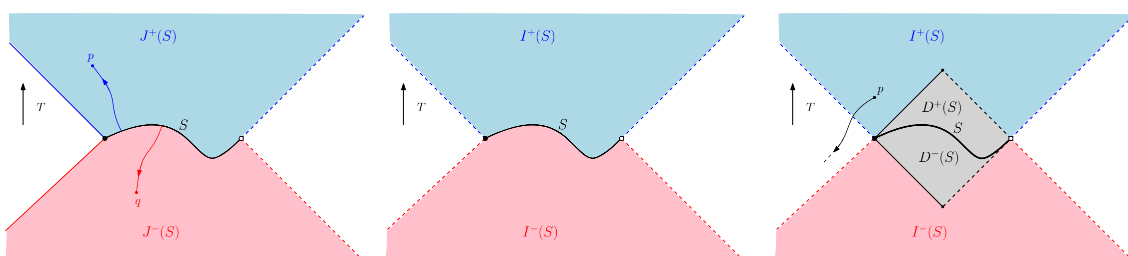
$$I^{\pm}(S) = \{p \in \mathcal{M}\,,\ \exists\ \textbf{timelike}\ \text{future-/past-directed curve}\ \gamma : [0,1] \to \mathcal{M},\ \text{with}\ \gamma(0) \in S\ \text{and}\ \gamma(1) = p\}\,.$$

Note that the constant curve $\gamma(s) = p$ for all s is not timelike, null or spacelike.

The future/past domain of dependence $D^{+}(S)$ *of S is the following subset of $\mathcal{M}$:*

$$D^{\pm}(S) = \{p \in \mathcal{M}\,,\ \underline{\text{all}}\ \text{causal past/future directed, past/future inextendible causal curves containing}\ p\ \text{intersect}\ S\}\,.$$

Note that automatically $S \subset D^{\pm}(S)$.



(a) The causal future and past $J^{\pm}(S)$ of a set $S \subset \mathbb{R}^{1+1}$, with $(\mathbb{R}^{1+1}, m)$ the 1+1-dimensional Minkowski spacetime.

(b) The chronological future and past $I^{\pm}(S)$ of a set $S \subset \mathbb{R}^{1+1}$, with $(\mathbb{R}^{1+1}, m)$ the 1+1-dimensional Minkowski spacetime.

(c) The future/past domains of dependence $D^{\pm}(S)$ of a set $S \subset \mathbb{R}^{1+1}$, with $(\mathbb{R}^{1+1}, m)$ the 1+1-dimensional Minkowski spacetime. Here, $p \in J^+(S)$ but $p \notin D^+(S) \cup D^-(S)$.

Figure 6: Examples of chronological/causal futures/pasts and future/past domains of dependence in the 2-dimensional Minkowski spacetime $(\mathbb{R}^{1+1}, m)$.

Definition 2.20. *A set $S \subset \mathcal{M}$ is* achronal *if $I^+(S) \cap S = \emptyset$.*

A particularly important class of spacetimes are *globally hyperbolic* spacetimes. As we will later see, their relevance is motivated by the fact that they constitute the spacetimes that can be obtained from the time evolution of initial data in the context of the initial value problem for the Einstein equations.

Definition 2.21. *A* Cauchy hypersurface Σ *is an achronal hypersurface of a spacetime* $(\mathcal{M}, g)$ *such that*

$$D^+(\Sigma) \cup D^-(\Sigma) = \mathcal{M}.$$

If a spacetime admits a Cauchy hypersurface, we say it is globally hyperbolic.

If $D^+(\Sigma) \cup D^-(\Sigma) \neq \mathcal{M}$*, then the future boundary of* $D^+(\Sigma)$ *in* $\mathcal{M}$*,*

$$\mathcal{CH}^+(\Sigma) := \overline{D^+(\Sigma)} \setminus \Big(I^-(D^+(\Sigma)) \cap \overline{D^+(\Sigma)}\Big)$$

is called the future Cauchy horizon of Σ *and the past boundary of* $D^-(\Sigma)$ *in* $\mathcal{M}$*,*

$$\mathcal{CH}^-(\Sigma) := \overline{D^-(\Sigma)} \setminus \Big(I^+(D^-(\Sigma)) \cap \overline{D^-(\Sigma)}\Big)$$

is called the past Cauchy horizon of Σ.

[Exercise: Show that $\mathcal{CH}^+(\Sigma)$ must be achronal. *Hint*: Show that the boundary of $S = D^+(\Sigma) \cup I^-(D^+(\Sigma))$ is achronal by using that $I^-(S) \subset S$.]

The presence of a future Cauchy horizon of a achronal hypersurface Σ indicates the *end of predictability* of the spacetime arising from initial data on Σ.

Lemma 2.1. *If* Σ *is a Cauchy hypersurface of a spacetime* $(\mathcal{M}, g)$*, then any inextendible timelike curve in* $\mathcal{M}$ *must intersect* Σ *exactly once.*

Proof. Let γ be an inextendible timelike curve. Suppose γ intersects Σ at times $s, t \in \Sigma$, with $s \neq t$. Then $\gamma(s) \in I^+(\gamma(t))$ or $\gamma(t) \in I^+(\gamma(s))$, which is in contradiction with achronality. □

[Exercise: Explain whether any inextendible null curve needs to intersect a Cauchy hypersurface exactly once.]

It turns out for globally hyperbolic spacetimes $\mathcal{M} \cong \mathbb{R} \times \Sigma$, with Σ a Cauchy hypersurface.

Theorem 2.2 (A spacetime splitting theorem)**.** *Let* $(\mathcal{M}, g)$ *be a globally hyperbolic spacetime, with* $\mathcal{M}$ *a smooth manifold and* g *a* C^k*-metric, with* $1 \leq k \leq \infty$*. Let* Σ *be a Cauchy hypersurface.*

Then $\mathcal{M}$ *is diffeomorphic to* $\mathbb{R} \times \Sigma$*.*

Proof. (*non-examinable*) By the time-orientability property, there exists a smooth timelike vector field T. At each $x \in \Sigma$, consider the inextendible curve $\tilde{\gamma}_x : I_x \to \mathcal{M}$ satisfying $\tilde{\gamma}_x(0) = x$ and $\dot{\tilde{\gamma}} = T$. We can then reparametrize $\tilde{\gamma}$ to obtain $\gamma_x : \mathbb{R} \to \mathcal{M}$, with $\gamma_x(0) = x$ and $\dot{\gamma}$ proportional to T. Since $\mathcal{M}$ is smooth, $x \mapsto \gamma_x(t)$ is smooth for all $t \in \mathbb{R}$.

Now let $\Phi : \mathbb{R} \times \Sigma \to \mathcal{M}$ be the smooth map defined as $\Phi(t, x) = \gamma_x(t)$. We will now show that this map is bijective.

- "*Surjectivity*": For any $p \in \mathcal{M}$, the timelike curve obtained by flowing along T must intersect Σ at some $x \in \Sigma$, using the Cauchy hypersurface property of Σ. After possible reparametrization, we therefore have that $p = \gamma_x(t)$ for some $t \in \mathbb{R}$.
- "*Injectivity*": Let $p = \gamma_x(t) = \gamma_y(s)$, with $s, t \in \mathbb{R}$ and $x, y \in \Sigma$. By Lemma 2.1, we must have that $x = y$. Suppose $s \neq t$, then there exists a closed timelike curve emanating from p. Suppose that this curve does not intersect Σ. Then this is in contradiction with the Cauchy surface property of Σ. Suppose the curve intersects Σ at two points. Then this is in

contradiction with Lemma 2.1. Finally, suppose the curve touches Σ at once point. Then we can perturb it a little bit to obtain a closed timelike curve that either does not intersect Σ, which is a case that leads to a contradiction, as we have shown above.

The inverse map Φ^{-1} corresponds to flowing along the integral curves of a rescaled T and therefore also is smooth. Hence, Φ is a differemorphism between $\mathbb{R}\times\Sigma$ and $\mathcal{M}$.

[Exercise: Suppose Σ and Σ' are two Cauchy hypersurfaces. Show that Σ is diffeomorphic to Σ'.] □

2.5.2 Musical isomorphisms: "raising and lowering indices"

On a manifold equipped with a metric $(\mathcal{M}, g)$, we can use the metric g to identify vector fields and 1-forms.

Let $X \in \mathcal{T}(\mathcal{M})$. Then we define

$$X^\flat := g(X,\cdot) \in \Omega^1(\mathcal{M}).$$

The symbol $\flat$ is called "flat" and is inspired by musical notation. Indeed $g(X,\cdot) : \mathcal{T}(\mathcal{M}) \to C^0(\mathcal{M})$ is multilinear over $C^0(\mathcal{M})$ so it defines a 1-form. We therefore have the existence of a map:

$$\flat : \mathcal{T}(\mathcal{M}) \to \Omega^1(\mathcal{M})$$

such that $\flat(X) = X^\flat$. Note that the associated maps $\flat_x : T_x\mathcal{M} \to T_x^*\mathcal{M}$, $\flat_x(v) = g_x(v,\cdot)$, with $x \in \mathcal{M}$, are linear maps between two vector spaces of equal dimension. Furthermore, by the non-degeneracy of g, $\flat_x$ must be injective. By the rank–nullity theorem of linear algebra, we therefore have that:

$$\dim\ker\flat_x + \dim\operatorname{ran}\flat_x = \dim T_x\mathcal{M}$$

and hence $\dim\operatorname{ran}\flat_x = \dim T_x^*\mathcal{M}$, so $\flat_x$ must also be surjective. It therefore has a well-defined inverse $\sharp_x : T_x^*\mathcal{M} \to T_x\mathcal{M}$ at each $x \in \mathcal{M}$ and we can make sense of the associated map:

$$\sharp : \Omega^1(\mathcal{M}) \to \mathcal{T}(\mathcal{M}).$$

For $\omega \in \Omega^1(\mathcal{M})$, we denote $\omega^\sharp := \sharp(\omega)$. The maps $\flat$ and $\sharp$ are called musical isomorphisms.

It is instructive to investigate the above maps with respect to a coordinate chart $(U, \{x^\mu\})$. We can then express:

$$g = g_{\mu\nu}dx^\mu \otimes dx^\nu =: g_{\mu\nu}dx^\mu dx^\nu,$$

with $g_{\mu\nu} \in C^0(U)$. Let $X = X^\alpha\partial_{x^\alpha} \in \mathcal{T}(U)$ and $Y = Y^\alpha\partial_{x^\alpha} \in \mathcal{T}(U)$. Then we can express in U:

$$X^\flat(Y) = g_{\mu\nu}X^\mu Y^\nu.$$

hence, $X^\flat = (g_{\nu\mu}X^\nu)dx^\mu = (g_{\mu\nu}X^\nu)dx^\mu$. In particular,

$$\left(\frac{\partial}{\partial x^\mu}\right)^\flat = g_{\mu\beta}dx^\beta.$$

We will denote the components of $X^\flat$ with an index in the subscript, i.e.

$$X_\mu := g_{\mu\nu}X^\nu.$$

For this reason, it is said that the map $\flat$, "lowers indices".

We now define $\mathfrak{g} \in \mathcal{T}^{(2,0)}(\mathcal{M})$ in the following way: for all $\omega, \theta \in \Omega^1(\mathcal{M})$,

$$\mathfrak{g}(\omega,\theta) := g(\omega^\sharp, \theta^\sharp).$$

We can express in U:

$$\mathfrak{g} = \mathfrak{g}^{\mu\nu}\partial_{x^\mu} \otimes \partial_{x^\nu} =: \mathfrak{g}^{\mu\nu}\partial_{x^\mu}\partial_{x^\nu}.$$

By the definition of $\mathfrak{g}$, we have that:

$$g_{\mu\nu} = g(\partial_{x^\mu}, \partial_{x^\nu}) = \mathfrak{g}((\partial_{x^\mu})^\flat, (\partial_{x^\nu})^\flat) = g_{\mu\alpha}g_{\nu\beta}\mathfrak{g}(dx^\alpha, dx^\beta) = g_{\mu\alpha}g_{\nu\beta}\mathfrak{g}^{\alpha\beta}$$

and hence

$$g_{\nu\beta}\mathfrak{g}^{\alpha\beta} = \delta^\alpha_\nu.$$

This means that the matrix with components $\mathfrak{g}^{\mu\nu}$ is the inverse of the matrix with components $g_{\mu\nu}$. We will therefore use the notation:

$$g^{-1} := \mathfrak{g}.$$

We can use g^{-1} to "raise indices". Let $\omega, \theta \in \Omega^1(\mathcal{M})$. We have that

$$\omega^\sharp(\theta) = g^{-1}(\omega, \theta) = (g^{-1})^{\mu\nu}\omega_\mu\theta_\nu.$$

Define $\omega^\mu := (g^{-1})^{\mu\nu}\omega_\nu$.

Let $T \in \mathcal{T}^{(r,s)}(\mathcal{M})$. With respect to the coordinate chart $(U, \{x^\mu\})$, we can write:

$$T = T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\frac{\partial}{\partial x^{\mu_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\mu_r}} \otimes dx^{\nu_1} \otimes \dots \otimes dx^{\nu_s}.$$

We can now create an $(r-1, s+1)$ tensor field with the components

$$T^{\mu_1\dots\mu_{k-1}}{}_{\mu_k}{}^{\mu_{k+1}\dots\mu_r}{}_{\nu_1\dots\nu_s} := g_{\mu_k\alpha}T^{\mu_1\dots\mu_{k-1}\alpha\mu_{k+1}\dots\mu_r}{}_{\nu_1\dots\nu_s}.$$

Similarly, we can create a $(r+1, s-1)$ tensor fields with the components:

$$T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_{k-1}}{}^{\nu_k}{}_{\nu_{k+1}\dots\nu_s} := (g^{-1})^{\nu_k\alpha}T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_{k-1}\alpha\nu_{k+1}\dots\nu_s}.$$

2.5.3 The Levi–Civita connection

A priori, there is no way to compare vectors in different tangent spaces on a general manifold $\mathcal{M}$. The Levi–Civita connection is a construction that provides a natural way to *transport* vectors from one tangent space to the other in the setting of manifolds equipped with a metric. For the sake of convenience, we will restrict ourselves to smooth tensor fields.

Definition 2.22. *An* affine connection *is a bilinear (over $\mathbb{R}$) map $\nabla : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \to \mathcal{T}(\mathcal{M})$ with $(X, Y) \mapsto \nabla_X Y$, such that for all $X, Y \in \mathcal{T}(\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$*

1. *(linearity over $C^\infty(\mathcal{M})$) $\nabla_{fX}Y = f\nabla_X Y$,*
2. *(Leibniz rule) $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.*

A Levi-Civita connection *with respect to a metric g is an affine connection that satisfies additionally: for all $X, Y, Z \in \mathcal{T}(\mathcal{M})$:*

1. *(metric-preserving) $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$,*
2. *(torsion free) $\nabla_X Y - \nabla_Y X = [X, Y]$.*

We define the corresponding *covariant derivative* as the map:

$$\nabla : \mathcal{T}(\mathcal{M}) \to \mathcal{T}^{(1,1)}(\mathcal{M}),$$
$$(\nabla Y)(X,\omega) := \omega(\nabla_X Y).$$

Note that ∇Y is a well-defined tensor field because it is bilinear over $C^\infty(\mathcal{M})$.

[EXERCISE: Verify this by considering $(\nabla Y)(fX+Z, h\omega+\alpha)$ for $f,h \in C^\infty(\mathcal{M})$, $X,Y,Z \in \mathcal{T}(\mathcal{M})$ and $\omega,\alpha \in \Omega^1(\mathcal{M})$.]

We state without proof (the proof can be found in several of the suggested texts in the first section of these notes):

Theorem 2.3. *Let $(\mathcal{M},g)$ be a manifold equipped with a metric. Then there exists a unique Levi–Civita connection with respect to g.*

We can express the Levi–Civita connection as follows with respect to a coordinate chart $(U,\{x^\mu\})$:

$$(\nabla_{\partial_\mu}\partial_\nu)^\sigma = \frac{1}{2}(g^{-1})^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}\right) =: \Gamma^\sigma_{\mu\nu}.$$

The expressions $\Gamma^\sigma_{\mu\nu}$ are called the *Christoffel symbols*. Hence,

$$(\nabla_X Y)^\sigma = (\nabla_{X^\mu\partial_\mu}(Y^\nu\partial_\nu))^\sigma = X^\mu\partial_\mu Y^\sigma + \Gamma^\sigma_{\mu\nu}X^\mu Y^\nu,$$
$$(\nabla Y)^\sigma{}_\mu = \partial_\mu Y^\sigma + \Gamma^\sigma_{\mu\nu}Y^\nu.$$

We would like to extend ∇ to act on more general tensor fields: $\nabla : \mathcal{T}^{(r,s)}(\mathcal{M}) \to \mathcal{T}^{(r,s+1)}(\mathcal{M})$.

Definition 2.23. *Define $\nabla f := df$ for $f \in C^\infty(\mathcal{M}) = T^{(0,0)}(\mathcal{M})$. This implies that $C^\infty(\mathcal{M}) \ni \nabla_X f = df(X) = X(f)$.*

Consider $\omega \in \Omega^1(\mathcal{M}) = \mathcal{T}^{(0,1)}(\mathcal{M})$ and define $\nabla\omega$ as follows:

$$(\nabla\omega)(X,Y) := \nabla_X(\omega(Y)) - \omega(\nabla_X Y)$$

for all $X,Y \in \mathcal{T}(\mathcal{M})$. Note that linearity over $C^\infty(\mathcal{M})$ in the first argument follows immediately, and linearity in the second argument follows from:

$$(\nabla\omega)(X,fY) = \nabla_X(\omega(fY)) - \omega(\nabla_X(fY)) = f(\nabla\omega)(X,Y) + (\nabla_X f)\omega(Y) - (\nabla_X f)\omega(Y) = f(\nabla\omega)(X,Y).$$

This definition ensures that ∇ is compatible with tensor contraction. With respect to a coordinate chart $(U,\{x^\mu\})$, we obtain for all $0 \leq \nu \leq n+1$

$$\nabla_\nu(\omega_\rho Y^\rho) = (\nabla\omega)_{\nu\rho}Y^\rho + \omega_\rho(\nabla Y)^\rho{}_\nu.$$

We can therefore also express:

$$(\nabla\omega)_{\nu\mu} = \partial_\nu\omega_\mu - \Gamma^\rho_{\mu\nu}\omega_\rho.$$

We will use the following notational conventions:

$$\nabla_\nu\omega_\mu := (\nabla\omega)_{\nu\mu},$$
$$\nabla_\nu Y^\mu := (\nabla Y)^\mu{}_\nu$$

We now extend ∇ as a map on general tensor fields: $\nabla : \mathcal{T}^{(r,s)}(\mathcal{M}) \to \mathcal{T}^{(r,s+1)}(\mathcal{M})$ is defined as

$$(\nabla T)(X,\omega^1,\ldots,\omega^r,Y_1,\ldots,Y_s) := (\nabla_X T)(\omega^1,\ldots,\omega^r,Y_1,\ldots,Y_s),$$

with

$$\begin{aligned}(\nabla_X T)(\omega^1,\ldots,\omega^r,Y_1,\ldots,Y_s) &= X(T(\omega^1,\ldots,\omega^r,Y_1,\ldots,Y_s))\\ &\quad - T(\nabla_X\omega^1,\ldots,\omega^r,Y_1,\ldots,Y_s) - \ldots - T(\nabla_X\omega^1,\ldots,\nabla_X\omega^r,Y_1,\ldots,Y_s)\\ &\quad - T(\omega^1,\ldots,\omega^r,\nabla_X Y_1,\ldots,Y_s) - \ldots - T(\omega^1,\ldots,\omega^r,Y_1,\ldots,\nabla_X Y_s).\end{aligned}$$

[EXERCISE: Show that $\nabla g = 0$.]

With respect to a coordinate chart $(U, \{x^\mu\})$, we obtain:

$$\nabla_\mu T^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_s} = \partial_\mu T^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_s} - \Gamma^\rho_{\mu\mu_1} T^{\nu_1\ldots\nu_r}{}_{\rho\ldots\mu_s} - \Gamma^\rho_{\mu\mu_2} T^{\nu_1\ldots\nu_r}{}_{\mu_1\rho\mu_3\ldots\mu_s} - \ldots - \Gamma^\rho_{\mu\mu_s} T^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_{s-1}\rho}$$
$$+ \Gamma^{\nu_1}_{\mu\rho} T^{\rho\nu_2\ldots\nu_r}{}_{\mu_1\ldots\mu_s} + \Gamma^{\nu_2}_{\mu\rho} T^{\nu_1\rho\nu_3\ldots\nu_r}{}_{\mu_1\ldots\mu_s} + \ldots + \Gamma^{\nu_r}_{\mu\rho} T^{\nu_1\ldots\nu_{r-1}\rho}{}_{\mu_1\ldots\mu_s}.$$

Proposition 2.4. *Covariant derivatives and Lie derivatives are related in the following way. Let $(U, \{x^\mu\})$ be an arbitrary coordinate chart. Let $X \in \mathcal{T}(\mathcal{M})$ and $T \in \mathcal{T}^{(r,s)}(\mathcal{M})$, then*

$$(\mathcal{L}_X T)^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_s} = X^\alpha \nabla_\alpha T^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_s} - T^{\alpha\nu_2\ldots\nu_r}{}_{\mu_1\ldots\mu_s} \nabla_\alpha X^{\nu_1} - \ldots - T^{\nu_1\ldots\nu_{r-1}\alpha}{}_{\mu_1\ldots\mu_s} \nabla_\alpha X^{\nu_r}$$
$$+ T^{\nu_1\ldots\nu_r}{}_{\alpha\mu_2\ldots\mu_s} \nabla_{\mu_1} X^\alpha + \ldots + T^{\nu_1\ldots\nu_r}{}_{\mu_1\ldots\mu_{s-1}\alpha} \nabla_{\mu_s} X^\alpha$$

The Lie derivative plays an important role when considering *isometries*.

Definition 2.24. *An isometry between manifolds equipped with metric, $(\mathcal{M}, g)$ and $(\widetilde{\mathcal{M}}, \widetilde{g})$, is a diffeomorphism $\psi : \mathcal{M} \to \widetilde{\mathcal{M}}$ satisfying*

$$\psi^* \widetilde{g} = g.$$

Definition 2.25. *A vector field $X \in \mathcal{T}(\mathcal{M})$ is called a* Killing vector field *if it satisfies:*

$$\mathcal{L}_X g = 0.$$

In particular, if $X(f) = \frac{d}{ds}|_{s=0}(f \circ \psi_s)$ for a 1-parameter group of diffeomorphisms $\psi_s : \mathcal{M} \to \mathcal{M}$ that are isometries, i.e. $\psi_s^* g = g$, then X is a Killing vector field. [EXERCISE: Show that $\mathcal{L}_X g = 0$ implies the following equation with respect to an arbitrary coordinate chart:

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0.]$$

2.5.4 Geodesics

Geodesics can be defined compactly using the covariant derivative.

Definition 2.26. *A curve $\gamma : \mathbb{R} \supseteq I \to \mathbb{R}$, with I an interval, is an* affinely parametrized geodesic, *if for $X \in \mathcal{T}(\mathcal{M})$ an extension of the tangent vectors to the curve, $\gamma'(s) \in T_{\gamma(s)}\mathcal{M}$, away from the curve γ:*

$$\nabla_X X = 0.$$

[EXERCISE: Convince yourself that the above definition is invariant under the choice of extension X.]

With respect to a coordinate chart $(U, \{x^\mu\})$, we can write

$$(\gamma'')^\sigma + \Gamma^\sigma_{\mu\nu} \gamma'^\mu \gamma'^\nu = 0,$$

where we really mean $\gamma'^\mu(s) = (x^\mu \circ \gamma)'(s)$ and $(\gamma'')^\mu(s) = (x^\mu \circ \gamma)''(s)$.

[EXERCISE: 1) Show that for affinely parametrized spacelike geodesics γ there exists an $a \in \mathbb{R}$ such that $\ell[\gamma](s) = as$. This explains the term "affinely parametrized".] 2) Show that for affinely parametrized timelike geodesics γ there exists an $a \in \mathbb{R}$ such that $\tau[\gamma](s) = as$.]

It can be shown that spacelike geodesics are *locally arc-length minimizing* and timelike geodesics are *locally proper time maximizing*. Null geodesics do not admit such a variational interpretation.

Proposition 2.5. *Let Y be a Killing vector field. Let X be a vector field such that $X|_{\gamma(s)} = \gamma'(s)$, with $\gamma : I \to \mathcal{M}$ an affinely parametrized geodesic. Then*

$$X(g(X, Y))|_\gamma = 0.$$

Hence, the "inner product" of a Killing vector field with the tangent to an affinely parametrized geodesic constitutes a conserved quantity *along the geodesic.*

Proof. EXERCISE ☐

2.6 The Riemann tensor, Ricci tensor and Ricci scalar

The Riemann tensor field is a geometric object that captures the "intrinsic curvature" of a manifold. That is to say, it provides a notion of curvature that is independent of how the manifold may be embedded inside some bigger manifold. It is the central object of study in general relativity.

Definition 2.27. *We define the* Riemann tensor field *as the following map:*

$$\begin{aligned}\mathrm{Riem}[g] : \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \times \mathcal{T}(\mathcal{M}) \times \Omega^1(\mathcal{M}) \to C^\infty(\mathcal{M})\\ \mathrm{Riem}[g](Z, X, Y, \omega) = \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right).\end{aligned}$$

We will also refer to this map as the Riemann tensor *and denote its components with respect to a coordinate chart by* $R_{\mu\nu\rho}{}^{\sigma}$.

[Exercise: Show that $\mathrm{Riem}[g]$ is multilinear over $C^\infty(\mathcal{M})$ and hence, it is a smooth tensor field: $\mathrm{Riem}[g] \in \mathcal{T}^{(1,3)}(\mathcal{M})$.]

With respect to a coordinate chart $(U, \{x^\mu\})$ we obtain:

$$\begin{aligned}R_{\mu\nu\rho}{}^{\sigma} := \mathrm{Riem}[g](\partial_\mu, \partial_\nu, \partial_\rho, dx^\sigma) &= dx^\sigma(\nabla_{\partial_\nu}\nabla_{\partial_\rho}\partial_\mu - \nabla_{\partial_\rho}\nabla_{\partial_\nu}\partial_\mu)\\ &= dx^\sigma(\nabla_{\partial_\nu}(\Gamma^\alpha_{\rho\mu}\partial_\alpha) - \nabla_{\partial_\rho}(\Gamma^\alpha_{\nu\mu}\partial_\alpha))\\ &= dx^\sigma(\partial_\nu\Gamma^\alpha_{\rho\mu}\partial_\alpha - \partial_\rho\Gamma^\alpha_{\nu\mu}\partial_\alpha + \Gamma^\alpha_{\rho\mu}\Gamma^\beta_{\nu\alpha}\partial_\beta - \Gamma^\alpha_{\nu\mu}\Gamma^\beta_{\rho\alpha}\partial_\beta)\\ &= \partial_\nu\Gamma^\sigma_{\rho\mu} - \partial_\rho\Gamma^\sigma_{\nu\mu} + \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\nu\alpha} - \Gamma^\alpha_{\nu\mu}\Gamma^\sigma_{\rho\alpha}.\end{aligned}$$

By acting with $\flat$, we obtain a $(0,4)$-tensor field with components $R_{\mu\nu\rho\sigma} = g_{\sigma\alpha}R_{\mu\nu\rho}{}^{\alpha}$. We will also refer to this tensor field as "the Riemann tensor".

Definition 2.28. *The* Ricci tensor *(field) is a* $(0,2)$*-tensor field* $\mathrm{Ric}[g]$ *with components* $R_{\mu\nu}$ *with respect to an arbitrary coordinate chart satisfying:*

$$R_{\mu\nu} = R_{\mu\sigma\nu}{}^{\sigma} = (g^{-1})^{\sigma\rho}R_{\mu\sigma\nu\rho}.$$

Definition 2.29. *The* Ricci scalar $R[g] \in C^\infty(\mathcal{M})$ *is defined as follows:*

$$R = (g^{-1})^{\mu\nu}R_{\mu\nu}.$$

The Riemann tensor and its tensor contractions ("summing over upper and lower indices") are natural quantities to consider on $(\mathcal{M}, g)$ because they are invariant under isometries $\psi : \mathcal{M} \to \widetilde{\mathcal{M}}$. Indeed, it can be shown that:

$$\mathrm{Riem}[\psi^*\widetilde{g}] = \psi^*(\mathrm{Riem}[\widetilde{g}]), \tag{2.3}$$
$$\mathrm{Ric}[\psi^*\widetilde{g}] = \psi^*(\mathrm{Ric}[\widetilde{g}]), \tag{2.4}$$
$$R[\psi^*\widetilde{g}] = \psi^*(R[\widetilde{g}]) = R[\widetilde{g}] \circ \psi \tag{2.5}$$

and we can apply in addition the isometry property $\psi^*\widetilde{g} = g$.

The Riemann tensor enjoys various symmetry properties.

Proposition 2.6. *With respect to an arbitrary coordinate chart, the components of the Riemann tensor satisfy:*

(i) $R_{(\mu\nu)\rho}{}^{\sigma} = 0$,

(ii) $R_{[\mu\nu\rho]}{}^{\sigma} = 0$,

(iii) $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$,

(iv) $\nabla_{[\alpha} R_{\sigma\mu]\nu\rho} = 0$ *(Bianchi identity).*

Here, we take the round brackets to mean a symmetrization over the indices: for example, $T_{(\mu\nu)} := \frac{1}{2}(T_{\mu\nu} + T_{\nu\mu})$.

From the above symmetry properties, we can deduce symmetry properties at the level of the Ricci tensor and Ricci scalar. We first introduce the Einstein tensor.

Definition 2.30. *The* Einstein tensor *(field)* $G[g] \in \mathcal{T}^{(0,2)}(\mathcal{M})$ *is defined as follows:*

$$G[g] = \mathrm{Ric}[g] - \frac{1}{2}R[g]g.$$

Corollary 2.7. *The Ricci and Einstein tensors are symmetric and, with respect to an arbitrary coordinate chart,* $G[g]$ *satisfies the* contracted Bianchi identity

$$\nabla^\mu G_{\mu\nu} = 0.$$

Proof. Symmetry of $G[g]$ follows immediately from symmetry of $\mathrm{Ric}[g]$. Recall,

$$R_{\mu\nu} = R^\sigma{}_{\mu\sigma\nu} = (g^{-1})^{\sigma\alpha} R_{\alpha\mu\sigma\nu}.$$

By property (iv) of Proposition 2.6, we have that

$$(g^{-1})^{\sigma\alpha} R_{\alpha\mu\sigma\nu} = (g^{-1})^{\sigma\alpha} R_{\sigma\nu\alpha\mu} = R_{\nu\mu}.$$

To derive the contracted Bianchi identity, we combine (i) and (iii) to infer that $R^\sigma{}_{\mu\nu\rho} = R^\sigma{}_{[\mu\nu]\rho}$ and

$$\nabla_\alpha R_{\sigma\mu\nu\rho} + \nabla_\sigma R_{\mu\alpha\nu\rho} + \nabla_\mu R_{\alpha\sigma\nu\rho} = 0.$$

Now we contract the above identity with $(g^{-1})^{\sigma\nu}(g^{-1})^{\mu\rho}$ to obtain:

$$\begin{aligned} 0 &= (g^{-1})^{\sigma\nu}(g^{-1})^{\mu\rho}\nabla_\alpha(R_{\sigma\mu\nu\rho} + \nabla_\sigma R_{\mu\alpha\nu\rho} + \nabla_\mu R_{\alpha\sigma\nu\rho}) \\ &= \nabla_\alpha R - \nabla^\nu R_{\alpha\nu} - \nabla^\rho R_{\alpha\rho} \\ &= \nabla_\alpha R - 2\nabla^\nu R_{\alpha\nu} \\ &= -2\nabla^\nu G_{\alpha\nu}. \end{aligned}$$

□

2.7 The Einstein equations

The Einstein equations in $(\mathcal{M}, g)$ (with respect to physical units in which $c = G = 1$):

$$G[g] = 8\pi\mathbb{T}.$$

Here, $\mathbb{T} \in \mathcal{T}^{(0,2)}(\mathcal{M})$ is the *stress-energy tensor* (field) or *energy-momentum tensor* (field), whose expression depends on the physical matter that one is interested in studying. By the contracted Bianchi identity $\nabla^\mu G_{\mu\nu} = 0$, the stress-energy tensor must be divergence-free:

$$\nabla^\mu \mathbb{T}_{\mu\nu} = 0.$$

The Einstein equations are coupled with appropriate equations for the matter model of interest.

Let $\psi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism. By (2.4) and (2.5), $G[g] = 8\pi\mathbb{T}$ if and only if

$$G[\psi^* g] = 8\pi(\psi^*\mathbb{T}) = 0.$$

This property is sometimes also referred to as *general covariance* ("the laws of general relativity take the same form when expressed with respect to a different coordinate chart).

As we will see later in the course, it will be useful to restrict to stress-energy tensors satisfying additional *energy conditions*, which are supposed to describe necessary conditions on "physically reasonable" matter.

Definition 2.31. *A stress-energy tensor $\mathbb{T}$ satisfies:*

- *the* null energy condition *(NEC) if for all* <u>null</u> *vector fields $X \in \mathcal{T}(M)$*
$$\mathbb{T}(X, X) \geq 0.$$
- *the* weak energy condition *(WEC) if for all* <u>causal</u> *vector fields $X \in \mathcal{T}(M)$*
$$\mathbb{T}(X, X) \geq 0.$$
If we interpret the integral curve of X as representing an observer, this means that the energy density measured by the observer is non-negative.
- *the* strong energy condition *(SEC) if for all* <u>causal</u> *vector fields $X \in \mathcal{T}(M)$*
$$\mathbb{T}(X, X) - \frac{1}{2} \operatorname{tr} \mathbb{T} g(X, X) \geq 0.$$
In the context of the Einstein equations, this means that $\mathrm{Ric}[g](X, X) \geq 0$. *By considering the integral curves of a timelike X, it can be shown that this implies that observers get closer together towards the future, i.e.* gravity is attractive. *For null curves, this is already encoded in the null energy condition.*
- *the* dominant energy condition *(DEC) if for all* <u>future-directed timelike</u> *vector fields $X \in \mathcal{T}(M)$*
$$-\mathbb{T}(X, \cdot)^\sharp$$
is a future-directed causal vector field. We can interpret $-\mathbb{T}(X, \cdot)^\sharp$ as as a momentum 4-vector field according to the observer represented by the integral curve of X. Hence, the "speed of matter" is less or equal to the speed of light.

[Exercise: Show that
$$\begin{aligned} \text{DEC} \Rightarrow \text{WEC} &\Rightarrow \text{NEC}, \\ \text{SEC} &\Rightarrow \text{NEC}.] \end{aligned}$$

Despite what its name might suggest, SEC $\not\Rightarrow$ WEC. For example, consider the following example of a valid stress-energy tensor:
$$\mathbb{T} = g.$$[11]
Then $\mathbb{T} - \frac{1}{2} \operatorname{tr} \mathbb{T} g = g - \frac{1}{2} \cdot 4g = -2g$, so the SEC is satisfied, but the WEC fails.

2.7.1 Vacuum

If $\mathbb{T} = 0$, we refer to the corresponding Einstein equations as the *vacuum Einstein equations*. In that case, they reduce to
$$\mathrm{Ric}[g] = 0.$$
Let $\psi : \mathcal{M} \to \mathcal{M}$ be a diffeomorphism. By (2.4), $\mathrm{Ric}[g] = 0$ iff
$$\mathrm{Ric}[\psi^* g] = \psi^*(\mathrm{Ric}[g]) = 0,$$
so $\psi^* g$ is automatically also a solution to the vacuum Einstein equations. Nevertheless, we would like to think of g and $\psi^* g$ as the "same" solution. This becomes relevant in the context of the initial value problem, where "uniqueness" of solutions corresponding to the same initial data can only hold "up to diffeomorphism".

[11] Note that g is symmetric and $\nabla g = 0$.

2.7.2 Lagrangian field theories

Given a choice of matter model, how do we determine the stress-energy tensor? In the context of matter models described by *Lagrangian field theories*, one can follow the prescription that we outline below.

Suppose the equations of motion of the matter is described by a Lagrangian $\mathcal{L} \circ (\phi, \nabla^g \phi, g)$,[12] with a *field* $\phi \in \mathcal{T}^{(r,s)}(\mathcal{M})$, where ∇^g denotes the Levi–Civita covariant derivative with respect to g.

For example, in the case of a Klein–Gordon scalar field $\phi \in C^\infty(\mathcal{M})$

$$\mathcal{L}_{KG} \circ (\phi, \nabla^g \phi, g) = \frac{1}{2} g^{-1}(\nabla^g \phi, \nabla^g \phi) + \frac{\mathfrak{m}^2}{2}\phi^2,$$

or in the case of an electromagnetic potential $A \in \Omega^1(\mathcal{M})$

$$\mathcal{L}_{EM} \circ (A, \nabla^g A, g) = \frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta},$$

with $F = dA$.

Then the stress energy tensor $\mathbb{T}$ can be defined as follows:

$$\mathbb{T}_{\mu\nu}[\phi, g] := 2\left(\frac{\partial \mathcal{L}}{\partial (g^{-1})^{\mu\nu}} \circ (\phi, \nabla^g \phi, g) - \frac{1}{2} g_{\mu\nu} \mathcal{L} \circ (\phi, \nabla^g \phi, g)\right).$$

Hence,

$$\begin{aligned}
\mathbb{T}^{\text{KG}}_{\mu\nu}[\phi, g] &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}((g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \mathfrak{m}^2 \phi^2), \\
\mathbb{T}^{\text{EM}}_{\mu\nu}[A, g] &= \frac{1}{4\pi}\left[(g^{-1})^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g_{\mu\nu}\right].
\end{aligned}$$

Note that the equation of motion for ϕ (which corresponds to the Euler–Lagrange equations) can be expressed as follows:

$$\mathfrak{m}^2 \phi = \Box_g \phi := (g^{-1})^{\alpha\beta} (\nabla^2 \phi)_{\alpha\beta} = \frac{1}{\sqrt{-\det g}} \partial_\alpha (\sqrt{-\det g} (g^{-1})^{\alpha\beta} \partial_\beta \phi) = 0.$$

The differential operator on the very right-hand side is called the *Laplace–Beltrami operator* associated to g. When $\mathfrak{m} = 0$, we refer to this equation as the (geometric) wave equation. Combining it with the Einstein equations, we obtain the *Einstein-scalar field system* of equations.

The equations of motion for F correspond to the Maxwell equations on a curved background with no charges or currents:

$$\begin{aligned}
dF &= 0, \\
\nabla_\mu F^{\mu\nu} &= 0.
\end{aligned}$$

If $\mathcal{M} = \mathbb{R}^{1+3}$ or if we restrict to a sufficiently small neighbourhood U of a point $x \in \mathcal{M}$, F is moreover an exact 2-form, so there exist a $A \in \Omega^1(\mathcal{M})$, such that $dA = F$.

Exercise: Show that the stress-energy tensors corresponding to the Klein–Gordon equation and the Maxwell equations satisfy the dominant energy condition. Do they satisfy the strong energy condition?]

[12] The Lagrangian is a map $\mathcal{L} : \mathcal{N} \to \mathbb{R}$, where $\mathcal{N}$ is manifold that can be chosen appropriately so that the composition $\mathcal{L} \circ (\phi, \nabla^g \phi, g) : \mathcal{M} \to \mathbb{R}$ is a well-defined map and the partial derivatives of $\mathcal{L}$ make sense.

2.7.3 Perfect fluids

Suppose we would like to describe the gravitational properties of fluids, modelling for example stars. If we ignore properties of fluids like their viscosity and heat conduction, they are described be so-called *perfect fluids*.

The equations of motion of perfect fluids can be obtained by starting with the following stress-energy tensor: let $U \in \mathcal{T}(\mathcal{M})$ with $g(U,U) = -1$ and $\rho, p \in C^\infty(\mathcal{M})$. Then

$$\mathbb{T}^{\text{fluid}}_{\mu\nu} := (\rho + p)U_\mu U_\nu + p g_{\mu\nu}.$$

With respect to an orthononormal basis $\{e_\mu\}$ at $x \in \mathcal{M}$, such that $e_0 = U_x$, we can represent $\mathbb{T}$ by the following matrix with components $\mathbb{T}^{\text{fluid}}_x(e_\mu, e_\nu)$:

$$\begin{pmatrix} \rho(x) & 0 & 0 & 0 \\ 0 & p(x) & 0 & 0 \\ 0 & 0 & p(x) & 0 \\ 0 & 0 & 0 & p(x) \end{pmatrix}.$$

Then it can be shown that the equation $\nabla_\mu \mathbb{T}^{\mu\nu}_{\text{fluid}} = 0$ gives the *relativistic Euler equations* on $(\mathcal{M}, g)$:

$$\begin{aligned} U^\alpha \nabla_\alpha \rho + (\rho + p)\nabla_\alpha U^\alpha &= 0, \\ (\rho + p)U^\alpha \nabla_\alpha U^\mu &= -\left((g^{-1})^{\mu\nu} + U^\mu U^\nu\right)\nabla_\nu p, \end{aligned}$$

where ρ can be interpreted as the mass-energy density of a fluid, p can be interpreted as the fluid pressure and U is the 4-velocity of the fluid.

Using that $g(U,U) = -1$, U has three independent components, so the relativistic Euler equations constitute four equations for five variables, three components of U^μ, p and ρ. To obtain a closed system of equations, one needs to prescribe an *equation of state* relating p and ρ. For example, p can be written as a function of ρ and a temperature T: $p(\rho, T)$.

The special case $p \equiv 0$ is called *dust*. Note that in this case $\nabla_U U = 0$, so the integral curves of U are affinely parametrized timelike geodesics. [Exercise: Show that the perfect fluid stress-energy tensor satisfies:

- the null energy condition if and only if $\rho + p \geq 0$,
- the weak energy condition if and only if $\rho + p \geq 0$ and $\rho \geq 0$,
- the dominant energy condition if and only if $\rho \geq |p|$,
- the strong energy condition if and only if $\rho + p \geq 0$ and $\rho + 3p \geq 0$.]

3 The Minkowski spacetime

Before we introduce examples of black hole spacetimes, we will first review the Minkowski spacetime solution to the vacuum Einstein equations (Minkowski 1905). This is sometimes also referred to as the "Minkowski space". This can be thought of as the "trivial solution" to the vacuum Einstein equations (which, as we will see, admit highly non-trivial solutions!). The study of causal curves on the Minkowski spacetime encompasses Einstein's theory of special relativity.

The Minkowski spacetime $(\mathbb{R}^{3+1}, m)$ is defined as follows:

$$m = -dt^2 + dx^2 + dy^2 + dz^2,$$

with (t, x, y, z) Cartesian coordinates on $\mathbb{R}^{3+1}$, which cover the spacetime globally.

In order to obtain a compact, two-dimensional representation of Minkowski, it will be useful to switch to spherical coordinates (r, θ, φ) with

$$\begin{aligned} x &= r \sin\theta \cos\varphi, \\ y &= r \sin\theta \sin\varphi, \\ z &= r \cos\theta, \end{aligned}$$

where $\theta \in (0, \pi)$, $\varphi \in (0, 2\pi)$ and $r \in (0, \infty)$. Note that these coordinates do not cover $\mathbb{R}^{3+1}$ globally due to a degeneracy at the origin $0 \in \mathbb{R}^{n+1}$, as well as at a great circle segment on $\mathbb{S}^2$ connecting the north and south poles.

[Exercise: Show that with respect to standard spherical coordinates (t, r, θ, φ), we can express:

$$m = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2).]$$

We denote:

$$\mathring{\not{g}} = d\theta^2 + \sin^2\theta d\varphi^2.$$

And observe that $\mathring{\not{g}}$ is the induced Riemannian metric of the unit round sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

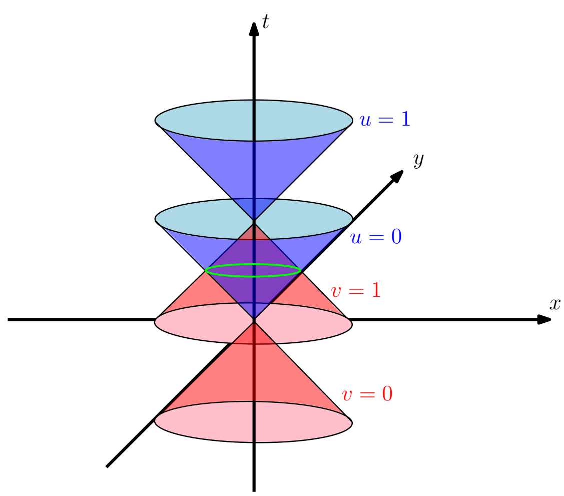


Figure 7: A depiction of the u- and v-level sets for $(\mathbb{R}^{2+1}, m)$. The intersections of the level sets are in this case round circles.

We now introduce the ingoing null coordinate u and the outgoing null coordinate v, which are defined as follows:

$$u = t - r,$$

$$v = t + r.$$

Then:

$$m = -\frac{1}{2}(du \otimes dv + dv \otimes du) + r^2 \mathring{\not{g}} =: -dudv + r^2 \mathring{\not{g}}.$$

Note that the level sets $\{u = u_0\}$ and $\{v = v_0\}$ are outgoing and ingoing cones with vertices at the t-axis and an opening angle of 45° or $\frac{\pi}{2}$ rad. The ingoing null coordinate is also called the *retarded time function* and the outgoing null coordinate v is called the *advanced time function*.

We can depict $(\mathbb{R}^{3+1}, m)$ as a half plane $\{u - v > 0\}$ by suppressing the spheres of intersection of u- and v-level sets.

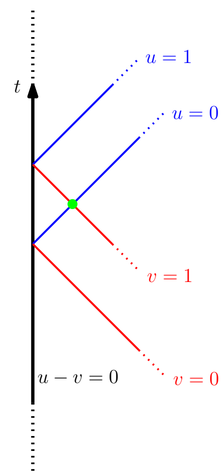


Figure 8: A depiction of the u- and v-level sets in $(\mathbb{R}^{3+1}, m)$ supressing the spherical directions. Every point on away from the t-axis represents a round sphere of radius $r(u, v) = \frac{v-u}{2}$.

To be able to depict $(\mathbb{R}^{3+1}, m)$ more compactly, we introduce the following rescaled double null coordinates.

$$\tilde{u} = \arctan u,$$
$$\tilde{v} = \arctan v.$$

[EXERCISE: Show that the metric takes the form $m = -\frac{1}{\cos^2 \tilde{u} \cos^2 \tilde{v}} d\tilde{u}d\tilde{v} + \frac{(\tan \tilde{v} - \tan \tilde{u})^2}{4} \mathring{\not{g}}$.]

We can represent $(\mathbb{R}^{3+1}, m)$ by the following subset of $\mathbb{R}^2$:

$$\mathcal{Q} = \left\{(\tilde{u}, \tilde{v}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \, , \, \tilde{v} - \tilde{u} > 0\right\}$$

We denote

$$\Gamma := \{\tilde{v} - \tilde{u} = 0\}.$$

Note that Γ represent the t-axis and it is a centre of spherical symmetry, i.e. it consists of all points that stay fixed with respect to spatial rotations around the t-axis.

The closure $\overline{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathbb{R}^2$ is called a *(Carter–)Penrose diagram* of the Minkowski spacetime. Changing the function arctan to a different function results in a different diagram. However, lightcones emanating from the t-axis will always be represented by straight lines at a 45° angle with the vertical. Note that Γ need not not be a straight line in all Penrose diagrams! We will give a definition of these diagrams in a more general setting later in the course.

We label elements of the boundary $\partial\mathcal{Q}\setminus\mathcal{Q}$ in the following way:

$$\begin{aligned}
\mathcal{I}^+ &:= \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\left\{\frac{\pi}{2}\right\},\\
\mathcal{I}^- &:= \left\{-\frac{\pi}{2}\right\}\times\left(-\frac{\pi}{2},\frac{\pi}{2}\right),\\
i^+ &:= \left\{\frac{\pi}{2}\right\}\times\left\{\frac{\pi}{2}\right\},\\
i^- &:= \left\{-\frac{\pi}{2}\right\}\times\left\{-\frac{\pi}{2}\right\},\\
i^0 &:= \left\{-\frac{\pi}{2}\right\}\times\left\{\frac{\pi}{2}\right\}.
\end{aligned}$$

We refer to $\mathcal{I}^\pm$ as *future/past null infinity*, $i^\pm$ as *future/past timelike infinity* and i^0 as *spacelike infinity*. This nomenclature is motivated by the fact that null cones in Minkowski emanating from the t-axis are represented by lines at $45°$ with Γ (or with the lines $v-u=$ const.) with future/past limit points on $\mathcal{I}^\pm$, timelike curves with bounded acceleration are represented by curves whose tangent is at an angle uniformly smaller than $45°$ with Γ with future/past limit points at $i^\pm$ and spacelike curves with tangents that have a norm bounded away from zero are represented by curves whose tangent is at an angle uniformly larger than $45°$ with Γ and which have endpoints at i^0.

Similarly, spherically symmetric null/ timelike/ spacelike hypersurfaces are represented by curves at/below/above $45°$ with γ.

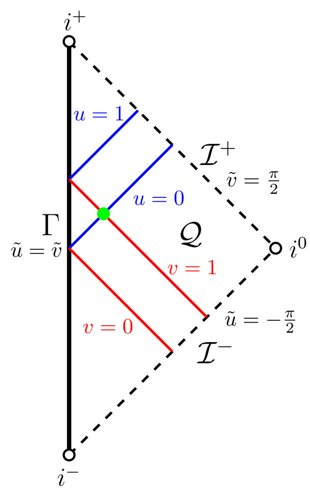


Figure 9: A Penrose diagram of the Minkowski spacetime.

[Exercise: Prove that all geodesics in Minkowski must be straight lines.]

[Exercise: Draw the null line $\{x-t=1,z=0,y=1\}$ in a Minkowski Penrose diagram.]

[Exercise: Draw the hyperboloids $\{t^2-r^2=s^2\}$ and the hyperboloids $\{(t+s)^2-r^2=1\}$ in a Minkowski Penrose diagram, for different values of $s\in\mathbb{R}$.]

[Exercise: Consider the sphere $S^2_{u_0,v_0}:=\{u=u_0\}\cap\{v=v_0\}$ in Minkowski, with $u_0,v_0\in\mathbb{R}$. Draw $J^+(S^2_{u_0,v_0})$, $J^-(S^2_{u_0,v_0})$ in a Penrose diagram.]

[Exercise: Let $\Sigma=\{t=0\}\cap\{r\leq 1\}$ be a subset of Minkowski. Draw Σ, $D^+(\Sigma)$ and $D^-(\Sigma)$ in a Penrose diagram.]

4 Schwarzschild black hole spacetimes

Let $M \in (0, \infty)$. Then the Schwarzschild spacetimes (SCHWARZSCHILD 1915) are pairs $(\mathcal{M}_{\text{ext}}, g_M)$, with $\mathcal{M}_{\text{ext}} = \mathbb{R} \times (2M, \infty) \times \mathbb{S}^2$ and

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \mathring{\slashed{g}}, \tag{4.1}$$

where $t \in \mathbb{R}$, $r \in (0, 2M)$ and $(\theta, \varphi) \in \mathbb{S}^2$. Note that this expression features the usual degeneration of spherical coordinates at a great circle segment connecting the north and south pole and hence multiple coordinate charts are required to fully cover the spheres of constant t and r.

It can be shown that the Schwarzschild spacetimes are solutions to the vacuum Einstein equations (we will revisit this later). We will also refer to them as *Schwarzschild exteriors*, since we will show in the next section that they can be extended to obtain a Schwarzschild *interior*.

The parameter M is called the *mass* of the Schwarzschild spacetime. Somewhat paradoxically, it is sensible to assign a notion of mass or energy to spacetimes even when they solve the vacuum Einstein equations, i.e. *matter is absent*.

We will encounter the concepts of mass and energy in more generality later in the course. Loosely speaking, you may think of this as a measure of the total energy contained in the "gravitational field" at any fixed time.

It may seem like the Schwarzschild metric is singular at $r = 2M$. Indeed, the metric components in (4.1) become ill-defined at $r = 2M$. This fact was a source of significant confusion after Schwarzschild wrote down his metric. It was LEMAÎTRE who first realized in 1932 that the spacetime $(\mathcal{M}_{\text{ext}}, g_M)$ can actually be extended across $r = 2M$ and any problems at $r = 2M$ are purely an artefact of the particular choice of coordinates (t, r, θ, φ). They can therefore be compared to the issues at $\theta = 0, \pi$ in spherical coordinates.

To see that nothing goes wrong at $r = 2M$, we will first introduce null coordinates (u, v) that generalize the null coordinates that we already encounter in the Minkowski spacetime. First, we introduce the *tortoise coordinate* $r_* : (2M, \infty) \to \mathbb{R}$, which satisfies

$$\frac{dr_*}{dr} = \frac{1}{1 - \frac{2M}{r}}.$$

[EXERCISE: Show that we can write

$$r_*(r) = c_0 + r + 2M \log\left(\frac{r - 2M}{2M}\right)$$

for an arbitrary constant $c_0 \in \mathbb{R}$ and hence the range of r_* is the full real line $\mathbb{R}$.]

For the sake of convenience, we will take $c_0 = 0$ and define:

$$r_* = r + 2M \log\left(\frac{r - 2M}{2M}\right). \tag{4.2}$$

With respect to $(t, r_*, \theta, \varphi)$ coordinates, we can express:

$$g_M = \left(1 - \frac{2M}{r}\right)\left[-dt^2 + dr_*^2\right] + r^2 \mathring{\slashed{g}}.$$

We now define the null functions u and v as follows:

$$u = t - r_*,$$
$$v = t + r_*.$$

Note that u, v can take on any value in $\mathbb{R}$ and: [Exercise: Show that $du^\sharp$ and $dv^\sharp$ are null vector fields.]

Then

$$g_M = -\left(1 - \frac{2M}{r}\right) du dv + r^2 \mathring{\not{g}},$$

where we view r as a function of u and v in the following way: $r = r(r_*) = r\left(\frac{1}{2}(v-u)\right)$. The coordinates (u, v, θ, φ) are called *Eddington–Finkelstein double null coordinates*.

[Exercise: Show that g_M takes the following form with respect to (v, r) coordinates (*ingoing Eddington–Finkelstein coordinates*) and (u, r) coordinates (*outgoing Eddington–Finkelstein coordinates* or *Bondi coordinates*):

$$g_M = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 \mathring{\not{g}},$$
$$g_M = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2 \mathring{\not{g}}.]$$

As in the Minkowski case, we can obtain a Penrose diagram of the Schwarzschild spacetime by rescaling:

$$\tilde{u} = \arctan u,$$
$$\tilde{v} = \arctan v.$$

Then $\mathcal{M}_{\text{ext}}$ is represented by the bounded subset $\mathcal{Q} = (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \subset \mathbb{R}^2$.

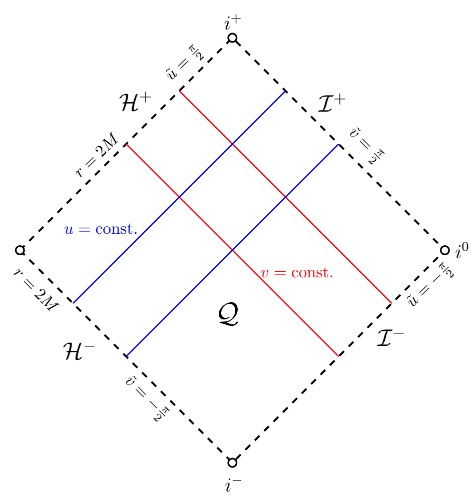


Figure 10: A Penrose diagram of the Schwarzschild exterior spacetime.

Note that, we can define, as in the Minkowski case, future/past null infinity $\mathcal{I}^\pm$, future/past timelike infinity $i^\pm$ and spacelike infinity i^0. We moreover have two additional boundary components $\mathcal{H}^\pm$ and their intersection, which we will give a name to later.

$$\mathcal{I}^+ := \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left\{\frac{\pi}{2}\right\},$$
$$\mathcal{H}^+ := \left\{\frac{\pi}{2}\right\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$\mathcal{I}^- := \left\{-\frac{\pi}{2}\right\} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\begin{aligned}\mathcal{H}^- &:= \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\left\{-\frac{\pi}{2}\right\},\\ i^+ &:= \left\{\frac{\pi}{2}\right\}\times\left\{\frac{\pi}{2}\right\},\\ i^- &:= \left\{-\frac{\pi}{2}\right\}\times\left\{-\frac{\pi}{2}\right\},\\ i^0 &:= \left\{-\frac{\pi}{2}\right\}\times\left\{\frac{\pi}{2}\right\}.\end{aligned}$$

4.1 Maximally-extended Schwarzschild

Now we will show that we can extend the spacetime smoothly across the boundaries $\mathcal{H}^+$ and $\mathcal{H}^-$ of $\mathcal{Q}$ in $\mathbb{R}^2$, where r approaches the value $2M$. We will do so by first switching from the ingoing null coordinate u to an affine parameter of appropriate ingoing null geodesics.

Note first that ∂_u is a null vector field. Furthermore: [Exercise: Show that

$$\nabla_{\partial_u}\partial_u = \Gamma^u_{uu} = \Omega^{-2}(\partial_u\Omega^2)\partial_u,$$

with $\Omega^2(u,v) := 1-\frac{2M}{r(u,v)}$.]

Consider an integral curve $\gamma:\mathbb{R}\to\mathcal{M}$ of ∂_u, with

$$\gamma(u)\cong\begin{pmatrix}\gamma^u\\ \gamma^v\\ \gamma^\theta\\ \gamma^\varphi\end{pmatrix}(u)=\begin{pmatrix}u\\ v_0\\ \theta_0\\ \varphi_0\end{pmatrix},$$

for some $v_0\in\mathbb{R}$, $\theta_0\in(0,\pi)$ and $\varphi_0\in(0,2\pi)$.

To show that γ is a (non-affinely parametrized) null geodesic, we rescale:

$$\frac{dU'}{du} = \Omega^2(u,v_0) = 1-\frac{2M}{r(u,v_0)}.$$

Then along γ:

$$\nabla_{\partial'_U}\partial'_U|_\gamma = \nabla_{\Omega^{-2}\partial_u}(\Omega^{-2}\partial_u)|_\gamma = -\Omega^{-6}\partial_u\Omega^2|_\gamma\partial_u|_\gamma + \Omega^{-4}\nabla_{\partial_u}\partial_u|_\gamma = 0,$$

so $\partial_{U'}|_\gamma$ satisfies the defining property of a tangent vector field to an affinely-parametrized null geodesic. We will show below that $U'(\infty)=0$, so γ reaches the boundary $\mathcal{H}^+$ at a finite affine time.

By (4.2), we can express:

$$e^{\frac{v-u}{4M}} = e^{\frac{r_*}{2M}} = e^{\frac{r}{2M}}\left(\frac{r-2M}{2M}\right),$$

so that

$$\frac{dU'}{du} = 1-\frac{2M}{r(u,v_0)} = \frac{2M}{r(u,v_0)}e^{-\frac{r(u,v_0)}{2M}}e^{\frac{v_0-u}{4M}} = e^{-1+\frac{v_0}{4M}}e^{-\frac{u}{4M}}(1+O(e^{-\frac{u}{4M}}))$$

and therefore, as $u\to\infty$,

$$U'(u) = -\frac{1}{4M}e^{-1+\frac{v_0}{4M}}e^{-\frac{u}{4M}}(1+O(e^{-\frac{u}{4M}})).$$

Note the exponential relation between U and u and $U'(\infty)=0$. To simplify matters a bit, we consider instead the following ingoing null coordinate:

$$U(u) = -e^{-\frac{u}{4M}}.$$

We can similarly define

$$\frac{dV'}{dv} = 1 - \frac{2M}{r(u_0, v)},$$

to obtain

$$\frac{dV'}{dv} = \frac{2M}{r(u_0, v)} e^{-\frac{r(u_0,v)}{2M}} e^{\frac{v-u_0}{4M}} = e^{-1-\frac{u_0}{4M}} e^{\frac{v}{4M}} (1 + O(e^{\frac{v}{4M}}))$$

and therefore, as $v \to -\infty$,

$$V'(v) = \frac{1}{4M} e^{-1-\frac{u_0}{4M}} e^{\frac{v}{4M}} (1 + O(e^{\frac{v}{4M}})),$$

so $V'(-\infty) = 0$.

Again, we consider the simplified outgoing null coordinate:

$$V(v) = e^{\frac{v}{4M}}.$$

We refer to the coordinates (U, V, θ, φ) as Kruskal–Szekeres coordinates. Note that

$$U \cdot V = -e^{\frac{v-u}{4M}} = -e^{\frac{r_*}{2M}} = -e^{\frac{r}{2M}} \frac{r - 2M}{2M}. \tag{4.3}$$

We can therefore express

$$\begin{aligned} g_M = -r^{-1}(r - 2M) \frac{du}{dU} \frac{dv}{dV} dU dV + r^2 \mathring{\not{g}} \\ = 16M^2 r^{-1}(r - 2M)(UV)^{-1} dU dV + r^2 \mathring{\not{g}} \\ = -32M^3 r^{-1} e^{-\frac{r}{2M}} dU dV + r^2 \mathring{\not{g}}, \end{aligned}$$

with $U < 0$ and $V > 0$.

Since nothing singular occurs at $U = 0$ or $V = 0$, we can *extend* $(\mathcal{M}_{\text{ext}}, g_M)$ to obtain the bigger spacetime $(\mathcal{M}_{\text{Krus}}, g_M)$, which is called a *maximally-extended Schwarzschild spacetime* or *Kruskal spacetime*. We proceed as follows:

- First, we will extend $r(U, V) = r(U \cdot V)$ to $U \geq 0$ and $V \leq 0$ by requiring (4.3) to hold, i.e. let $f : (0, \infty) \to (-\infty, 1)$ with $f(x) = -e^{\frac{x}{2M}} \frac{x-2M}{2M}$, then:
  - ⋆ f is smooth (in fact, analytic),
  - ⋆ $\frac{df}{dx}(x) = -\frac{1}{2M} \left(\frac{x-2M}{2M} + 1 \right) < 0$ for all $x \in (0, \infty)$,
  - ⋆ f is bijective (injective by $\frac{df}{dx}(x) < 0$ and surjective by the fact that $f(x) \to 1$ as $x \downarrow 0$ and $f(x) \to -\infty$ as $x \to \infty$ and the intermediate value theorem.

  This implies that the inverse $f^{-1} : (-\infty, 1) \to (0, \infty)$ is well-defined and, by the Inverse Function Theorem, f^{-1} is smooth (in fact, analytic). We can therefore extend: $r(U, V) := f^{-1}(U \cdot V)$ smoothly to the set $\{(U, V) \in \mathbb{R}^2 \,,\, 0 \leq UV < 1\}$, where it will take on values in $(0, 2M]$.

- We will define the *maximally-extended Schwarzschild* or *Kruskal* manifold by

  $$\mathcal{M}_{\text{Krus}} =: \{(U, V) \in \mathbb{R}^2 \,,\, UV < 1\} \times \mathbb{S}^2$$

  and equip it with the metric

  $$g_M = -32M^3 r^{-1} e^{-\frac{r}{2M}} dU dV + r^2 \mathring{\not{g}}.$$

It is clear that $(\mathcal{M}_{\text{Krus}}, g_M)$ agrees with the Schwarzschild exterior spacetime in the subset $\{U < 0, V > 0\}$ and that it remains well-defined in the region $\{0 \leq UV < 1\}$. We will use the shorthand notation $g_M = -\Omega^2_{\text{Krus}}(U,V)dUdV + r^2\mathring{\not{g}}$, with

$$\Omega^2_{\text{Krus}}(U,V) = -32M^3r^{-1}e^{-\frac{r}{2M}}.$$

[Exercise: Show that the region $\{0 \leq UV < 1\}$ can be covered by coordinates (t', r, θ, φ), such that

$$g_M = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt'^2 + r^2\mathring{\not{g}}.$$

for a suitable choice of $t'(U,V)$.]

The set $UV = 1$ consists of two disconnected hyperbolas. Is it possible to extend the spacetime across $UV = 1$?

One can show that the following contraction of the Riemann tensor, the so-called *Kretschmann scalar* $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ satisfies [Excercise: Show this (at your own peril!).]:

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{48M^2}{r^6}.$$

and hence $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ blows up at $r = 0$.

Proposition 4.1. *The spacetime $(\mathcal{M}_{\text{Krus}}, g_M)$ cannot be extended across $\{UV = 1\}$ as a spacetime with a C^2 metric.*

Proof. Suppose such an extension did exist. Then $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ (the Kretschmann scalar) would be well-defined. But since $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is a function that does not depend on the choice of coordinates on the manifold, this is in contradiction with the fact that $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ blows up at $UV = 1$, since $r \to 0$. □

In fact, it can be shown (Sbierski 2016) that $(\mathcal{M}_{\text{Krus}}, g_M)$ is inextendible with a C^0 metric across $\{UV = 1\}$. Observers experience an "infinite tidal deformation" as they approach $UV = 1$, rather than merely an infinite tidal force (consistent with blow-up of the Kretschmann scalar).

We can represent the maximally-extended Schwarzschild spacetime $(\mathcal{M}_{\text{Krus}}, g_M)$ via a Penrose diagram. We redefine:

$$\tilde{U} = \arctan U,$$
$$\tilde{V} = \arctan V.$$

Note that $UV = 1$ corresponds to $\tan\tilde{U} \cdot \tan\tilde{V} = 1$. Hence $(\tilde{U}, \tilde{V})$ have the following range:

$$\{(\tilde{U},\tilde{V}) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\ ,\ \tan\tilde{U} \cdot \tan\tilde{V} < 1\}.$$

The set $\{r = 0\}$ is therefore represented by the curves:

$$\left\{\tilde{U} = \arctan\left(\frac{1}{\tan\tilde{V}}\right)\right\} = \left\{\tilde{U} = \frac{\pi}{2} - \tilde{V} \mid 0 < \tilde{V} < \frac{\pi}{2}\right\} \cup \left\{\tilde{U} = -\frac{\pi}{2} - \tilde{V} \mid -\frac{\pi}{2} < \tilde{V} < 0\right\},$$

where we used that

$$\arctan\left(\frac{1}{\tan x}\right) = \arctan\left(\frac{\cos x}{\sin x}\right) = \arctan\left(\frac{\sin(x+\frac{\pi}{2})}{-\cos(x+\frac{\pi}{2})}\right) = \arctan\left(-\tan(x+\frac{\pi}{2})\right)$$
$$= -\arctan\left(\tan(x+\frac{\pi}{2})\right)$$

$$= \begin{cases} -(x + \frac{\pi}{2}) & (-\frac{\pi}{2} < x < 0), \\ -(x + \frac{\pi}{2} - \pi) & (0 < x < \frac{\pi}{2}). \end{cases}$$

We can immediately see from the Penrose diagram that future-directed causal curves in the region where $\{U \geq 0, V \geq 0\}$ cannot enter the region $\{U < 0, V \geq 0\}$, since $J^-(\{U < 0, V \geq 0\}) \cap \{U \geq 0, V \geq 0\} = \emptyset$. We refer to the region $\mathcal{BH} := \{U \geq 0, V \geq 0\}$ as the *Schwarzschild black hole region*. The intersection of $\mathcal{H}_L^{\pm}$ and $\mathcal{H}_R^{\pm}$ at $(U, V) = (0, 0)$ is called the *bifurcation sphere*.

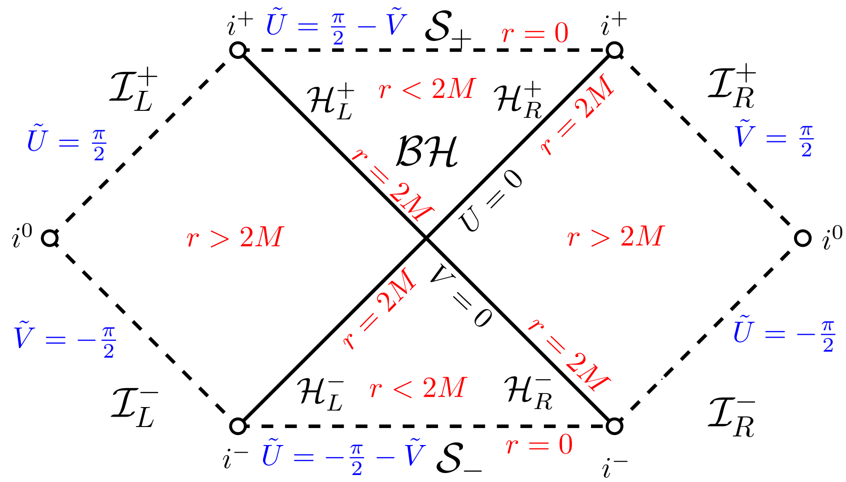


Figure 11: A Penrose diagram of the maximally-extended Schwarzschild spacetime.

The lines $\mathcal{S}_+$ and $\mathcal{S}_-$ in $\partial\mathcal{Q}$ represents the Schwarzschild singularity, where $r = 0$. It is a spacelike curve in $\mathbb{R}^{1+1}$, so we say the Schwarzschild singularity is spacelike.

[EXERCISE: 1) Construct a Penrose diagram of Schwarzschild spacetimes with $M < 0$. 2) Are these spacetimes globally hyperbolic?]

4.2 Isometries and Killing vector fields

Consider the region $(\mathcal{M}_{\text{ext}}, g_M)$. Since the metric coefficients of g_M do not depend on t, it follows that

$$(\mathcal{L}_{\partial_t} g_M)_{\mu\nu} = \partial_t (g_M)_{\mu\nu} = 0,$$

so ∂_t is a Killing vector field. With respect to (U, V) coordinates, we can express

$$\partial_t = \partial_t U \partial_U + \partial_t V \partial_V = \partial_t u \partial_u U \partial_U + \partial_t v \partial_v V \partial_U = \frac{1}{4M}(-U\partial_U + V\partial_V).$$

We can therefore define the vector field $T = \frac{1}{4M}(-U\partial_U + V\partial_V)$ in the full spacetime $(\mathcal{M}_{\text{Krus}}, g_M)$ and observe that it agrees with ∂_t in $(\mathcal{M}_{\text{ext}}, g_M)$.

Along $\mathcal{H}_R^+ = \{U = 0\} \cap \{V > 0\}$, we have that $T = \frac{1}{4M} V \partial_V$, so T is an outgoing null vector field that is tangent to $\mathcal{H}_R^+$.

Note that we can cover the region $\{V > 0\} \cap \mathcal{M}_{\text{Krus}}$ by ingoing Eddington–Finkelstein coordinates (v, r, θ, φ), with

$$g_M = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 \mathring{\slashed{g}}.$$

[EXERCISE: Show that $g_M^{-1} = (\partial_v \otimes \partial_r + \partial_r \otimes \partial_v) + \left(1 - \frac{2M}{r}\right) \partial_r \otimes \partial_r + r^{-2} \mathring{\slashed{g}}^{-1}$.]

Note that in ingoing Eddington–Finkelstein coordinates, we can express along $\mathcal{H}^+_R$:

$$T = \frac{1}{4M} V\partial_V = \frac{1}{4M} V(v)(\partial_V v)\partial_v = \partial_v.$$

Since g_M in (v, r) coordinates is also v-independent in the region $r \leq 2M$, we conclude that T must remain a Killing vector field when $r \leq 2M$.

Since T is null along $\mathcal{H}^+_R$, we must have that $\nabla_T T|_{\mathcal{H}^+_R} = f(v)T|_{\mathcal{H}^+_R}$ for some function f (Exercise). Furthermore, in (v, r) coordinates, using that $g_{vr} = 1$, the Levi-Civita connection is metric-preserving and $\nabla_X Y - \nabla_Y X = [X, Y]$ for $X, Y \in \mathcal{T}(\mathcal{M})$:

$$\begin{aligned} f(v) = g(f(v)\partial_v, \partial_r)|_{\mathcal{H}^+_R} &= g(\partial_r, \nabla_{\partial_v}\partial_v)|_{\mathcal{H}^+} = \partial_v(g(\partial_v, \partial_r)) - g(\nabla_{\partial_v}\partial_r, \partial_v)|_{\mathcal{H}^+_R} \\ &= -g([\partial_v, \partial_r], \partial_v)|_{\mathcal{H}^+_R} - g(\nabla_{\partial_r}\partial_v, \partial_v)|_{\mathcal{H}^+_R} = -\frac{1}{2}\partial_r(g_{vv})|_{\mathcal{H}^+_R} = \frac{M}{r^2}|_{\mathcal{H}^+_R} = \frac{1}{4M}. \end{aligned}$$

In particular, $\nabla_T T|_{\mathcal{H}^+_R} = \kappa_+ T$, where $\kappa_+ = \frac{1}{4M}$ is called the *surface gravity* of the Schwarzschild event horizon.

The vector field ∂_φ is clearly also a Killing vector field. In fact:

[Exercise: Show that the following *angular momentum vector fields* are all Killing vector fields of $(\mathcal{M}_{\mathrm{Krus}}, g_M)$:

$$\begin{aligned} L_1 &= -\sin\varphi\partial_\theta - \cot\theta\cos\varphi\partial_\varphi, \\ L_2 &= \cos\varphi\partial_\theta - \cot\theta\sin\varphi\partial_\varphi, \\ L_3 &= \partial_\varphi.] \end{aligned}$$

Definition 4.1. *A spacetime* $(\mathcal{M}, g)$ *is* static *if it admits a timelike Killing vector field* T*, which satisfies:*

$$T^\flat \wedge dT^\flat = 0.$$

This is equivalent[13] *to the statement that around any point* $p \in \mathcal{M}$*, there exist local coordinates* $(\tau, x^1, \ldots, x^n)$*, such that we can express:*

$$g = -f d\tau^2 + h_{ij} dx^i dx^j,$$

where $T = \partial_\tau$*,* f, h_{ij} *are functions of* $x^1, \ldots, x^n$ *and* $h_{ij}dx^i dx^j$ *is a Riemannian metric on the level sets of* τ*.*

Clearly, the Schwarzschild exteriors $(\mathcal{M}_{\mathrm{ext}}, g_M)$ are static.

The existence of angular momentum Killing vector fields L_1, L_2, L_3 is a consequence of the *spherical symmetry* on the (extended) Schwarzschild spacetimes.

4.3 Dynamics of geodesics

In Einstein theory of general relativity, idealized observers in a spacetime are represented by timelike geodesics and the path of photons is represented by null geodesics. This is known as the *geodesic hypothesis.*

Independently of its relation with observers, the behaviour of causal geodesics also serves as a convenient tool for probing important geometric properties of the spacetime.

[13] This equivalence follows from the Frobenius theorem.

4.3.1 Black hole exterior

Let $\gamma : I \to \mathcal{M}_{\text{ext}}$ be an affinely parametrized causal geodesic, which is either timelike or null. In the case of timelike geodesics, we will take the affine parameter to be the proper time along the geodesic. Then $g(\dot\gamma, \dot\gamma) = -\sigma$, with $\sigma = 0$ if γ is null and $\sigma = 1$ if γ is timelike.

We denote

$$\mathcal{M}_{\text{ext}} \ni \gamma(s) \cong \begin{pmatrix} t(s) \\ r(s) \\ \theta(s) \\ \varphi(s) \end{pmatrix}, \qquad T_{\gamma(s)}\mathcal{M} \ni \dot\gamma(s) \cong \begin{pmatrix} \dot t \\ \dot r \\ \dot\theta \\ \dot\varphi \end{pmatrix}(s).$$

Then $g(\dot\gamma, \dot\gamma) = -\sigma$ implies that:

$$-\sigma = -\left(1 - \frac{2M}{r}\right)\dot t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot r^2 + r^2\dot\theta^2 + r^2\sin^2\theta\dot\varphi^2. \tag{4.4}$$

We can simplify this equation a bit by redefining our spherical coordinates (θ, φ) so that we are guaranteed that $\theta(0) = \frac{\pi}{2}$ and $\dot\theta(0) = 0$. We can now apply the transformation $\theta' : \theta \mapsto \pi - \theta$, which leaves the Schwarzschild metric and hence the geodesic equation invariant, and which also leaves the geodesic initial data $(\gamma(s), \dot\gamma(s))$ invariant. But uniqueness of solutions to the geodesic equation, we must have that $\theta'(s) = \theta(s)$ for all $s \in I$, so that necessarily $\theta(s) = \frac{\pi}{2}$. That is to say, the geodesic will stay restricted to the equator.

We can therefore simplify (4.4) to obtain:

$$-\sigma = -\left(1 - \frac{2M}{r}\right)\dot t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot r^2 + r^2\dot\varphi^2.$$

Now we use the Killing property of ∂_t and ∂_φ to conclude that $g_{\gamma(s)}(\partial_t, \dot\gamma)$ and $g_{\gamma(s)}(\partial_\varphi, \dot\gamma)$ are conserved in s (Proposition 2.5). We define:

$$\begin{aligned} E &= -\, g_{\gamma(s)}(\partial_t, \dot\gamma)(s) = -\left(1 - \frac{2M}{r(s)}\right)\dot t(s), \\ L &= g_{\gamma(s)}(\partial_\varphi, \dot\gamma)(s) = r^2(s)\dot\varphi(s) \end{aligned}$$

and refer to E as the energy of the geodesic with respect to ∂_t and to L as the angular momentum of the geodesic (in the z-direction). Then we obtain:

$$-\sigma = -\left(1 - \frac{2M}{r}\right)^{-1}E^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot r^2 + r^{-2}L^2$$

Rearranging the above equation, we obtain:

$$E^2 = \dot r^2 + \left(1 - \frac{2M}{r}\right)(L^2 r^{-2} + \sigma) =: \dot r^2 + V_\sigma(r).$$

The above equation resembles the conservation of energy property of a 1-D dynamical system, where V_σ plays the role of a potential energy term and $\dot r^2$ plays the role of a kinetic energy term.

By studying the qualitative properties of the potential function V_σ, we can determine the qualitative properties of the dynamics of causal geodesics.

Null geodesics We consider first the case $\sigma = 0$. Then $V_0(r) = L^2\left(1 - \frac{2M}{r}\right) r^{-2}$. If $L = 0$, then $V_0 \equiv 0$. Suppose that $L \neq 0$.

Then we have that $V_0(2M) = 0$ and $V_0(r) \to 0$ as $r \to \infty$. Furthermore,

$$V_0'(r) = -2L^2 r^{-4}(r - 3M)$$

Hence, V_0 has one extremum at $r = 3M$, which is a maximum.

Suppose $r(0) > 3M$, $\dot{r}(0) < 0$. Then we distinguish three cases:

- $E^2 > V(3M)$: the geodesic enters the black hole region;
- $E^2 < V(3M)$: $r(s)$ decreases until $V_0(r(s)) = E^2$ and then $r(s)$ increases and $r(s) \to \infty$;
- $E^2 = V(3M)$: the geodesic approaches $r = 3M$.

The timelike hypersurface $\{r = 3M\}$ is called the *photon sphere* or *light ring* and it admits solutions $\gamma_{3M}(s)$ with constant radius (circular orbits):

$$\gamma_{3M}(s) \cong \begin{pmatrix} 3EM^{-1}s \\ 3M \\ \frac{\pi}{2} \\ \frac{1}{9}M^{-2}Ls \mod 2\pi \end{pmatrix}.$$

Consider a geodesic with $E^2 = V(3M)$. Since a generic small perturbation of the initial data $(\gamma(0), \dot{\gamma}(0))$ for the geodesic equation results in $E^2 \neq V(3M)$ and therefore corresponds to a geodesic that either enters the black hole region or escapes to infinity, we have that the (asymptotically) circular orbits are *unstable*.

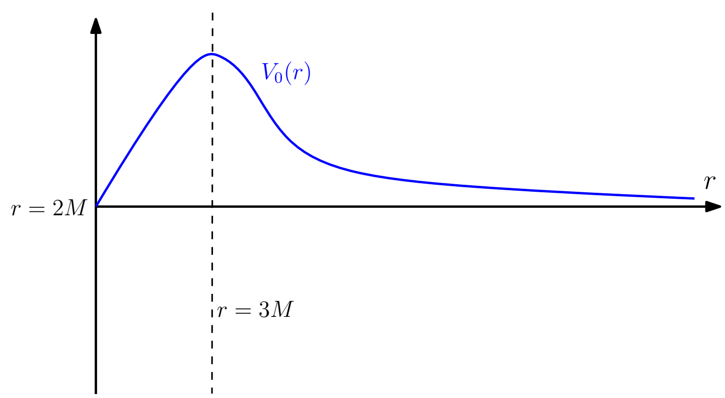


Figure 12: The potential $V_0(r)$.

Timelike geodesics Now consider the case $\sigma = 1$. Then

$$V_1(r) = 1 - \frac{2M}{r} + \frac{L^2}{r^2} - \frac{2L^2M}{r^3}.$$

We can compare $V_1(r)$ to the potential appearing in the 1-dimensional reduction of the following problem: a test particle of mass 1 in a gravitational force field created by a body of mass M in Newton's theory of gravity. In that case, the distance r between the test particle and the body satisfies $\frac{1}{2}\dot{r}^2 + V_{\text{Newt}}(r) = \tilde{E}$, with $\tilde{E}$ the total energy and

$$2V_{\text{Newt}}(r) + 1 = 1 - \frac{2M}{r} + \frac{L^2}{r^2}.$$

Label $E^2 = 2\tilde{E} + 1$. Then $\dot{r}^2 + (2V_{\text{Newt}}(r) + 1) = E^2$. The key difference with $V_1(r)$ is the term $-\frac{2L^2 M}{r^3}$, which becomes dominant for small r.

We have that $V_1(2M) = 0$ and $V_1(r) \to 1$ as $r \to \infty$. Furthermore,

$$V_1'(r) = \frac{2M}{r^2} - \frac{2L^2}{r^3} + \frac{6L^2 M}{r^4} = 2r^{-4}(Mr^2 - L^2 r + 3L^2 M).$$

Hence, $V_1(r)$ admits at most two extrema at radii $r = R_+$ and $r = R_+$, with

$$R_\pm = \frac{L^2}{2M}(1 \pm \sqrt{1 - 12M^2 L^{-2}}).$$

If $L^2 < 12M^2$, then there are no extrema. If $L^2 = 12M^2$, then there is exactly one extremum at $r = \frac{L^2}{2M} = 6M$. Since

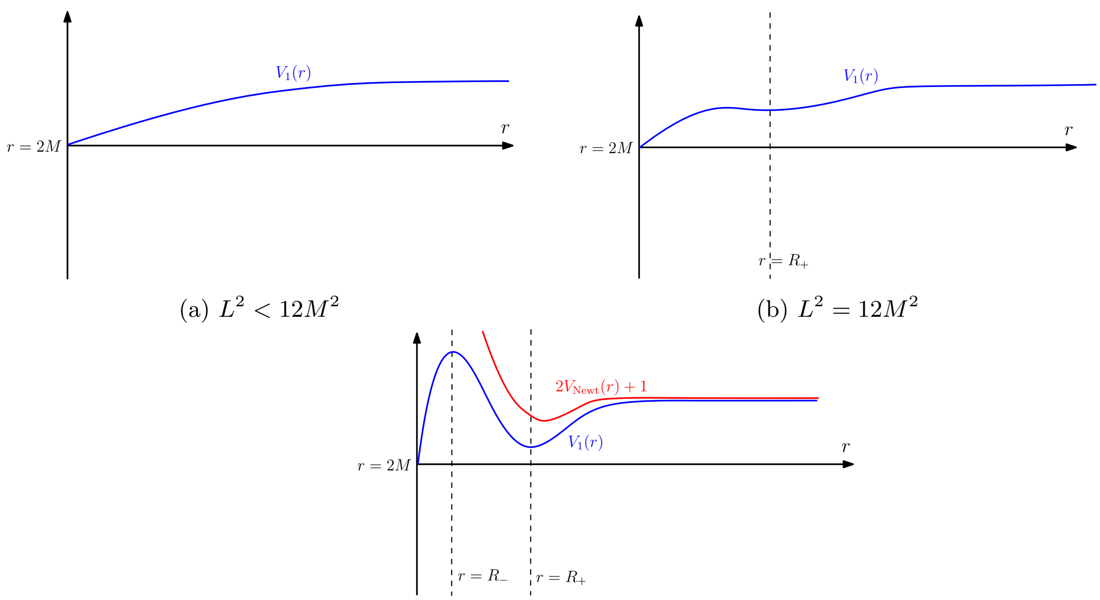
$$V_1''(r) = 2r^{-5}(-2Mr^2 + 3L^2 r - 12L^2 M),$$

one can verify that in the case $L^2 = 12M^2$:

$$V_1''(6M) = 0.$$

Hence, the extremum is a saddle point.

If $L^2 > 12M^2$, then there are exactly two extrema: R_- is a maximum and R_+ is a minimum.



(a) $L^2 < 12M^2$

(b) $L^2 = 12M^2$

(c) $L^2 > 12M^2$

Figure 13: The potential $V_1(r)$.

As in the case of null geodesics, when $E = V_1(R_\pm)$, there are geodesics that stay fixed at $r = R_\pm$. Since $r = R_-$ is a maximum of the potential, these geodesics correspond to unstable circular orbits. As $r = R_+$ is a maximum, for any $\epsilon > 0$, all sufficiently small perturbations of the initial data $(\gamma(0), \dot{\gamma}(0))$ will result in geodesics which stay in the region $R_+ - \epsilon \le r \le R_+ + \epsilon$ and therefore correspond to bounded orbits (that need not be circular).[14]

[14] In the Newtonian two-body problem, such orbits are closed and form ellipses. In the Schwarzschild case, this

4.3.2 Red-shift

Consider the region of the maximally-extended Schwarzschild spacetime that is spanned by ingoing Eddington–Finkelstein coordinates (v, r, θ, φ).

A global red-shift effect There is a *global red shift effect* present in the Schwarzschild black hole exterior. Consider an observer, Alice, entering the black hole and an observer Bob, who stays outside of the black hole. Suppose Alice emits a photon at the spacetime point $(u, v_A, \theta_0, \varphi_0)$ moving in the radial direction, which is intercepted by Bob at the spacetime point $(u, v_B, \theta_0, \varphi_0)$.

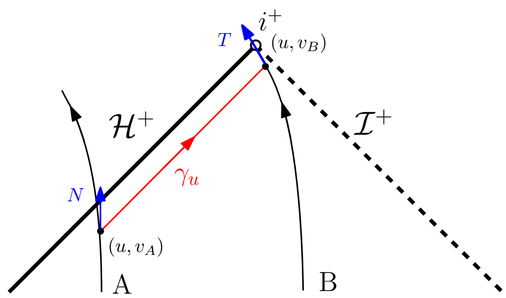


Figure 14: Alice emitting a photon travelling at constant (θ, φ), represented by γ_u, that is intercepted by Bob, who stays in the black hole exterior. Bob measures a red-shift of the energy of the photon.

We describe the photon by an outgoing null geodesic γ_u. As we already saw, V', defined via:

$$\frac{dV'}{dv} = 1 - \frac{2M}{r(u,v)},$$

defines an affine parameter along γ_u. Hence,

$$\dot{\gamma}_u = \partial_{V'} = \frac{1}{1 - \frac{2M}{r(u,v)}} \partial_v.$$

Let us assume for the sake of simplicity that the timelike curve describing Alice's spacetime path is tangent to the vector field

$$N = \partial_v - \partial_r$$

in (v, r) coordinates, in a neighbourhood of the event horizon.

[Exercise: Show that N is timelike and that with respect to Eddington–Finkelstein double null coordinates (u, v):

$$N = \left(1 + \frac{2}{(1 - 2Mr^{-1})}\right) \partial_u + \partial_v = \frac{1}{(1 - 2Mr^{-1})} \left(3 - 2Mr^{-1}\right) \partial_u + \partial_v.]$$

Then the energy of the photon, according to Alice is given by the expression:

$$E_{\gamma_u, A} = -g(\dot{\gamma}_u, N)(u, v_A) = -\frac{1}{(1 - \frac{2M}{r})^2} \left(3 - 2Mr^{-1}\right) g_{uv}\Big|_{r = r(u, v_A)}$$

is no longer the case. It can be shown that for $R_+ M^{-1} \gg 1$, the orbits trace to leading-order ellipses whose periapsis (the shortest distance between the two massive bodies) precesses (rotates). Historically, this deviation from Newtonian theory was used to explain the anomalous precession of the perihelion (the periapsis where the larger body is the sun) of Mercury orbiting around the sun, which could not be accounted for using solely Newtonian theory and the effects of the other planets in the solar system.

$$= \frac{1}{2(1-\frac{2M}{r(u,v_A)})}\left(3-2Mr^{-1}\right)(u,v_A).$$

We assume also, for the sake of simplicity that the timelike curve describing Bob's spacetime path is tangent to $T=\partial_u+\partial_v$. Then the energy of the photon, according to Bob is:

$$E_{\gamma_u,B} = -g(\dot{\gamma}_u,T)(u,v_B) = -\frac{1}{1-\frac{2M}{r}}g_{uv}\Big|_{r=r(u,v_B)} = \frac{1}{2}.$$

The ratio of the two energies is given by

$$\frac{E_{\gamma_u,B}}{E_{\gamma_u,A}} = \frac{1-\frac{2M}{r}}{3-\frac{2M}{r}}(u,v_A).$$

Observe:

1. $E_{\gamma_u,B} < E_{\gamma_u,A}$,
2. $\frac{E_{\gamma_u,B}}{E_{\gamma_u,A}} \to 0$ as $u\to\infty$.

We say the energy of the photon is *red-shifted* and there is an *infinite red-shift* in the limit of the photon emission occurring as Alice crosses the event horizon of the black hole.

This terminology is motivated by the following relation between the energy E of a photon and its frequency ω (in radians/second): $E=\hbar\omega$, with $\hbar$ the reduced Planck constant. That is to say, a decrease in energy corresponds to a lowering in frequency, which in turn means a shift to the red part of the spectrum.

A horizon red-shift effect Recall that the Killing vector field $T=\partial_v$ in (v,r) coordinates is tangential to $\mathcal{H}^+$.

Recall also that, in (v,r) coordinates,

$$\nabla_{\partial_v}\partial_v|_{\mathcal{H}^+} = \kappa_+\partial_v|_{\mathcal{H}^+}.$$

Since $\kappa_+\neq 0$, $\partial_v|_{\mathcal{H}^+}$ does not correspond to $\dot{\gamma}$, with γ an affinely parametrized null geodesic. However, we have that

$$\nabla_{e^{-\kappa_+ v}\partial_v}(e^{-\kappa_+ v}\partial_v)|_{\mathcal{H}^+} = 0.$$

Hence, we can reparametrize the integral curves γ of ∂_v along $\mathcal{H}^+$ to obtain an affinely parametrized null geodesic γ_{2M} with parameter $s(v)=e^{-\kappa_+ v}-1$. More explicitly, we obtain the following null geodesics $\gamma_{2M}:[0,\infty)\to\mathcal{M}_{\mathrm{Krus}}$ that are tangent to the future-event horizon $\mathcal{H}^+$:

$$\gamma_{2M}(s) \cong \begin{pmatrix} -\kappa_+^{-1}\log(s+1) \\ 2M \\ \theta_0 \\ \varphi_0 \end{pmatrix},$$

with $\theta_0\in(0,\pi)$ and $\varphi_0\in(0,2\pi)$ arbitrary.

Although the geodesics γ_{2M} are "trapped" at $r=2M$, it turns out their energy decays exponentially with respect to v. We define energy with respect to the timelike vector field $N=\partial_v-\partial_r$.

Note that $\mathcal{L}_T N=[T,N]=0$, so N is time-translation invariant. We can think of the integral curves of N as representing timelike observers crossing $\mathcal{H}^+$. The energy of γ_{2M} with respect to N is as follows:

$$E_{\gamma_{2M}}(v) = -g_{\gamma_{2M}(s(v))}(\dot{\gamma}_{2M}(s(v)),N_{\gamma_{2M}(s(v))}) = -g_{\gamma_{2M}(s(v))}(e^{-\kappa_+ v}\partial_v|_{\gamma_{2M}(s(v))},(\partial_v-\partial_r)|_{\gamma_{2M}(s(v))}) = e^{-\kappa_+ v}.$$

The exponential decay of the above energy is called the *horizon red-shift effect* and it is closely related to the non-vanishing of the surface gravity κ_+. We will later see that there a black hole spacetimes which have a vanishing κ_+ and a non-vanishing horizon red shift effect, but, due to the presence of an event horizon, they *do* have a global red-shift effect.

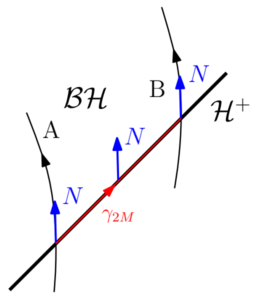


Figure 15: Alice emitting a photon travelling along the event horizon, which is intercepted by Bob as he crosses the event horizon, who measures a red-shift of the energy of the photon.

4.3.3 Black hole interior

Consider the black hole interior region, where $r < 2M$. With respect to (t', r, θ, φ) coordinates, we can express:

$$g_M = -\left(\frac{2M}{r} - 1\right)^{-1} dr^2 + \left(\frac{2M}{r} - 1\right) dt'^2 + r^2 \mathring{\not{g}}.$$

As in the black hole exterior, we obtain the following equation for timelike curves, parametrized by their proper time s:

$$-1 = \left(\frac{2M}{r} - 1\right) \dot{t'}^2 - \left(\frac{2M}{r} - 1\right)^{-1} \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta \dot{\varphi}^2, \tag{4.5}$$

[EXERCISE: Show that ∂_r, with respect to (t', r, θ, φ) coordinates is a past-directed timelike vector field.]

Since ∂_r past-directed and timelike, future-directed causal curves must satisfy the following inequality:

$$0 < g_{\gamma(s)}(\partial_r, \dot{\gamma})(s) = -\left(\frac{2M}{r(s)} - 1\right)^{-1} \dot{r}(s)$$

and therefore, $\dot{r}(s) < 0$. We will now show that $r = 0$ is attained at a finite value of s denoted $s_{\max}$.

For $\sigma = 1$, we have by (4.5) that

$$\dot{r}^2 \geq \left(\frac{2M}{r} - 1\right),$$

with equality if the curve is an integral curve of ∂_r. Since moreover $\dot{r} \leq 0$, we in fact have that:

$$\dot{r} \leq -\sqrt{\frac{2M}{r} - 1}$$

and therefore

$$s = -\int_{r(s)}^{r(0)} \frac{ds}{dr}(r)\, dr \leq \int_{r(s)}^{r(0)} \frac{1}{\sqrt{2Mr^{-1} - 1}}\, dr \leq \frac{1}{\sqrt{2Mr^{-1}(0) - 1}}(r(0) - r(s)),$$

so $s_{\max} \leq \frac{r(0)}{\sqrt{2Mr^{-1}(0)-1}}$.

We conclude that observers, represented by timelike curves, must reach the Schwarzschild singularity at $r = 0$ in finite proper time, bounded above by $s_{\max}$. No matter how much they try to accelerate away from the singularity, they are doomed to reach the end of the spacetime in finite time!

4.4 Reissner–Nordström spacetimes

Schwarzschild spacetimes can be embedded into the larger Reissner–Nordström family of spacetimes (Reissner 1916, Weyl 1917, Nordström 1918), which solve the Einstein–Maxwell equations with $F = \frac{Q}{r^2} dt \wedge dr + P \sin\theta d\theta \wedge d\varphi$, where $Q \in \mathbb{R}$ is called the *electric charge* of the spacetime and $P \in \mathbb{R}$ is called the *magnetic charge*. We define $e = \sqrt{Q^2 + P^2}$. Reissner–Nordström spacetimes are pairs $(\mathcal{M}_{\text{ext}}, g_{M,e})$, with $\mathcal{M}_{\text{ext}} = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$ and

$$\begin{aligned} g_{M,e} &= -Ddt^2 + D^{-1}dr^2 + r^2 \mathring{\not{g}}, \\ D(r) &= 1 - \frac{2M}{r} + \frac{e^2}{r^2}, \end{aligned}$$

with $r_\pm = M \pm \sqrt{M^2 - e^2}$ the roots of the polynomial $r^2 D(r)$ in the case $M \geq e$, otherwise $r_+ = 0$. If $M = e$ (so $r_+ = r_-$) we say the spacetime is **extremal**, if $M > e$, we say the spacetime is **sub-extremal**, if $M > e$, then we says the spacetime is **super-extremal**.

Just like Schwarzschild spacetimes, Reissner–Nordström spacetimes are static and spherically symmetric.

In order to extend the spacetime from $r_+ < r < \infty$ to $r_- < r < \infty$, we will apply the same strategy as in Schwarzschild. First, we can introduce double null coordinates (u, v) with respect to which:

$$g_{M,e} = -\Omega^2 du dv + r^2 \mathring{\not{g}},$$

with $\Omega^2 = 1 - \frac{2M}{r} + \frac{e^2}{r^2}$ by defining

$$\begin{aligned} u &= t - r_*, \\ v &= t + r_*, \\ \frac{dr_*}{dr} &= \frac{1}{1 - \frac{2M}{r} + \frac{e^2}{r^2}}. \end{aligned}$$

[Exercise: Verify that r_* is given by the following expression (up to the addition of a constant):

$$\begin{aligned} r_*(r) &= r + \frac{r_+^2}{2\sqrt{M^2 - e^2}} \log\left(\frac{r - r_+}{r_+}\right) - \frac{r_-^2}{2\sqrt{M^2 - e^2}} \log\left(\frac{r - r_-}{r_-}\right) \quad \text{if} \quad e < M, \\ r_*(r) &= r - \frac{M^2}{r - M} + 2M \log(r - M) \quad \text{if} \quad e = M, \\ r_*(r) &= r + \frac{\left(2M^2 - e^2\right) \arctan\left(\frac{r - M}{\sqrt{e^2 - M^2}}\right)}{\sqrt{e^2 - M^2}} + M \log\left(r^2 - 2Mr + e^2\right) \quad \text{if} \quad e > M.] \end{aligned}$$

Before defining Kruskal coordinates, we first switch to ingoing Eddington–Finkelstein coordinates (v, r, θ, φ) to obtain:

$$g_{M,e} = -Ddv^2 + 2dvdr + r^2 \mathring{\not{g}}.$$

for $0 < r < \infty$.

As in Schwarzschild, we define in these coordinates the null hypersurface $\mathcal{H}_R^+ := \{r = r_+\}$ and $T = \partial_v$, and we observe that $\nabla_T T|_{\mathcal{H}_R^+} = \kappa_+ T|_{\mathcal{H}_R^+}$, with the surface gravity κ_+ taking on the values:

$$\kappa_+ = -\frac{1}{2}\partial_r(g_{vv})|_{\mathcal{H}_R^+} = \frac{1}{2r_+^2}(2M - 2e^2 r_+^{-1}) \overbrace{=}^{e^2 = 2Mr_+ - r_+^2} \frac{1}{2r_+^2}(2r_+ - 2M) = \frac{\sqrt{M^2 - e^2}}{r_+^2} = \frac{r_+ - r_-}{2r_+^2}.$$

Note in particular, the in the extremal case, $\kappa_+ = 0$. Defining moreover $\kappa_- = \frac{r_- - r_+}{2r_+^2}$, we can rewrite r_* as follows in the sub-extremal case:

$$r_*(r) = r + \frac{1}{2\kappa_+}\log\left(\frac{r-r_+}{r_+}\right) + \frac{1}{2\kappa_-}\log\left(\frac{r-r_-}{r_-}\right).$$

As in the Schwarzschild case, we consider the null curves $\{v = v_0, \theta = \theta_0, \varphi = \varphi_0\}$ and observe that the affine parameter is given by: $\frac{dU'}{du}(u) = \Omega^2(u, v_0)$. We will now restrict ourselves to the sub-extremal case. We can express: Furthermore,

$$e^{\kappa_+(v-u)} = e^{2\kappa_+ r_*} = e^{\kappa_+ r}\left(\frac{r-r_+}{r_+}\right)\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{\kappa_-}}. \tag{4.6}$$

Hence

$$\begin{aligned}\frac{dU'}{du}(u) = \Omega^2(u,v_0) = r^{-2}(r-r_+)(r-r_-) = & \left[r^{-2}r_+e^{-\kappa_+ r}\left(\frac{r-r_-}{r_-}\right)^{-\frac{\kappa_+}{\kappa_-}}(r-r_-)\right](u,v_0)e^{\kappa_+(v_0-u)}\\ = & \, r_+^{-2}r_+(r_+-r_-)e^{-\kappa_+ r_+}\left(\frac{r_+-r_-}{r_-}\right)^{-\frac{\kappa_+}{\kappa_-}}e^{\kappa_+ v_0}e^{-\kappa_+ u} + O(e^{-2\kappa_+ u}).\end{aligned}$$

This motivates the following Krushkal coordinate:

$$U(u) = -e^{-\kappa_+ u}.$$

Similarly, we define $V(v) = e^{\kappa_+ v}$. Note that

$$-UV = e^{\kappa_+(v-u)} = r^2\Omega^2(u,v)r_+^{-1}r_-^{-1}e^{\kappa_+ r}\left(\frac{r-r_-}{r_-}\right)^{\frac{\kappa_+}{\kappa_-}-1}$$

and $\frac{dU}{du} = -\kappa_+ U$, $\frac{dV}{dv} = \kappa_+ V$.

We can then express:

$$g_{M,e} = -\Omega^2(u,v)\frac{du}{dU}\frac{dv}{dV}dUdV + r^2\mathring{\not{g}} = -\frac{r_+r_-}{\kappa_+^2 r^2 e^{\kappa_+ r}}\left(\frac{r-r_-}{r_-}\right)^{1-\frac{\kappa_+}{\kappa_-}}dUdV + r^2\mathring{\not{g}}.$$

We can extend $r(U,V)$ analytically to $U \geq 0$ and $V \leq 0$ and therefore extend $(\mathcal{M}_{\rm ext}, g_{M,e})$ analytically to the manifold:

$$\mathcal{M}_{\rm Krusk} = \mathbb{R}_U \times \mathbb{R}_V \times \mathbb{S}^2_{\theta,\varphi}.$$

Rescaling $\tilde{U} = \arctan U$ and $\tilde{V} = \arctan V$, we obtain a Penrose diagram. Compared to the Schwarzschild diagram, we have the following additional boundary components:

$$\begin{aligned}\mathcal{CH}_L^{\pm} :=& \{\tilde{U} = \pm\frac{\pi}{2}, \pm U > 0\},\\ \mathcal{CH}_R^{\pm} :=& \{\tilde{V} = \pm\frac{\pi}{2}, \pm V > 0\}.\end{aligned}$$

Note that $\mathcal{CH}_L^{\pm}$ and $\mathcal{CH}_R^{\pm}$ are the Cauchy horizons corresponding to the Cauchy hypersurface $\Sigma = \{U + V = 0\}$. The spacetime $(\mathcal{M}_{\rm Krusk}, g_{M,e})$ is therefore globally hyperbolic. Furthermore, since they correspond to $\pm U \to \infty$ and constant V or $\pm V \to \infty$ and constant U, we can apply (4.6) to conclude that $r \to r_-$ as we approach the Cauchy horizons.

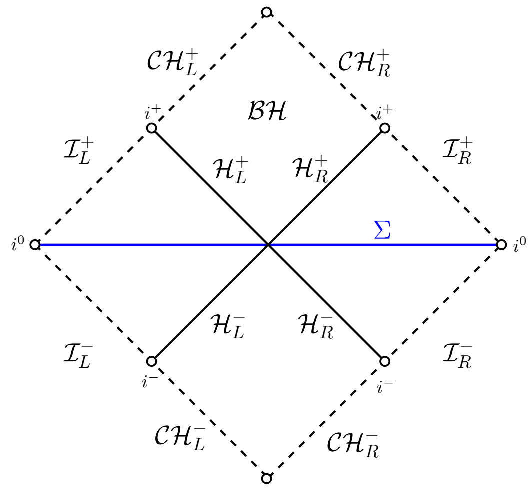


Figure 16: Sub-extremal Reissner–Nordström with Cauchy hypersurface $\Sigma = \{U + V = 0\}$.

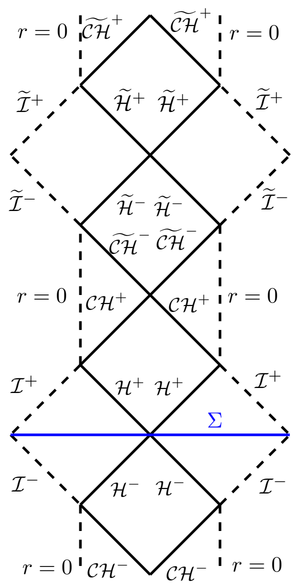


Figure 17: A Penrose diagram of the analytic extension of the Reissner–Nordström spacetime.

In (v, r, θ, φ) coordinates, we were able to consider also $0 < r < r_-$. Indeed, in these coordinates $\mathcal{CH}_L^+ = \{r = r_-\}$ is a smooth (in fact analytic) hypersurface of an extended manifold. Similarly, in (u, r, θ, φ) coordinates $\mathcal{CH}_R^+ = \{r = r_-\}$ is a smooth (in fact analytic) hypersurface of an extended manifold obtained by considering $(u, v, r, \theta, \varphi)$ coordinates in the region $r < r_+$.

We can similarly draw the Penrose diagram for the analytic extensions of $\mathcal{M}_{\text{Krus}}$ across $\mathcal{CH}_L^\pm$ and $\mathcal{CH}_R^\pm$.

We will later see that a globally hyperbolic spacetime, like $(\mathcal{M}_{\text{Krusk}}, g_{M,e})$, arises uniquely from the time evolution of appropriate initial data posed along a Cauchy hypersurface, like Σ. Extensions across the Cauchy horizons are independent of the initial data are therefore highly non-unique.[15] Uniqueness can be obtained by restricting the analytic extensions, but this is a very

[15]Note that in the particular case of Reissner–Nordström imposing the additional restriction of spherical sym-

unphysical regularity class to restrict to as it is incompatible with the finite speed of propagation property embedded in the Einstein equations. Indeed, knowledge of the metric on a open subset $S \subset \mathcal{M}$, would completely determine the spacetime globally, rather than only affect the part of the spacetime in the causal future $J^+(S)$.

[Exercise: Consider the null hypersurface $H = \{v = v_0, 0 < r < \infty\}$ in (v, r, θ, φ) coordinates. Determine the globally hyperbolic spacetime region in the analytic extension of $\mathcal{M}_{\mathrm{Krusk}}$ which has H as a Cauchy hypersurface and draw this region in a Penrose diagram.]

[Exercise: Construct the analytic extension of extremal Reissner–Nordström exterior spacetimes $(\mathcal{M}_{\mathrm{ext}}, g_{M,e})$ across $\{r = r_+\}$. Construct a Penrose diagram of this extended spacetime and discuss the singularity properties at the boundary of the Penrose diagram.]

[Exercise: Construct a Penrose diagram of super-extremal Reissner–Nordström exterior spacetime $(\mathcal{M}_{\mathrm{ext}}, g_{M,e})$. Describe the properties of possible analytic extensions of these spacetimes.]

metry of smooth extensions actually singles out the unique analytic extension as the only possibility satisfying the Einstein–Maxwell equations. We will encounter a similar "rigidity" property in vacuum in the context of Birkhoff's theorem. When allowing for extensions that are not spherically symmetric, however, we can construct infinitely many extensions.

5 Spherically symmetric spacetimes

In this section, we will focus on spherically symmetric spacetimes. We will see that the vacuum Einstein equations have a strong *rigidity* property in spherical symmetry, i.e. the only possible spherically symmetric vacuum spacetimes are isometric to a subset of the Schwarzschild spacetime family or the Minkowski spacetime.

Via the inclusion of suitable matter, however, we can nevertheless see many of the interesting phenomena characteristic to dynamical black holes spacetimes.

We will give a precise definition of what we mean by a "spherically symmetric" spacetime. First, we need to introduce some notation.

Consider the manifold $\mathbb{R}^2 \times \mathbb{S}^2$ and the associated canonical projection maps:

$$\begin{aligned} \pi &: \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{R}^2, \\ \pi(x, y) &= x, \\ \pi_{\mathbb{S}^2} &: \mathbb{R}^2 \times \mathbb{S}^2 \to \mathbb{S}^2, \\ \pi_{\mathbb{S}^2}(x, y) &= y. \end{aligned}$$

Now consider the manifold $\mathbb{R}_t \times \mathbb{R}^3$. Then we define $\pi : \mathbb{R}_t \times \mathbb{R}^3 \to \mathbb{R} \times [0, \infty)$ as the map that sends all points in the spheres $S^2_{t',r}$ of area radius $r = \sqrt{x^2 + y^2 + z^2}$ in $\{t = t'\}$, to the point $(t', r) \in \mathbb{R} \times (0, \infty)$ and $\pi(t', 0) = (t', 0)$. We also define $\pi_{\mathbb{S}^2} : \mathbb{R}_t \times (\mathbb{R}^3 \setminus \{0\}) \to \mathbb{S}^2$ as the projection map to the unit round sphere $S^2_{t,1}$.

Definition 5.1. *Let $(\mathcal{M}, g)$ be a spacetime and assume that there exists a diffeomorphism:* ***A)*** *$\psi : \mathcal{M} \to \mathbb{R}^2 \times \mathbb{S}^2$ or* ***B)*** *$\psi : \mathcal{M} \to \mathbb{R} \times \mathbb{R}^3$.*

In case B) we denote

$$\Gamma = \psi^{-1}(\mathbb{R} \times \{0\}).$$

We will refer to Γ as the centre of spherical symmetry.

We say $(\mathcal{M}, g)$ is a spherically symmetric spacetime *if ψ and g satisfy the following properties:*

1. *The metric g on $\mathcal{M} \setminus \Gamma$ can be decomposed as follows:*

$$g = (\pi \circ \psi)^* \overline{g} + (r^2 \circ \pi) \cdot (\pi_{\mathbb{S}^2} \circ \psi)^* \mathring{\not{g}}, \tag{5.1}$$

where $\overline{g}$ is a C^2 Lorentzian metric on A) $\mathbb{R}^2$ or B) $\mathbb{R} \times [0, \infty)$ and where A) $r : \mathbb{R}^2 \to (0, \infty)$ is C^2, or B) $r : \mathbb{R} \times [0, \infty) \to [0, \infty)$ is C^2 with $r(t, 0) = 0$.[16]

2. *Γ is a timelike curve in $(\mathcal{M}, g)$.*

We will denote by $\mathcal{Q}$ the 2-dimensional manifolds: A) $\mathbb{R}^2$ and B) $\mathbb{R} \times (0, \infty)$. In case B), $\partial\mathcal{Q} = \mathbb{R} \times \{0\} \subset \mathbb{R}^2 \neq \emptyset$, with respect to $\mathbb{R}^2$.

In slight abuse of notation we will also denote by Γ the curve $\partial\mathcal{Q}$ in case B). To simplify the notation, we write the requirement (5.1) as follows:

$$g = \overline{g} + r^2 \mathring{\not{g}}.$$

Lemma 5.1. *There exists global coordinates $u, v : \mathcal{Q} \to \mathbb{R}$ and a C^2 function $\Omega^2 : \mathcal{Q} \to (0, \infty)$, such that*

$$\overline{g} = -\Omega^2 du dv.$$

[16] A metric of the above form is called a *warped product metric*. We write A) $\mathcal{M} \setminus \Gamma \cong \mathbb{R}^2 \times_r \mathbb{S}^2$ or B) $\mathcal{M} \cong (\mathbb{R} \times (0, \infty)) \times_r \mathbb{S}^2$. The regularity requirements on $\overline{g}$ and r can be relaxed, but we will assume C^2 for the sake of convenience.

Proof. (*non-examinable*) Let $p \in \mathcal{Q}$ be arbitrary. Let $\underline{L}_p$ and L_p be null tangent vectors at p, such that $\overline{g}_p(\underline{L}_p, L_p) = -1$ and consider corresponding affinely parametrized null geodesics $\gamma_{\underline{L}}$ ("ingoing"') and γ_L ("outgoing") emanating from p, where the respective affine parameters u and v are fixed by taking: $\dot{\gamma}_{\underline{L}}(p) = \underline{L}_p$ and $\dot{\gamma}_L(p) = L_p$ and setting $(u, v) = 0$ at p.

Denote by $\overline{\nabla}$ the covariant derivative associated to $\overline{g}$. We extend $\underline{L}_p$ to the curve $\gamma_{\underline{L}}$ by defining $\underline{L} := \dot{\gamma}$, or $\overline{\nabla}_{\underline{L}}\underline{L} = 0$. Similarly, we extend L_p to γ_L by demanding $\overline{\nabla}_L L = 0$. We also uniquely extend L to $\gamma_{\underline{L}}$ by demanding $\overline{\nabla}_{\underline{L}} L = 0$ along $\gamma_{\underline{L}}$, and we extend $\underline{L}$ to γ_L by demanding $\overline{\nabla}_L \underline{L} = 0$ along γ_L. This guarantees that $\overline{g}(L, \underline{L}) = -1$ along $\gamma_{\underline{L}} \cup \gamma_L$.

Now consider the points $\gamma_{\underline{L}}(u)$. Then we can consider affinely parametrized null geodesics $\tilde{\gamma}_u$ with initial tangent vector L emanating from the point $\gamma_{\underline{L}}(u)$. This provides a further extension of L that we will denote by L', with $L' = \dot{\tilde{\gamma}}_u$ along $\tilde{\gamma}_u$. We can carry out an analogous procedure for affinely parametrized null geodesics emanating from $\gamma_L(v)$ with initial tangent vector $\underline{L}$ to obtain a vector field $\underline{L}'$.

Denote $\Omega^{-2} := -\overline{g}(L', \underline{L})$. Note that $\Omega^2 > 0$, since L' and $\underline{L}'$ cannot be proportional anywhere. In order to define double null coordinates u and v we rescale the vector fields L' and $\underline{L}'$ to obtain the following alternative extensions of L_p and $\underline{L}_p$ to a neighbourhood of p:

$$L = \Omega^2 L' \quad \text{and} \quad \underline{L} = \Omega^2 \underline{L}'.$$

[Exercise: Show that $\overline{g}([L, \underline{L}], L) = \overline{g}([L, \underline{L}], \underline{L}) = 0$ and hence $[L, \underline{L}] = 0$. Hint: Show first that $\overline{g}([L', \underline{L}'], L') = \Omega^{-4} L'(\Omega^2)$.]

We now extend u and v away from $\gamma_{\underline{L}} \cup \gamma_L$ as functions defined on a neighbourhood of p, by requiring $L(u) = 0$ and $\underline{L}(v) = 0$. Since $[L, \underline{L}] = 0$, it follows that (u, v) must be well-defined coordinates in a neighbourhood of p and that $L = \partial_v$ and $\underline{L} = \partial_u$.[17] Furthermore, since $\overline{g}(\underline{L}, L) = -\Omega^2$, we can write

$$\overline{g} = -\Omega^2(u, v)\, du dv.$$

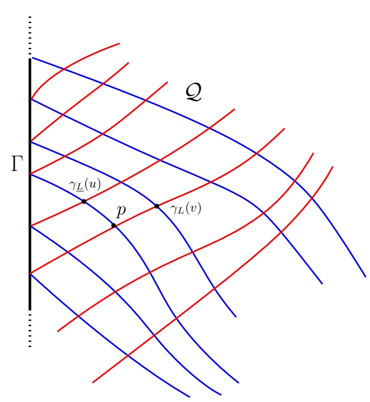


Figure 18: The set $\mathcal{Q}$ (with possible centre Γ) and the global foliation by ingoing and outgoing null geodesic.

In dimensions greater than 2, two ingoing (or two outgoing) null geodesic could in principle intersect. This happens, for example, at the vertex of a lightcone in Minkowski, so the corresponding double null coordinates would be ill-defined past the intersection point. In two dimensions,

[17] On a general n-manifold $\mathcal{M}$, if $X_1, \ldots, X_n$ are n linearly independent smooth vector fields in a neighbourhood of $p \in \mathcal{M}$, such that $[X_i, X_j] = 0$ for all $i, j \in \{1, \ldots, n\}$, then there exists a coordinate chart $(y^1, \ldots, y^n)$ in a neighbourhood of p such that $X_i = \frac{\partial}{\partial y^i}$. The idea of the proof is to use that the local flows corresponding to these vector fields commute.

this cannot happen, since at each point $x \in \mathcal{Q}$ the subspace of $T_x\mathcal{Q}$ spanned by either ingoing or outgoing null geodesics is 1-dimensional.

We provide below a precise argument for concluding that the domain of the coordinates (u, v) constructed above in a neighbourhood of $p \in \mathcal{Q}$ is the full $\mathcal{Q}$. Consider the subset

$$\mathcal{D} = \{q \in \mathcal{Q} \mid \text{there exists a neighbourhood of } q \text{ covered by the } (u, v) \text{ coordinates}\}.$$

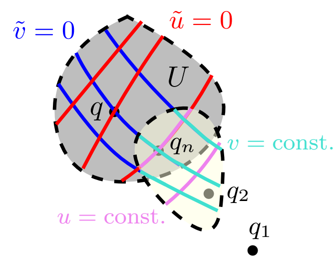


Figure 19: Showing that $\mathcal{D}$ is closed.

- Since $p \in \mathcal{D}$, $\mathcal{D} \neq \emptyset$.
- Furthermore, if $q \in \mathcal{A}$, then we automatically get the existence of an open neighbourhood of q that is also contained in $\mathcal{A}$ by the way we defined $\mathcal{A}$, so $\mathcal{A}$ is open.
- Let $q_n \in \mathcal{A}$, such that $q_n \to q$ (convergence with respect to the $\mathbb{R}^2$ topology). We can repeat the above local construction of double null coordinates around q to obtain a neighbourhood U of q covered by different double null coordinates $(\tilde{u}, \tilde{v})$, such that $q_n \in U$ for n suitably large. Since the tangent spaces $T_x\mathcal{Q}$ are 2-dimensional, the ingoing null geodesics in U with respect to $(\tilde{u}, \tilde{v})$ must agree with the null geodesics in a neighbourhood of q_n associated to (u, v) coordinates, and similarly, the outgoing null geodesics must agree. So $(\tilde{u}, \tilde{v})$ and (u, v) are related by a rescaling in u and v and a shift of the origin $(\tilde{u}, \tilde{v}) = 0$. Therefore the coordinates (u, v) can be extended to also cover q, so $q \in \mathcal{D}$. The set $\mathcal{D}$ is therefore also closed in $\mathcal{Q}$.

Since $\mathcal{D}$ is a non-empty, open and closed subset of the connected set $\mathcal{Q}$, it must be the whole $\mathcal{Q}$. □

The u-level sets and v-level sets in a spherically symmetric spacetime constitute a *global double null foliation* of the spacetime.

Remark 5.1. *The statement $\overline{g} = -\Omega^2 dudv$ implies that the Lorentzian spacetime $(\mathcal{Q}, \overline{g})$ is conformally isometric to the 1+1-dimensional Minkowski metric.*[18] *This is a Lorentzian analogue of the Uniformization Theorem from Riemannian geometry, which can be stated as follows: let $(\mathcal{Q}, \overline{g})$ be an orientable compact 2-dimensional Riemannian manifold. Then $(\mathcal{Q}, \overline{g})$ is conformally isometric to a surface of constant Gauss curvature equation to 1, 0 or -1, i.e. a quotient of the unit round sphere $\mathbb{S}^2$, the Euclidean plane $\mathbb{E}^2$ or the hyperbolic plane $\mathbb{H}^2$.*

[18] A conformal isometry between $(\mathcal{M}, g)$ and $(\mathcal{N}, \tilde{g})$ is a diffeomorphism $\psi : \mathcal{M} \to \mathcal{N}$ such that $\psi^*\tilde{g} = \Omega^2 g$ for some (smooth) function $\Omega^2 : \mathcal{M} \to (0, \infty)$.

5.1 Penrose diagrams

We now use the existence of global double null coordinates on spherically symmetric spacetimes to represent the spacetimes as bounded subsets of $\mathbb{R}^2$ via Penrose diagrams. As we will later see, Penrose diagrams remain an invaluable (schematic) tool even when considering more general spacetimes that are not spherically symmetric. The special property of spherically symmetric spacetimes is that they *always* admits global double null coordinates and it is this property that underlies the definition of Penrose diagrams, as we already saw in the examples of Minkowski, Schwarzschild and Reissner–Nordström.

Definition 5.2. *A* (Carter–)Penrose diagram *of a spherically symmetric spacetime* $(\mathcal{M}, g)$ *is a map* Φ *together with a bounded subset* $\mathcal{PD} \subset \mathbb{R}^2$*, such that*

1.
$$\Phi : \mathcal{Q} \to \mathcal{PD}$$
is a C^2 *diffeomorphism.*

2. *Null curves in* $\mathcal{Q}$ *are mapped to null curves in* $(\mathcal{PD}, -dudv)$ *with* $-dudv$ *the 1+1-dimensional Minkowski metric in standard double null coordinates.*

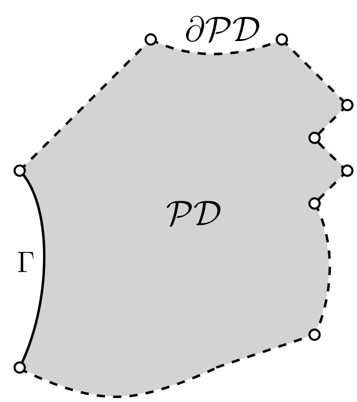


Figure 20: Example of a Penrose diagram of a spherically symmetric spacetime.

Corollary 5.2. *Let* $\phi : \mathcal{Q} \to \phi(\mathcal{Q}) \subset \mathbb{R}^2$ *be a global double null coordinate chart. Define* $(\Phi \circ \phi^{-1})(u, v) = (\arctan u, \arctan v)$*. Then* $(\Phi, \Phi(\mathcal{Q}))$ *is a Penrose diagram of* $(\mathcal{M}, g)$*.*

- We will now fix the time orientation on $(\mathcal{M}, g)$ so that the vector field $\partial_u + \partial_v$ is future-directed!
- We will also refer to ∂_v (and its integral curves) as "outgoing" and ∂_u (and its integral curves) as "ingoing".

A main advantage of Penrose diagrams is that they depict causal curves in $(\mathcal{M}, g)$ by causal curves in $(\mathcal{PD}, -dudv)$.

Proposition 5.3. 1. *Timelike curves in* $(\mathcal{M}, g)$ *are represented by timelike curves in* $(\mathcal{PD}, -dudv)$*.*

2. *Null curves in* $(\mathcal{M}, g)$ *are represented by causal curves in* $(\mathcal{PD}, -dudv)$*. Radial null curves are represented by null curves in* $\mathcal{PD}$*.*

3. Γ *is represented by a timelike curve in* $(\mathbb{R}^2, -dudv)$*.*

4. A null curves in $(\mathcal{M}, g)$ that intersects with Γ at a single point $p \in \Gamma$ is represented by two causal curve segments in $(\mathcal{PD}, -dudv)$, such that one segment is tangent to line of constant v at p and the other segment is tangent to a line of constant u at p (see Figure 21).

Proof. Let γ be a timelike curve. Then we can express in (u, v, θ, φ) coordinates:

$$\gamma(s) \cong \begin{pmatrix} \gamma^u \\ \gamma^v \\ \gamma^\theta \\ \gamma^\varphi \end{pmatrix}(s)$$

and by the timelike property, we have that

$$0 > g(\dot{\gamma}, \dot{\gamma}) = -\Omega^2 \dot{\gamma}^u \dot{\gamma}^v + r^2((\dot{\gamma}^\theta)^2 + \sin^2\theta(\dot{\gamma}^\varphi)^2).$$

Hence, the associated curve in $\mathcal{Q}$:

$$\overline{\gamma}(s) = \begin{pmatrix} \gamma^u \\ \gamma^v \end{pmatrix}(s)$$

must also satisfy $-\Omega^2 \gamma^u \gamma^v < 0$, so $-\gamma^u \gamma^v < 0$, which means that $\overline{\gamma}$ is timelike with respect to $(\mathcal{Q}, -dudv)$ so also with respect to $(\mathcal{PD}, -dudv)$ (after appropriately rescaling u and v).

Similarly, if γ is null, then by the above argument $-\Omega^2 \gamma^u \gamma^v \leq 0$, so we can only say that $\overline{\gamma}$ is causal with respect to $(\mathbb{R}^2, -dudv)$, unless $\dot{\gamma}^\theta = \dot{\gamma}^\varphi = 0$, in which case $\overline{\gamma}$ is null.

[Exercise: Show that Γ is represented by a timelike curve in $(\mathbb{R}^2, -dudv)$. *Hint:* Suppose that $\Gamma \subset \mathcal{PD}$ has a spacelike or null segment and use that for any $p, q \in \Gamma$, $q \in I^+(p)$ or $p \in I^+(q)$ to reach a contradiction.]

Now let $\gamma : I \to \mathcal{M}$, with $0 \ni I \subset \mathbb{R}$ open, be a future-directed null curve in $\mathcal{M}$, such that $p = \gamma(0) \in \Gamma$. Let $\overline{\gamma}$ represent γ for $s < 0$ and $\tilde{\gamma}$ represent γ for $s > 0$.

By the null property of γ, we have that $g_p(\dot{\gamma}(0), \dot{\gamma}(0)) = 0$. Furthermore, without loss of generality, we can rescale our null coordinates, so that at $\Omega^2(u_p, v_p) = 1$, with (u_p, v_p) the limiting value of the (u, v) coordinates at p. Then, using moreover that $r|_\Gamma \equiv 0$,

$$-\dot{\overline{\gamma}}^u(0)\dot{\overline{\gamma}}^v(0) = g_p(\dot{\gamma}(0)\dot{\gamma}(0) = -\dot{\tilde{\gamma}}^u(0)\dot{\tilde{\gamma}}^v(0).$$

Hence, $\dot{\overline{\gamma}}^u(0) = 0$ or $\dot{\overline{\gamma}}^v(0) = 0$, and $\dot{\tilde{\gamma}}^u(0) = 0$ or $\dot{\tilde{\gamma}}^v(0) = 0$.

Since γ is future-directed and by 3. we can assume WLOG that Γ bounds $\mathcal{Q}$ to the left. Suppose that $\dot{\overline{\gamma}}^u(0) = 0$. Then $\dot{\overline{\gamma}}^v(0) > 0$, which means that $0 < v_p - \overline{\gamma}^v(s) = O(s)$ and $u_p - \overline{\gamma}^v(s) = O(s^2)$, which is in contradiction with the fact that Γ is a left-boundary. Hence, $\dot{\tilde{\gamma}}^v(0) = 0$. Similarly, it follows that $\dot{\tilde{\gamma}}^u(0) = 0$. This concludes that $\overline{\gamma}$ must be ingoing null at p and $\tilde{\gamma}$ must be outgoing null at p. □

[Exercise: Show that spacelike curves in $(\mathcal{M}, g)$ need not be represented by spacelike curves in $(\mathcal{PD}, -dudv)$.]

An important role will be played by the boundary $\partial\mathcal{PD} = \overline{\mathcal{PD}} \setminus \mathcal{PD}$ with respect to the ambient space $\mathbb{R}^2$. The boundary $\partial\mathcal{PD}$ includes Γ, but it will also have additional components. Since they are represented by curves in $(\mathbb{R}^2, -dudv)$, we can also investigate the spacelike, timelike or null nature of the boundary segments.

Remark 5.2. *Note that a change in the map $\Phi : \mathcal{Q} \to \mathcal{PD}$ can lead to a change in the shape of timelike and spacelike segments of $\mathcal{PD}$, but the null segments stay the same, up to a rescaling. When we talk about "the" Penrose diagram, we really mean a particular choice of $\mathcal{PD}$, where we do not care about the precise size of $\mathcal{PD}$ and the precise shape of the timelike and spacelike segments of $\partial\mathcal{PD}$.*

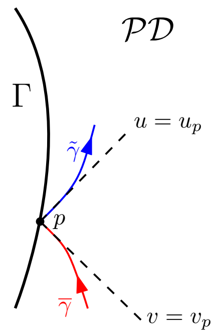


Figure 21: Representation of null curves intersecting Γ at $p \in \Gamma$.

An important type of Penrose diagram boundary segment is "null infinity", $\mathcal{I}$ (also called "scri"), which we already encountered in Minkowski, Schwarzschild and Reissner–Nordström.

Definition 5.3. *Let $p \in \partial\mathcal{PD}$. Choose coordinates (u, v) such that $p = (0, 0)$. We say $p \in \mathcal{I}$ if the following holds: let $(0, v)$ and $(0, u)$ denote points in $\mathcal{Q}$, then*

$$\limsup_{v\to 0} r(0,v) := \lim_{V\to 0}\left[\sup_{|v|\leq V} r(0,v)\right] = \infty \quad \text{or} \quad \limsup_{u\to 0} r(u,0) := \lim_{U\to 0}\left[\sup_{|u|\leq U} r(u,0)\right] = \infty.$$

We refer to the boundary subset $\mathcal{I} \subseteq \partial\mathcal{PD}$ as null infinity*. Suppose that $\mathcal{I}$ is achronal. Then we say:*

- *$p \in \mathcal{I}^+ \subseteq \mathcal{I}$ if*
$$\limsup_{v\uparrow 0} r(0,v) = \infty \quad \text{or} \quad \limsup_{u\uparrow 0} r(u,0) = 0.$$
We refer to the subset $\mathcal{I}^+$ as future null infinity*.*

- *$p \in \mathcal{I}^- \subseteq \mathcal{I}$ if*
$$\limsup_{v\downarrow 0} r(0,v) = \infty \quad \text{or} \quad \limsup_{u\downarrow 0} r(u,0) = 0.$$
We refer to the subset $\mathcal{I}^- \subseteq \mathcal{I}$ as past null infinity*.*

Penrose diagrams are very convenient tools for representing causal properties of hypersurfaces and open subsets of the manifold. Let $S \subset \mathcal{Q} \cup \Gamma$. Then S represents the set $\mathcal{M} \supset \Sigma \cong (S \cap \mathcal{Q}) \times \mathbb{S}^2 \cup (\Gamma \cap S)$. If S is a finite union of smooth curves (with boundaries), then Σ is a smooth hypersurface(-with-boundary) of $\mathcal{M}$.

For such sets we can characterize the causal/chronological future/past and the future/past domain of dependence on the Penrose diagram $\mathcal{PD}$. Indeed,

- $J^\pm(\Sigma)$ is represented by $J^\pm(S)$ with respect to $(\mathcal{PD} \cup \Gamma, -dudv)$.
- $I^\pm(\Sigma)$ is represented by $I^\pm(S)$ with respect to $(\mathcal{PD} \cup \Gamma, -dudv)$.

Furthermore, since radial null geodesics in $(\mathcal{M}, g)$ are represented by lines in $\mathcal{Q}$ that are reflected at Γ, we have to be a little more careful when discussing the domain of dependence. We have that:

- If $S \cap \Gamma = \emptyset$, then $D^\pm(\Sigma)$ is simply represented by $D^\pm(S)$ with respect to $(\mathcal{PD}, -dudv)$.
- If $S \cap \Gamma \neq \emptyset$, then $D^\pm(\Sigma)$ is represented by $D^\pm(S \cup \Gamma)$ with respect to $(\mathcal{PD} \cup \Gamma, -dudv)$.

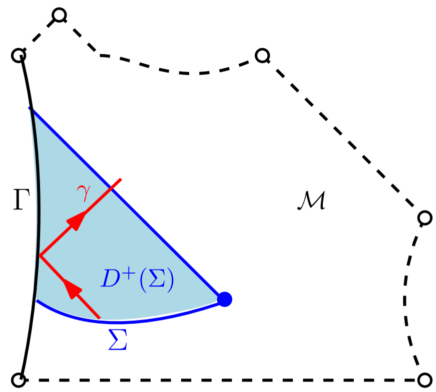


Figure 22: The representation of the domain of dependence of a hypersurface-with-boundary Σ in a spherically symmetric spacetime with $\Gamma \neq \emptyset$. The line γ represents a radial null curve passing through the centre of spherical symmetry Γ in $\mathcal{M}$.

In slight abuse of notation, we will use the notations $J^{\pm}(\Sigma)$ and $J^{\pm}(S)$, or $D^{\pm}(\Sigma)$ and $D^{\pm}(S)$ interchangeably.

The Penrose diagram allows us to consider also the sets $J^{\pm}(S)$, when $S \cap (\partial\mathcal{PD} \setminus \Gamma) \neq \emptyset$. Here, we interpret $J^{\pm}(S)$ as a subset of $\mathbb{R}^2$. For example, we can consider $J^-(\mathcal{I}^+)$.

We can now define what we mean by a "black hole" region in spherical symmetry.

Definition. *Let $(\mathcal{M}, g)$ be a spherically symmetric spacetime with $\mathcal{I}^+ \neq \emptyset$. Then we define the* black hole region *of $(\mathcal{M}, g)$ to be the subset of $\mathcal{M}$ represented by:*

$$\mathcal{BH} := \overline{\mathcal{PD}} \setminus J^-(\mathcal{I}^+).$$

5.2 Einstein equations in spherical symmetry

Consider the Christoffel symbols:

$$\Gamma^{\rho}_{\mu\nu}[g] = \frac{1}{2}(g^{-1})^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right).$$

Denote by:

- $\overline{\Gamma}^c_{ab}$, $a, b, c \in \{0, 1\}$ the Christoffel symbols of $(\mathcal{Q}, \overline{g})$,
- $\not{\Gamma}^C_{AB}$, $A, B \in \{2, 3\}$, the Christoffel symbols of $(\mathbb{S}^2, \mathring{\not{g}})$.

We will use the small Latin indices $a, b, c, \ldots$ to denote elements in $\{0, 1\}$ and the capital Latin indices $A, B, \ldots$ to denote components in $\{2, 3\}$. In view of the fact that $g = \overline{g} + r^2\mathring{\not{g}}$, the remaining Christoffel symbols take the following form:

$$\begin{aligned}
\Gamma^a_{bC} &= 0,\\
\Gamma^a_{BC} &= -\frac{1}{2}(g^{-1})^{ab}(\partial_b r^2)\mathring{\not{g}}_{BC},\\
\Gamma^A_{bc} &= 0,\\
\Gamma^A_{Bc} &= \frac{1}{2}r^{-2}\partial_c(r^2)\delta^A_B = (\partial_c \log r)\delta^A_B.
\end{aligned}$$

We denote with $\overline{R}_{abcd}$ the Riemann tensor associated to $\overline{g}$. Since $\mathcal{Q}$ is 2-dimensional, the symmetries of the Riemann tensor imply that the space of all Riemann tensors on $\mathcal{Q}$ is 1-dimensional (see Problem Sheet 3), and we can therefore write:

$$\overline{R}_{abcd} = K(\overline{g}_{ac}\overline{g}_{bd} - \overline{g}_{ad}\overline{g}_{bc}),$$

with K the Gaussian curvature of $(\mathcal{Q}, \overline{g})$. Similarly, let $\not{R}_{ABCD}$ be the Riemann tensor associated to $\mathring{\not{g}}$. Then:

$$\not{R}_{ABCD} = \mathring{\not{g}}_{AC}\mathring{\not{g}}_{BD} - \mathring{\not{g}}_{AD}\mathring{\not{g}}_{BC},$$

since the Gaussian curvature of the unit round sphere is equal to 1.

The corresponding Ricci tensors take the following form:

$$\begin{aligned}
\overline{R}_{ac} &= (\overline{g}^{-1})^{bd}\overline{R}_{abcd} = K\overline{g}_{ac},\\
\not{R}_{AC} &= (\mathring{\not{g}}^{-1})^{BD}\not{R}_{ABCD} = \mathring{\not{g}}_{AC}.
\end{aligned}$$

Now we will relate the Ricci tensor components R_{ab} to $\overline{R}_{ab}$. Recall first that

$$R_{\mu\nu\rho}{}^{\sigma} = \partial_\nu \Gamma^\sigma_{\rho\mu} - \partial_\rho \Gamma^\sigma_{\nu\mu} + \Gamma^\alpha_{\rho\mu}\Gamma^\sigma_{\nu\alpha} - \Gamma^\alpha_{\nu\mu}\Gamma^\sigma_{\rho\alpha}.$$

Hence,

$$\begin{aligned}
R_{ac} = R_{a\nu c}{}^{\nu} &= \partial_\nu \Gamma^\nu_{ca} - \partial_c \Gamma^\nu_{\nu a} + \Gamma^\alpha_{ca}\Gamma^\nu_{\nu\alpha} - \Gamma^\alpha_{\nu a}\Gamma^\nu_{c\alpha}\\
&= \overline{R}_{ac} + \partial_A \Gamma^A_{ac} - \partial_c \Gamma^A_{Aa} + \Gamma^b_{ca}\Gamma^A_{Ab} - \Gamma^b_{Aa}\Gamma^A_{cb} + \Gamma^B_{ca}\Gamma^A_{AB} - \Gamma^A_{Ba}\Gamma^B_{cA} + \Gamma^A_{ca}\Gamma^b_{bA} - \Gamma^A_{ba}\Gamma^b_{cA}\\
&= K\overline{g}_{ac} - \partial_c \Gamma^A_{Aa} + \Gamma^b_{ca}\Gamma^A_{Ab} - \Gamma^A_{Ba}\Gamma^B_{cA}\\
&= K\overline{g}_{ac} - 2(\partial_c\partial_a \log r) + 2\Gamma^b_{ca}(\partial_b \log r) - 2(\partial_a \log r)(\partial_c \log r)\\
&= K\overline{g}_{ac} - 2r^{-1}\overline{\nabla}_c\overline{\nabla}_a r,
\end{aligned}$$

with $\overline{\nabla}$ the Levi–Civita connection with respect to $\overline{g}$.

Similarly,

$$\begin{aligned}
R_{AC} = R_{A\nu C}{}^{\nu} &= \partial_\nu \Gamma^\nu_{CA} - \partial_C \Gamma^\nu_{\nu A} + \Gamma^\alpha_{CA}\Gamma^\nu_{\nu\alpha} - \Gamma^\alpha_{\nu A}\Gamma^\nu_{C\alpha}\\
&= \not{R}_{AC} + \partial_a \Gamma^a_{AC} - \partial_C \Gamma^a_{aA} + \Gamma^B_{CA}\Gamma^a_{aB} - \Gamma^a_{BA}\Gamma^B_{Ca} + \Gamma^b_{CA}\Gamma^a_{ab} - \Gamma^a_{bA}\Gamma^b_{Ca} + \Gamma^a_{CA}\Gamma^B_{Ba} - \Gamma^a_{BA}\Gamma^B_{Ca}\\
&= \mathring{\not{g}}_{AC} + \partial_a \Gamma^a_{AC} - \Gamma^a_{BA}\Gamma^B_{Ca} + \Gamma^b_{CA}\Gamma^a_{ab} + \Gamma^a_{CA}\Gamma^B_{Ba} - \Gamma^a_{BA}\Gamma^B_{Ca}\\
&= \mathring{\not{g}}_{AC} - \partial_a((g^{-1})^{ab} r\partial_b r)\mathring{\not{g}}_{AC} + (g^{-1})^{ab}(\partial_b r)(\partial_a r)\mathring{\not{g}}_{AC} + \Gamma^a_{ab}(g^{-1})^{ab} r(\partial_b r)\mathring{\not{g}}_{AC}\\
&\quad - 2(g^{-1})^{ab}(\partial_b r)(\partial_a r)\mathring{\not{g}}_{AC} + (g^{-1})^{ab}(\partial_b r)(\partial_a r)\mathring{\not{g}}_{AC}\\
&= (1 - \partial_a((g^{-1})^{ab} r(\partial_b r) + \Gamma^a_{ab}(g^{-1})^{ab} r(\partial_b r))\mathring{\not{g}}_{AC}\\
&= \left(1 - (\overline{g}^{-1})^{ab}\overline{\nabla}_a(r\overline{\nabla}_b r)\right)\mathring{\not{g}}_{AC}.
\end{aligned}$$

[Exercise: Show that $R_{aA} = 0$.]

From the Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 8\pi \mathbb{T}_{\mu\nu},$$

it follows that $-R = 8\pi \operatorname{tr}\mathbb{T}$ and hence

$$R_{\mu\nu} = 8\pi\left(\mathbb{T}_{\mu\nu} - \frac{1}{2}\operatorname{tr}\mathbb{T} g_{\mu\nu}\right).$$

Note also that $\operatorname{tr}\mathbb{T} = (\overline{g}^{-1})^{ab}\mathbb{T}_{ab} + r^{-2}(\mathring{\gamma}^{-1})^{AB}\mathbb{T}_{AB}$. It therefore follows immediately that

$$8\pi\mathbb{T}_{Ab} = R_{Ab} - \frac{1}{2}Rg_{Ab} = 0, \tag{5.2}$$

$$\left(1 - (\overline{g}^{-1})^{ab}\overline{\nabla}_a(r\overline{\nabla}_b r)\right)\mathring{\gamma}_{AB} = R_{AB} = 8\pi(\mathbb{T}_{AB} - \frac{1}{2}\operatorname{tr}\mathbb{T}r^2\mathring{\gamma}_{AB}) =: r^2 S\mathring{\gamma}_{AB}, \tag{5.3}$$

$$K\overline{g}_{ab} - 2r^{-1}\overline{\nabla}_a\overline{\nabla}_b r = 8\pi(\mathbb{T}_{ab} - \frac{1}{2}\operatorname{tr}\mathbb{T}\overline{g}_{ab}) \tag{5.4}$$

Equation (5.3) is equivalent to:

$$r^{-1}\Box_{\overline{g}}r = r^{-2} - S - (\overline{g}^{-1})^{ab}r^{-2}\partial_a r\partial_b r$$

and, taking the trace of the right-hand side of (5.3) with respect to $\mathring{\gamma}$, we obtain:

$$S = -4\pi(\overline{g}^{-1})^{ab}\mathbb{T}_{ab}.$$

Now we can take the trace of (5.4) with respect to $\overline{g}$ to obtain:

$$K = r^{-1}\Box_{\overline{g}}r - S - 4\pi\operatorname{tr}\mathbb{T} = r^{-2} - r^{-2}(\overline{g}^{-1})^{ab}\partial_a r\partial_b r + 8\pi(\overline{g}^{-1})^{ab}\mathbb{T}_{ab} - 4\pi\operatorname{tr}\mathbb{T}. \tag{5.5}$$

Then we can rearrange (5.4) to obtain:

$$r^{-1}\overline{\nabla}_a\overline{\nabla}_b r = \frac{1}{2r^2}\left[1 - (\overline{g}^{-1})^{cd}\partial_c r\partial_d r\right]\overline{g}_{ab} - 4\pi(\mathbb{T}_{ab} - (\overline{g}^{-1})^{cd}\mathbb{T}_{cd}\overline{g}_{ab}). \tag{5.6}$$

Now consider double null coordinates (u,v), so that $\overline{g}_{uv} = -\frac{1}{2}\Omega^2$ and $\overline{g}_{uu} = \overline{g}_{vv} = 0$. Furthermore, $(\overline{g}^{-1})^{uv} = -2\Omega^{-2}$ and $(\overline{g}^{-1})^{uu} = (\overline{g}^{-1})^{vv} = 0$. Then

$$\begin{aligned}
\Gamma^u_{uu} &= \frac{1}{2}(\overline{g}^{-1})^{uv}(\partial_u\overline{g}_{uv} + \partial_u\overline{g}_{uv} - \partial_v\overline{g}_{uu}) = \Omega^{-2}\partial_u\Omega^2,\\
\Gamma^v_{vv} &= \Omega^{-2}\partial_v\Omega^2,\\
\Gamma^u_{vv} &= \Gamma^u_{uv} = \Gamma^v_{uu} = \Gamma^v_{uv} = 0.
\end{aligned}$$

Hence,

$$-\frac{1}{2}K\Omega^2 = \overline{R}_{uv} = \partial_b\Gamma^b_{vu} - \partial_v\Gamma^b_{bu} + \Gamma^a_{vu}\Gamma^b_{ba} - \Gamma^a_{bu}\Gamma^b_{va} = -\partial_v(\Omega^{-2}\partial_u\Omega^2) = -\partial_v\partial_u\log\Omega^2.$$

From (5.5), it therefore follows that

$$\partial_v\partial_u\log\Omega^2 = \frac{\Omega^2}{2r^2}\left[1 + 4\Omega^{-2}(\partial_u r)(\partial_v r) - 4\pi r^2\operatorname{tr}\mathbb{T} - 32\pi r^2\Omega^{-2}\mathbb{T}_{uv}\right]. \tag{5.7}$$

From (5.6) we conclude the following equations:

$$\partial_u\partial_v r = -\frac{\Omega^2}{4r}\left[1 + 4\Omega^{-2}(\partial_u r)(\partial_v r)\right] + 4\pi r\mathbb{T}_{uv}, \tag{5.8}$$

$$\Omega^2\partial_u(\Omega^{-2}\partial_u r) = -4\pi r\mathbb{T}_{uu}, \tag{5.9}$$

$$\Omega^2\partial_v(\Omega^{-2}\partial_v r) = -4\pi r\mathbb{T}_{vv}. \tag{5.10}$$

We refer to (5.9) and (5.10) as the *Raychaudhuri equations.*
We define the *Hawking mass* $m : \mathcal{Q} \to \mathbb{R}$ as follows:

$$m = \frac{r}{2}(1 + 4\Omega^{-2}(\partial_u r)(\partial_v r)) = \frac{r}{2}(1 - (\overline{g}^{-1})^{ab}(\partial_a r)(\partial_b r)) = \frac{r}{2}(1 - \overline{g}^{-1}(dr, dr)).$$

We will later see, when discussing self-gravitating fluids, why the quantity m is compatible with a notion of mass typically attributed to matter.

Note that in the Schwarzschild case $g = g_M$, $m = M$ is constant. In general,

$$\partial_u m = -8\pi r^2 \Omega^{-2} (\partial_v r \mathbb{T}_{uu} - \partial_u r \mathbb{T}_{uv}), \tag{5.11}$$
$$\partial_v m = 8\pi r^2 \Omega^{-2} (-\partial_u r \mathbb{T}_{vv} + \partial_v r \mathbb{T}_{uv}). \tag{5.12}$$

[EXERCISE: Derive (5.11) and (5.12).]

Note that these equations imply immediately that in the case of vacuum ($\mathbb{T} \equiv 0$), the Hawking mass is constant on $\mathcal{Q}$. Note also that the metric tensor g can be reconstructed from the pairs (r, Ω^2) or (r, m). The Einstein equations are equivalent to (5.7)–(5.10).

Lemma 5.4. *If the stress-energy tensor $\mathbb{T}$ satisfies the null energy condition, then*

$$\mathbb{T}_{uu}, \mathbb{T}_{vv} \geq 0.$$

If the stress-energy tensor $\mathbb{T}$ satisfies the dominant energy condition, then additionally:

$$\mathbb{T}_{uv} \geq 0.$$

Proof. EXERCISE □

[EXERCISE: Show that the Schwarzschild family of metrics are solutions to the vacuum Einstein equations. Hint: Show that g_M corresponds to a solution to (5.7)–(5.10).]

5.3 The Einstein-scalar field system

In the case of the Einstein-scalar field system with $\mathfrak{m} = 0$, we recall that

$$\mathbb{T}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi$$

We can express the wave equation $\Box_g \phi = 0$ for a spherically symmetric ϕ as follows:

$$0 = \Box_g \phi = \frac{1}{\sqrt{-\det g}} \partial_\alpha (\sqrt{-\det g} (g^{-1})^{\alpha\beta} \partial_\beta \phi) = -2r^{-2} \Omega^{-2} [\partial_u (r^2 \partial_v \phi) + \partial_v (r^2 \partial_u \phi)].$$

Hence,

$$\partial_u \partial_v \phi = -r^{-1} \partial_u r \partial_v \phi - r^{-1} \partial_v r \partial_u \phi.$$

Furthermore, $\mathbb{T}_{uu} = (\partial_u \phi)^2$, $\mathbb{T}_{vv} = (\partial_v \phi)^2$, $\mathbb{T}_{uv} = 0$ and

$$\operatorname{tr} \mathbb{T} = 2 (g^{-1})^{uv} \mathbb{T}_{uv} + r^{-2} (\mathring{g}^{-1})^{AB} \mathbb{T}_{AB} = -(g^{-1})^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi = 4\Omega^{-2} \partial_u \phi \partial_v \phi.$$

The Einstein–scalar field system of equations is therefore equivalent to the following coupled system of equations for $r, \Omega^2 : \mathcal{Q} \to \mathbb{R}_+$ and $\phi : \mathcal{Q} \to \mathbb{R}$:

$$\partial_v \partial_u \log \Omega^2 = \frac{1}{2} \Omega^2 r^{-2} + 2r^{-2} (\partial_u r)(\partial_v r) - 8\pi \partial_u \phi \partial_v \phi, \tag{5.13}$$
$$\partial_u \partial_v r = -\frac{\Omega^2}{4r} \left[1 + 4\Omega^{-2} (\partial_u r)(\partial_v r) \right], \tag{5.14}$$
$$\Omega^2 \partial_u (\Omega^{-2} \partial_u r) = -4\pi r (\partial_u \phi)^2, \tag{5.15}$$
$$\Omega^2 \partial_v (\Omega^{-2} \partial_v r) = -4\pi r (\partial_v \phi)^2, \tag{5.16}$$
$$\partial_u \partial_v \phi = -r^{-1} \partial_u r \partial_v \phi - r^{-1} \partial_v r \partial_u \phi. \tag{5.17}$$

Remark 5.3. *The equations* (5.13), (5.14) *and* (5.17) *are called* propagation equations *and they form a system of* 1+1-dimensional semilinear[19] wave equations*. Note that the products of two derivative terms never feature two u or two v derivatives. This is called the* null structure of the Einstein-scalar field equations *and it is a very important structural property that also holds outside of spherical symmetry and is important for addressing global dynamics of small perturbations of spacetimes as well as the nature of singularities.*

This kind of structure does not apply, for example, to the relativistic (or non-relativistic and compressible) Euler fluid equations, where "singular" behaviour, in the form of shocks, can occur even if one considers initial data corresponding to a small perturbation of the constant state.

Remark 5.4. *The equations* (5.15) *and* (5.16) *are* constraint equations.

We moreover have that

$$\partial_u m = -8\pi r^2 \Omega^{-2} \partial_v r (\partial_u \phi)^2, \tag{5.18}$$

$$\partial_v m = 8\pi r^2 \Omega^{-2} (-\partial_u r)(\partial_v \phi)^2. \tag{5.19}$$

5.4 Local existence and uniqueness of the characteristic initial value problem for the Einstein-scalar field system

We will now show that we can sensibly study *dynamics* for the spherically symmetric Einstein–scalar field problem. First we will state a local existence and uniqueness statement for general systems of 1+1-dimensional wave equations.

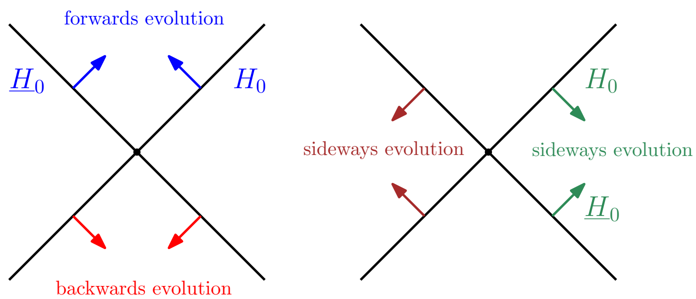


Figure 23: The evolution problem for systems of 1+1-dimensional wave equations spherical symmetry.

Theorem 5.5 (Local existence and uniqueness characteristic initial value problem for 1+1-dimensional wave equations (with null structure))**.** *Consider the following system of semilinear 1+1-dimensional wave equations: let $K \in \mathbb{N}$ and $\Psi : \mathbb{R}^2 \supseteq U \to \mathbb{R}^K$, with $0 \in U$ and*

$$\partial_u \partial_v \Psi^A = \sum_{B,C=1}^{K} N_{BC}(\Psi) \partial_u \Psi^B \partial_v \Psi^C + \sum_{B=1}^{K} \left[L_B(\Psi) \partial_u \Psi^B + R_B(\Psi) \partial_v \Psi^B \right] + f^A(\Psi), \tag{5.20}$$

with $N_{BC}, L_B, R_B, f^A : \mathbb{R}^K \to \mathbb{R}$ smooth functions.

Consider the union of line segments H_0 and $\underline{H}_0$ in $\mathbb{R}^2$, with:

$$H_0 := \{0\}_u \times [0, \epsilon)_v, \quad \underline{H}_0 := [0, \epsilon)_u \times \{0\}_v$$

[19] Semilinear PDE are nonlinear PDE that are linear in the highest derivative terms (second-order in this case).

and prescribe the following characteristic initial data*:*

$$\begin{aligned} \Psi_0 \in \mathbb{R}^K & \quad \text{on } H_0 \cap \underline{H}_0, \\ \underline{\Psi}'_0 : \underline{H}_0 \to \mathbb{R}^K & \quad \text{on } \underline{H}_0, \\ \Psi'_0 : H_0 \to \mathbb{R}^K & \quad \text{on } H_0. \end{aligned}$$

(i) *(Forwards and backwards evolution) Then, for $\epsilon > 0$ suitably small depending on $\Psi_0, \underline{\Psi}'_0, \Psi'_0$, there exists a unique smooth solution $\Psi : [0,\epsilon) \times [0,\epsilon) \to \mathbb{R}^K$ to (5.20), such that*

$$\begin{aligned} \Psi|_{H_0 \cap \underline{H}_0} &= \Psi_0, \\ \partial_u \Psi|_{\underline{H}_0} &= \underline{\Psi}'_0, \\ \partial_v \Psi|_{H_0} &= \Psi'_0. \end{aligned}$$

The same statement holds with ϵ replaced by $-\epsilon$ in the definitions of H_0 and $\underline{H}_0$.

(ii) *(Sideways evolution) The above statement also holds for $\Psi : (-\epsilon, 0] \times [0,\epsilon) \to \mathbb{R}^K$ with the following choices for H_0 and $\underline{H}_0$:*

$$H_0 := \{0\}_u \times [0,\epsilon)_v, \quad \underline{H}_0 := (-\epsilon, 0]_u \times \{0\}_v.$$

and also with ϵ replace by $-\epsilon$.

We can also compare two global solutions in the following way:

Theorem 5.6. (i) *(Cauchy stability) Let $U_0, V_0 \in (0,\infty)$ and let $\Psi : [0,U_0) \times [0,V_0) \to \mathbb{R}^K$ be a smooth solution to (5.20). Then, for any $\epsilon > 0$ and $N > 0$, there exists a $\delta > 0$ such that for any initial data*

$$\begin{aligned} \widetilde{\Psi}_0 \in \mathbb{R}^K & \quad \text{on } H_0 \cap \underline{H}_0, \\ \widetilde{\underline{\Psi}}'_0 : \underline{H}_0 \to \mathbb{R}^K & \quad \text{on } \underline{H}_0, \\ \widetilde{\Psi}'_0 : H_0 \to \mathbb{R}^K & \quad \text{on } H_0. \end{aligned}$$

with $|\widetilde{\Psi}_0 - \Psi|_{H_0 \cap \underline{H}_0}| < \delta$ and for all $k \leq N$, $\sup_u |\partial_u^k(\widetilde{\underline{\Psi}}'_0 - \partial_u \Psi|_{\underline{H}_0})| < \delta$, $\sup_v |\partial_v^k(\widetilde{\Psi}'_0 - \partial_v \Psi|_{H_0})| < \delta$, we have that the corresponding solution $\widetilde{\Psi} : [0,U_0) \times [0,V_0) \to \mathbb{R}^K$ is C^N and

$$\sup_{u,v} |\partial_u^{k_1} \partial_v^{k_2} (\widetilde{\Psi} - \Psi)| < \epsilon.$$

(ii) *(Global uniqueness) Solutions corresponding to the same initial data agree.*

Remark 5.5. *In higher dimensions, systems of nonlinear wave equations do not satisfy local well-posedness for "sideways evolution", only for "forwards" and "backwards" evolution!*

We can apply the above lemma to obtain local well-posedness (existence, uniqueness, Cauchy stability) for the spherically symmetric Einstein–scalar field equations.

Theorem 5.7 (Local well-posedness of the characteristic initial value problem for the spherically symmetric Einstein–scalar field system)**.** *Let $\epsilon > 0$. Consider $H_0 \cup \underline{H}_0 \subset \mathbb{R}^2$, with*

$$H_0 := \{0\}_u \times [0,\epsilon)_v, \quad \underline{H}_0 := [0,\epsilon)_u \times \{0\}_v.$$

Prescribe the following numbers and smooth functions:

$$(r_0, (r^{-1}\Omega^2)_0, \phi_0, \underline{r}'_0, r'_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 \quad \text{on } H_0 \cap \underline{H}_0,$$

$$\begin{aligned}
\underline{\phi}_0' : \underline{H}_0 \to \mathbb{R} \quad &\text{on } \underline{H}_0,\\
\phi_0' : H_0 \to \mathbb{R} \quad &\text{on } H_0,\\
(r^{-1}\underline{\Omega}^2)_0' : \underline{H}_0 \to \mathbb{R} \quad &\text{on } \underline{H}_0,\\
(r^{-1}\Omega^2)_0' : H_0 \to \mathbb{R} \quad &\text{on } H_0.
\end{aligned}$$

Then, for $\epsilon > 0$ suitably small, there exists a unique solution $(r, \Omega^2, \phi) : [0, \epsilon) \times [0, \epsilon) \to \mathbb{R}^2_+ \times \mathbb{R}$ to (5.13)–(5.17), such that

$$\begin{aligned}
(r, r^{-1}\Omega^2, \phi, \partial_u r, \partial_v r)(0,0) &= (r_0, (r^{-1}\Omega^2)_0, \phi_0, \underline{r}_0', r_0'),\\
\partial_u \phi|_{\underline{H}_0} &= \underline{\phi}_0',\\
\partial_v \phi|_{H_0} &= \phi_0',\\
\partial_u (r^{-1}\Omega^2)|_{\underline{H}_0} &= (r^{-1}\underline{\Omega}^2)_0',\\
\partial_u (r^{-1}\Omega^2)|_{H_0} &= (r^{-1}\Omega^2)_0'.
\end{aligned}$$

The above statement also holds with any of the following alternative choices for H_0 and $\underline{H}_0$ (with appropriately modified domain of (r, Ω^2, ϕ)):

$$H_0 := \{0\}_u \times (-\epsilon, 0]_v, \quad \underline{H}_0 := (-\epsilon, 0]_u \times \{0\}_v.$$

Proof. **Strategy**: We will show that the prescribed initial data determines the values of r, Ω^2, ϕ on $H_0 \cup \underline{H}_0$ via the Raychaudhuri equations. The statement of the theorem then follows from a direct application of (5.20) together with a verification that the Raychaudhuri equations remain valid away from the initial null lines.

Note that if $\phi(0,0) = \phi_0$, then we need

$$\begin{aligned}
\phi(0,v) &= \phi_0 + \int_0^v \phi_0'(v')\,dv',\\
\phi(u,0) &= \phi_0 + \int_0^u \underline{\phi}_0'(u')\,du'.
\end{aligned}$$

Similarly, if $r^{-1}\Omega^2(0,0) = (r^{-1}\Omega^2)_0$, then we need

$$\begin{aligned}
r^{-1}\Omega^2(0,v) &= (r^{-1}\Omega^2)_0 + \int_0^v (r^{-1}\Omega^2)_0'(v')\,dv',\\
r^{-1}\Omega^2(u,0) &= (r^{-1}\Omega^2)_0 + \int_0^u (r^{-1}\underline{\Omega}^2)_0'(u')\,du'.
\end{aligned}$$

Note that for $\epsilon > 0$ suitably small, depending on the choice $(r^{-1}\Omega^2)_0'$ and $(r^{-1}\underline{\Omega}^2)_0'$, we have that $r^{-1}\Omega^2(0,v) > 0$ and $r^{-1}\Omega^2(u,0) > 0$.

Let $r(0,0) = r_0$, $\partial_u r(0,0) = \underline{r}_0'$ and $\partial_v r(0,0) = r_0'$. We can rewrite (5.15) and (5.16) as follows:

$$\begin{aligned}
\partial_u\left(\frac{r}{\Omega^2} r^{-1}\partial_u r\right) &= -4\pi \frac{r}{\Omega^2}(\partial_u\phi)^2,\\
\partial_v\left(\frac{r}{\Omega^2} r^{-1}\partial_v r\right) &= -4\pi \frac{r}{\Omega^2}(\partial_v\phi)^2,
\end{aligned}$$

The left-hand sides we can further rewrite as $\partial_u(\frac{r}{\Omega^2}\partial_u \log r)$ and $\partial_v(\frac{r}{\Omega^2}\partial_v \log r)$.

Then integrating (5.15) and (5.16) gives for $\epsilon > 0$ suitably small:

$$\frac{r}{\Omega^2}(u,0)\partial_u \log r(u,0) = \frac{1}{(r^{-1}\Omega^2)_0} r_0^{-1}\underline{r}_0' - 4\pi \int_0^u \frac{1}{r^{-1}\Omega^2(u',0)}(\underline{\phi}_0')^2(u')\,du',$$

$$\frac{r}{\Omega^2}(0,v)\partial_v \log r(0,v) = \frac{1}{(r^{-1}\Omega^2)_0} r_0^{-1} r_0' - 4\pi \int_0^v \frac{1}{r^{-1}\Omega^2(0,v')} (\phi_0')^2(v')\, dv'$$

and determines $\partial_u \log r(u,0)$ and $\partial_v \log r(0,v)$. Integrating these then gives:

$$\log r(u,0)(0,v) = \log r_0 + \int_0^v \partial_v \log r(0,v')\, dv',$$
$$\log r(u,0)(u,0) = \log r_0 + \int_0^u \partial_u \log r(u',0)\, du'.$$

We take $r(u,0)$ and $r(0,v)$ to be the corresponding exponentials.

[EXERCISE: Show that, the solution (r, Ω^2, ϕ) to (5.13), (5.14) and (5.17) corresponding to the prescribed initial data satisfies also (5.15) and (5.16).] □

Corollary 5.8. *Two solutions $(r_1, \Omega_1^2, \phi_1)$ and $(r_2, \Omega_2^2, \phi_2)$ to (5.13)–(5.17) corresponding to the same initial data on H_0 and $\underline{H}_0$ must agree on the intersection of their domains.*

[EXERCISE: Show that any spherically symmetric spacetime obtained from solutions (r, Ω^2) to the characteristic initial value problem for (5.13)–(5.17) can also be constructed with the following restricted initial data: $(r^{-1}\Omega^2)_0 = 1$, $(r^{-1}\underline{\Omega}^2)_0' \equiv 0$ and $(r^{-1}\Omega^2)_0' \equiv 0$. *Hint*: Given a spherically symmetric spacetime with metric $g = -\Omega^2 dudv + r^2 \mathring{\slashed{g}}$, what happens to (r, Ω^2, ϕ) along $H_0 \cup \underline{H}_0$ under a rescaling of the u and v coordinates?]

5.5 Birkhoff's theorem

Using the uniqueness property from Corollary 5.8, we can show that all spherically symmetric spacetimes <u>in vacuum</u> must be isometric to a subset of a Schwarzschild or Minkowski spacetime. This result is called Birkhoff's theorem and it goes back to JEBSEN 1921, BIRKHOFF 1923.

Theorem 5.9 (Birkhoff's theorem)**.** *A spherically symmetric spacetime solution $(\mathcal{M}, g)$ to $\mathrm{Ric}(g) = 0$ is isometric to a spacetime region inside a (maximally-extended) Schwarzschild spacetime with $M \in \mathbb{R}$ or to a spacetime region inside the Minkowski spacetime.*

Remark 5.6. *Note that the region $\{r > 0\}$ of the Minkowski spacetime is isometric to the Schwarzschild spacetimes with $M = 0$.*

Proof. **Strategy**: Let $p \in \mathcal{Q}$ and let H_0 and $\underline{H}_0$ be null lines in $\mathcal{Q}$ of constant u and v, respectively, that intersect at p. For solutions to the system (5.13)–(5.17) with $\phi \equiv 0$, we can treat the values $r(p), \Omega^2(p), \partial_u r(p), \partial_v r(p)$ and $\partial_v \Omega^2|_{H_0}$, $\partial_u \Omega^2|_{\underline{H}_0}$ as characteristic initial data. We want to show that there exist double null coordinates in Schwarzschild with mass $M \in \mathbb{R}$ or Minkowski, such that $H_0 \cup \underline{H}_0$ are contained in Schwarzschild/Minkowski, such that $r(p), \Omega^2(p), \partial_u r(p), \partial_v r(p), \partial_v \Omega^2|_{H_0}$ and $\partial_u \Omega^2|_{\underline{H}_0}$ agree with the characteristic initial data in our general spacetime.

Then, by the global uniqueness part of Corollary 5.8, (r, Ω^2) must take on the corresponding Schwarzschild or Minkowski values everywhere in the region obtained by evolving the data on $H_0 \cup \underline{H}_0$ forwards, backwards and sideways. Since the full manifold $\mathcal{M}$ can be covered by regions arising from the evolution of characteristic data, the spacetime $(\mathcal{M}, g)$ must be isometric to a region in Schwarzschild or Minkowski.

Without loss of generality, we can take $p = (0,0)$ and we consider H_0 and $\underline{H}_0$ in a general solution to (5.13)–(5.17) with $\phi \equiv 0$, as in Theorem 5.7. We will fix $r_0 = r(0,0)$ and $M = m(0,0)$ and rescale our u and v coordinates $u \mapsto f(u)$, $v \mapsto h(v)$, so that $\Omega^2|_{H_0} \equiv 1$ and $\Omega^2_{\underline{H}_0} \equiv 1$, for the sake of convenience.

As we already observed, $m(u,v) = M$ for some constant $M \in \mathbb{R}$. By the definition of m, we therefore have that

$$4\Omega^{-2}(\partial_u r)(\partial_v r) = \frac{2M}{r} - 1.$$

In particular, this means that at $(0,0)$:

$$4(\partial_u r)(0,0)(\partial_v r)(0,0) = \frac{2M}{r_0} - 1.$$

This means that, given M, $\partial_v r(0,0)$ is determined by $\partial_u r(0,0) \neq 0$ (or the other way around) and the sign of $\frac{2M}{r_0} - 1$ determines the relative sign of $\partial_v r(0,0)$ and $\partial_u r(0,0)$.

By integrating the Raychaudhuri equations (5.15) and (5.16) , we have that $\partial_u r = \partial_u r(0,0)$ on $\underline{H}_0$ and $\partial_v r = \partial_v r(0,0)$ on H_0.

Case I : Suppose first that $\partial_u r(0,0) \neq 0$ or $\partial_v r(0,0) \neq 0$. Without loss of generality, we may assume that $\partial_u r(0,0) < 0$. Otherwise, we rescale: $u \mapsto -u$ and $v \mapsto -v$ and use that $(u,v) \mapsto -(u,v)$ is an isometry, or we simply interchange the roles of u and v in the argument.

Let $\tilde{u}, \tilde{v}$ be Eddington–Finkelstein double null coordinates on Schwarzschild such that the Schwarzschild radius function $r_S(\tilde{u},\tilde{v})$ satisfies $r_S(0,0) = r_0$. Then $g_S = -(1 - \frac{2M}{r_S}) d\tilde{u} d\tilde{v} + r_S^2 \mathring{\not{g}}$. Now let $u(\tilde{u})$ and $v(\tilde{v})$ be defined as follows. $u(0) = 0$, $v(0) = 0$ and

$$d\tilde{u} = \frac{\lambda}{1 - \frac{2M}{r_S}(\tilde{u},0)} du,$$

$$d\tilde{v} = \frac{1 - \frac{2M}{r_S}(0,0)}{\lambda(1 - \frac{2M}{r_S}(0,\tilde{v}))} dv.$$

Then $g = -\Omega_S^2(u,v) du dv + r_S^2 \mathring{\not{g}}$, with $\Omega_S^2(u,v) = \frac{(1-\frac{2M}{r_S}(\tilde{u}(u),\tilde{v}(v)))(1-\frac{2M}{r_S}(0,0))}{(1-\frac{2M}{r_S}(\tilde{u}(u),0))(1-\frac{2M}{r_S}(0,\tilde{v}(v)))}$. Hence $\Omega_S^2(u,0) = \Omega_S^2(0,v) = 1$, as required. Furthermore,

$$\partial_u r_S(0,0) = \frac{d\tilde{u}}{du}(0)(\partial_{\tilde{u}} r_S)(0,0) = -\frac{\lambda}{2},$$

$$\partial_v r_S(0,0) = \frac{d\tilde{v}}{dv}(0)(\partial_{\tilde{v}} r_S)(0,0) = \frac{1 - \frac{2M}{r_0}}{2\lambda}$$

Now we simply take $\lambda = -2\partial_u r(0,0)$.

Case II : Suppose now that $\partial_u r(0,0) = \partial_v r(0,0) = 0$. Then $r_0 = 2M$. Let $(\tilde{U},\tilde{V})$ be Kruskal coordinates on the maximally-extended Schwarzschild spacetimes with mass M, such that $(\tilde{U},\tilde{V}) = (0,0)$ is the bifurcation sphere. Then $\partial_{\tilde{V}} r_s(0,0) = \partial_{\tilde{U}} r_s(0,0) = 0$. After a suitable rescaling, we obtain null coordinates U and V such that $\Omega_S^2(U,0) = \Omega_S^2(0,V) = 1$. □

5.6 Global properties of general spherically symmetric spacetimes

It turns out that even with minimal information about the matter model under consideration, it is possible to make statements about global features of the spacetime.

Definition 5.4. *Let $(\mathcal{M}, g)$ be a spherically symmetric spacetime.*

1. *A sphere at $p \in \mathcal{Q}$ is* trapped *if $(\partial_u r)(p), (\partial_v r)(p) < 0$.*
2. *A sphere at $p \in \mathcal{Q}$ is* marginally trapped *if $(\partial_u r)(p) < 0$ and $(\partial_v r)(p) \geq 0$.*
3. *A sphere at $p \in \mathcal{Q}$ is* anti-trapped *if $(\partial_u r)(p), (\partial_v r)(p) > 0$ and* marginally anti-trapped *if $(\partial_u r)(p) = 0$ and $(\partial_v r)(p) > 0$.*

[Exercise: Determine which spheres in the maximally-extended Schwarzschild spacetime (expressed in Kruskal coordinates) are trapped, marginally trapped, anti-trapped and marginally anti-trapped.]

The existence of a trapped sphere is a local property in the spacetime. The Penrose incompleteness theorem shows that this nevertheless has global consequences: the spacetime must have future-directed and future-inextendible null geodesics whose affine parameters are bounded from above ("future-directed null geodesics that do not live forever"). A spacetime with such a property is said to be *future-null-geodesically incomplete.*

Proposition 5.10 (Spherically symmetric Penrose incompleteness theorem)**.** *Let* $(\mathcal{M}, g)$ *be a spherically symmetric spacetime, such that* $\mathrm{Ric}[g](X, X) \geq 0$ *for any null vector field* X*. Assume that* $(\mathcal{M}, g)$ *has a trapped sphere. Then* $(\mathcal{M}, g)$ *is future-null-geodesically incomplete.*

Remark 5.7. *If we define the stress energy tensor corresponding to* $(\mathcal{M}, g)$ *via* $8\pi\mathbb{T} = \mathrm{Ric}[g] - \frac{1}{2}R[g]g$*, then we require* $\mathbb{T}$ *to satisfy the null energy condition. Since this theorem does not involve the equations satisfied by matter coupled to the Einstein equations, it is really a result in Lorentzian geometry. We do not need to appeal to the Einstein equations!*

Proof. There exists a $p \in \mathcal{Q}$ such that $\partial_v r(p) < 0$ and $\partial_u r(p) < 0$. Let $p = (0, 0)$ and consider the outgoing null line $\ell = \{(0, v) \in \mathcal{Q}\}$. By rescaling v so that $\Omega^2|_\ell \equiv 1$, we have that

$$\nabla_{\partial_v}\partial_v = \Gamma^v_{vv}\partial_v = \Omega^{-2}\partial_v\Omega^2 = 0,$$

so ℓ must represent an affinely parametrized null geodesic γ in $(\mathcal{M}, g)$. We will show that v is bounded along ℓ, which implies that γ is future-geodesically incomplete or reaches Γ.

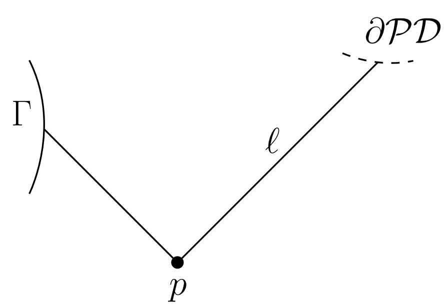


Figure 24: Null lines emanating from a trapped sphere at $p \in \mathcal{Q}$.

The key equation in this argument is the Raychaudhuri equation (5.10), which implies that

$$\partial_v^2 r(0, v) = -8\pi\mathbb{T}_{vv}(0, v),$$

where $\mathbb{T}_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$. Hence, $\mathbb{T}_{vv} = R_{vv} = \mathrm{Ric}[g](\partial_v, \partial_v) \geq 0$, by assumption. Hence $\partial_v^2 r(0, v) \leq 0$, so $\partial_v r(0, v) \leq \partial_v r(0, 0) < 0$.

We can now integrate $\partial_v r(0, v)$ to obtain:

$$r(0, v) = r(0, 0) + \int_0^v \partial_v r(0, v')\, dv' \leq r(0, 0) - v(-\partial_v r)(0, 0).$$

Since $r(0, v) > 0$, this means that

$$v < \frac{r(0, 0)}{(-\partial_v r)(0, 0)}.$$

We can repeat the same argument with v replaced by u to show that the future-directed ingoing line emanating from $(0, 0)$ must also either each $\partial\mathcal{PD} \setminus \Gamma$ in finite affine time or reach Γ. Since the future-directed ingoing and outgoing lines cannot both reach Γ, we conclude that the spacetime admits a future geodesically incomplete $\square$

[EXERCISE: Explain why in the case of spherically symmetric spacetimes satisfying the null energy condition, which contain an anti-trapped sphere, there are null geodesics did not exist for infinite affine time in the past.]

We already encountered future-directed null geodesics which did not exist for ever in the Schwarzschild spacetime, where such geodesics were hidden behind a black hole event horizon. We will now show that, this remains true in very general spherical symmetric spacetimes that have a non-empty future null infinity $\mathcal{I}^+$.

We will show that a trapped sphere implies the existence of a black hole.[20]

Definition 5.5. *We define the* regular set $\mathcal{R} \subseteq \mathcal{Q}$, *the* trapped set $\mathcal{T} \subseteq \mathcal{Q}$ *and the* apparent horizon $\mathcal{A} \subset \mathcal{Q}$ *as follows:*

$$\mathcal{R} := \{(u,v) \in \mathcal{Q} \;\; \partial_u r(u,v) < 0,\ \partial_v r(u,v) > 0\},$$
$$\mathcal{T} := \{(u,v) \in \mathcal{Q} \;\; \partial_u r(u,v) < 0,\ \partial_v r(u,v) < 0\},$$
$$\mathcal{A} := \{(u,v) \in \mathcal{Q} \;\; \partial_u r(u,v) < 0,\ \partial_v r(u,v) = 0\}.$$

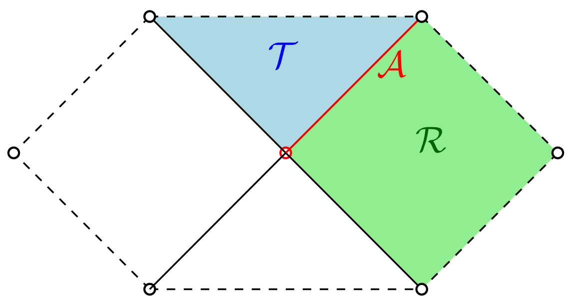


Figure 25: The regions $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{A}$ in a maximally-extended Schwarzschild spacetime.

We consider the following to set-ups:

Set-up 1: Suppose $\Gamma \neq \emptyset$. Let $\Sigma = H_0$ be an outgoing line emanating from Γ. Assume that the future limit point on H_0 is an element of $\mathcal{I}$ and that $\partial_u r < 0$ on H_0.

Set-up 2: Suppose $\Gamma = \emptyset$. Consider two intersecting null lines H_0 (outgoing) and $\underline{H}_0$ (ingoing) in $\mathcal{Q}$. Let $\Sigma = H_0 \cup \underline{H}_0$. Assume that the future limit point on H_0 is an element of $\mathcal{I}$ and that $\partial_u r < 0$ on Σ.

In both cases, we define $\mathcal{Q}^+ = D^+(\Sigma \cup \Gamma)$ (with respect to $(\mathcal{PD}, -dudv)$). We will choose (u,v) coordinates such that $H_0 \subset \{u = 0\}$ and $\underline{H}_0 \subset \{v = 0\}$. We will take v to have bounded range so that the future limit point of H_0 is given by $(0, v_{\mathcal{I}})$.

Lemma 5.11. *Assume that* $\mathbb{T} = \mathrm{Ric}[g] - \frac{1}{2}Rg$ *satisfies the null energy condition. Then*

$$\mathcal{Q}^+ = (\mathcal{R} \cup \mathcal{T} \cup \mathcal{A}) \cap \mathcal{Q}^+.$$

[20] This is a sufficient condition, but not a necessary one. EXERCISE: Show this by considering the extremal Reissner–Nordström black hole spacetimes.

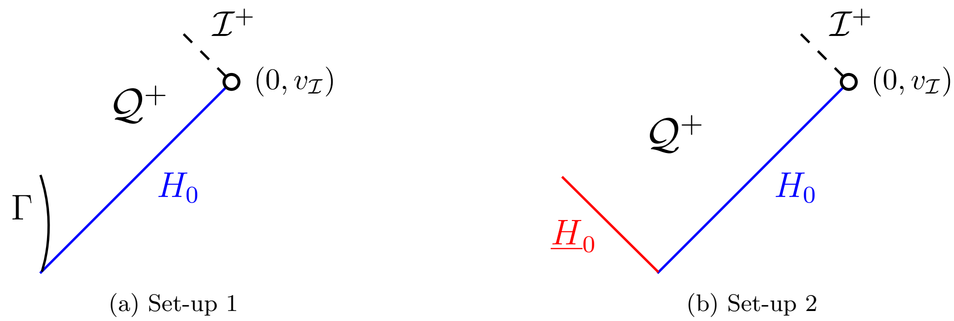


Figure 26: Penrose diagrams illustrating $\mathcal{Q}^+$ for the two different set-ups.

Proof. By construction, $H_0 \subset (\mathcal{R} \cup \mathcal{T} \cup \mathcal{A}) \cap \mathcal{Q}^+$. Since any point in $\mathcal{Q}^+$ can be connected to H_0 via an ingoing null line and by the Raychaudhuri equation (5.9) $\partial_u^2 r \leq 0$ along that null line. Hence $\partial_u r < 0$ everywhere in $\mathcal{Q}^+$. □

Proposition 5.12. *Null infinity $\mathcal{I}$ of $\mathcal{Q}^+$ satisfies either $\mathcal{I} = (0, v_{\mathcal{I}})$, or it is an ingoing null segment $\mathcal{I} = \{(u, v_{\mathcal{I}}) \in \mathbb{R}^2 \,|\, u < u_{\mathcal{I}}\}$, for some $u_{\mathcal{I}} > 0$. In the latter case, $\mathcal{I} = \mathcal{I}^+$.*

Proof. Consider the line segments $\ell_v = \{(u, v) \in \mathbb{R}^2 \,|\, u \geq 0\}$ with $v \leq v_{\mathcal{I}}$ that emanate from H_0. Since $\partial_u r < 0$ in $\mathcal{Q}^+$, we have that $r|_{\ell_v \cap \mathcal{Q}^+} \leq r(0, v)$. Hence, the lines ℓ_v do not intersect $\mathcal{I}$ for $v < v_{\mathcal{I}}$ and we conclude that $\mathcal{I} \subset \ell_{v_{\mathcal{I}}}$. Suppose that there exists a $u_0 > 0$ such that $(u_0, v_{\mathcal{I}}) \in \mathcal{I}$. Then, by $\partial_u r < 0$, we must have that $(u, v_{\mathcal{I}}) \in \mathcal{I})$ for all $0 \leq u \leq u_0$. Let $u_{\mathcal{I}}$ denote the supremum of such u_0. Since $\mathcal{I}$ is an ingoing future boundary of $\mathcal{Q}^+$, it follows immediately that $\mathcal{I} = \mathcal{I}^+$.

If there exist no such u_0, then $\mathcal{I}$ consists of the single point $(0, v_{\mathcal{I}})$. □

We will now assume that $\mathcal{I}$ consists of more than one point, so by the above proposition, $\mathcal{I}^+ = \mathcal{I} = \{(u, v_{\mathcal{I}}) \in \mathbb{R}^2 \,|\, u < u_{\mathcal{I}}\}$. We denote $i^+ := (u_{\mathcal{I}}, v_{\mathcal{I}})$. Then i^+ is the future limit point of $\mathcal{I}^+$ in $\mathbb{R}^2$.

Recall that a *black hole region* in $\mathcal{Q}^+$ is the set:

$$\mathcal{BH} := \mathcal{Q}^+ \setminus J^-(\mathcal{I}^+)$$

which can be empty. If $\mathcal{BH} \neq \emptyset$, then the future event horizon $\mathcal{H}^+$ is the future boundary of $J^-(\mathcal{I}^+)$ in $\mathcal{Q}^+$.

The set $J^-(\mathcal{I}^+)$ is called *the domain of outer communications*.

We will now show that the presence of a (marginally) trapped surface implies the non-emptyness of $\mathcal{BH}$.

Proposition 5.13. *Assume that $8\pi\mathbb{T} = \mathrm{Ric}[g] - \frac{1}{2}Rg$ satisfies the null energy condition. Suppose that $\mathcal{T} \cup \mathcal{A} \neq \emptyset$. Then:*

(i) *$\mathcal{T} \cup \mathcal{A} \subseteq \mathcal{BH}$. In particular, the black hole region of $\mathcal{Q}^+$ is non-empty.*

(ii) *$\mathcal{H}^+$ is an outgoing null line with future limit point i^+ and $\mathcal{H} \subset \mathcal{R} \cup \mathcal{A}$.*

(iii) *Let $(u_*, v_*) \in \mathcal{T} \cup \mathcal{A}$. Then $(u_*, v) \in \mathcal{T} \cap \mathcal{A}$ for all $v \geq v_*$.*

(iv) *On $\mathcal{A}$, we have that $1 - \frac{2m}{r} \equiv 0$.*

Proof. "(iii)" Let $p = (u_*, v_*) \in \mathcal{Q}^+$, such that $\partial_v r(p) \leq 0$. Since the null energy condition is satisfied, we have that $\partial_v^2 r(u_*, v) \leq 0$ after rescaling v, so $\partial_v r(u_*, v) \leq 0$.

"(i)": Suppose additionally that $p \in J^-(\mathcal{I}^+)$. Then the line $\{(u_*, v)\ ,\ v \geq v_*\}$ in $\mathbb{R}^2$ must intersect $\mathcal{I}^+$. By definition of $\mathcal{I}^+$, this means that $\limsup_{v \geq v_*} r(u_*, v) = \infty$. From the above, it also follows that $\limsup_{v \geq v_*} r(u_*, v) \leq r(u_*, v_*) < \infty$, which is a contradiction. We conclude that $p \notin J^-(\mathcal{I}^+)$, so $\mathcal{Q}^+ \setminus J^-(\mathcal{I}^+) \neq \emptyset$.

"(ii)": By definition, $\mathcal{H}^+$ is the future boundary of $J^-(\mathcal{I}^+)$, which is an outgoing null line with limit point i^+. Suppose $\mathcal{H}^+ \cap \mathcal{T} \neq \emptyset$. Then there would exist a $p \in \mathcal{H}^+ \cap \mathcal{T}$ and a q in neighbourhood of p such that $q \in J^-(\mathcal{I}^+)$ and $\partial_v r(q) < 0$. But this is a contradiction by (i). Hence $\mathcal{H}^+ \subset \mathcal{R} \cup \mathcal{A}$.

"(iv)": This is immediate from the definition of m and the fact that $\partial_v r = 0$ on $\mathcal{A}$. □

[Exercise: Suppose $\mathcal{H}^+ \cap \mathcal{A} \neq \emptyset$. Show that then $\mathbb{T}_{vv} \equiv 0$ on $\mathcal{H}^+ \cap \mathcal{A}$.]

We immediately obtain the following corollary.

Corollary 5.14 (Hawking's area theorem in spherical symmetry)**.** $\partial_v(4\pi r^2) \geq 0$ *on* $\mathcal{H}^+$*.*

This statement is known as the *second law of black hole mechanics*.

By imposing the dominant energy condition, we can obtain the following *monotonicity properties* for the Hawking mass.

Proposition 5.15. *Assume that* $8\pi\mathbb{T} = \mathrm{Ric}[g] - \frac{1}{2}Rg$ *satisfies the dominant energy condition. Then:*

(i) *In* $(\mathcal{R} \cup \mathcal{A}) \cap \mathcal{Q}^+$*:*

$$\partial_v m \geq 0,$$
$$\partial_u m \leq 0,$$

(ii) *Assume that m is uniformly bounded from above along H_0. Then m can be continuously extended to $\mathcal{I}^+$, i.e.*

$$M_{\mathrm{Bondi}}(u) := \lim_{\substack{v \to v_{\mathcal{I}} \\ u < u_{\mathcal{I}}}} m(u, v)$$

is well-defined. Furthermore,

$$M_{\mathrm{Bondi}}(u_2) \leq M_{\mathrm{Bondi}}(u_1).$$

for $0 \leq u_1 \leq u_2 < u_{\mathcal{I}}$*.*

(iii) *Assume that m is bounded from below on $\underline{H}_0 \cap \{u \leq u_{\mathcal{I}}\}$. Then the following limit is well-defined $M_f := \lim_{u \uparrow u_{\mathcal{I}}} M_{\mathrm{Bondi}}(u)$.*

(iv) *If $\Gamma \neq \emptyset$, then $m|_\Gamma \equiv 0$ and $m \geq 0$ in $\mathcal{R} \cup \mathcal{A}$.*

Proof. "(i)": The monotonicity properties follow directly from (5.11) and (5.12), using that $\mathbb{T}_{uu}, \mathbb{T}_{vv}, \mathbb{T}_{uv} \geq 0$ by Lemma 5.4. Since $\partial_u m \leq 0$ and m is uniformly bounded along H_0, we must have that m is uniformly bounded in $\mathcal{R}$.

"(ii)": Let $(u_*, v_*) \in J^-(\mathcal{I}^+)$. Then $(u, v) \in \mathcal{R}$ for all $v \geq v_*$ and $u \leq u_*$ by Proposition 5.13. Since $\partial_u m \leq 0$ in $\mathcal{R}$, this implies that $m(u_*, v) \leq m(0, v)$, which is uniformly bounded from above by assumption. Since moreover $\partial_v m(u_*, v) \geq 0$, $m(u, v)$ is monotonically non-decreasing and must attain a finite limit as $v \uparrow v_{\mathcal{I}}$.[21]. Hence, $M_{\mathrm{Bondi}}(u)$ is well-defined and using that the limit respects inequalities:

$$0 \geq \lim_{v \to v_{\mathcal{I}}} [m(u_2, v) - m(u_1, v)] = M_{\mathrm{Bondi}}(u_2) - M_{\mathrm{Bondi}}(u_1)$$

[21] This follows from the following fact from real analysis: bounded monotone sequences converge.

and we conclude (ii).

"(iv)": Consider $p \in \Gamma$. We can write:

$$m(p) = \frac{r}{2}(1 - \overline{g}^{-1}(dr, dr))(p).$$

Since r is certainly a C^1 function on $\mathbb{R} \times [0\infty)$ and $\overline{g}$ is certainly continuous on $\mathbb{R} \times [0\infty)$, the expression $\overline{g}^{-1}(dr, dr)(p)$ must be finite. Since $r(p) = 0$, we must therefore have that $m(p) = 0$.

Let $p = (u_*, v_*) \in \mathcal{R} \cup \mathcal{A} \cap \mathcal{Q}^+$. Then $p \in J^+(\Gamma)$. By (iii) of Proposition 5.13, $(u_*, v) \in \mathcal{R} \cup \mathcal{A}$ for all $v \leq v_*$. Since $\partial_v m \geq 0$ in $\mathcal{R} \cup \mathcal{A}$, we have that $m \geq 0$ at p.

"(iii)": To conclude (iii) we use that $M_{\text{Bondi}}(u)$ non-increasing and, if we can show that it is uniformly bounded from below, we can conclude that it must attain a limit as $u \uparrow u_{\mathcal{I}}$ via the argument in the proof of (ii). When $\Gamma \neq \emptyset$, we have that $m \geq 0$ by (iv), so $M_{\text{Bondi}}(u) \geq 0$ which is a uniform bound from below. When $\Gamma = \emptyset$, we use that assumption that m is uniformly bounded from below on $\underline{H}_0 \cap \{u \leq u_{\mathcal{I}}\}$ together with $\partial_v m \geq 0$ in $\mathcal{R} \cup \mathcal{A}$ to conclude that $M_{\text{Bondi}}(u) \geq \inf m|_{\underline{H}_0}$. □

We refer to M_f as the *final Bondi mass* of the spacetime and $M_i := M_{\text{Bondi}}(0)$ as the *initial Bondi mass.*

Proposition 5.16 (Positive Mass Theorem in spherical symmetry)**.** *Assume that* $8\pi\mathbb{T} = \text{Ric}[g] - \frac{1}{2}R[g]g$ *satisfies the dominant energy condition and that* $\Gamma \neq \emptyset$. *Then* $M_i \geq 0$, *with equality if and only if* $(\mathcal{M}, g)$ *is isometric to a region of the Minkowski spacetime.*

Proof. Let $\Gamma \neq \emptyset$, then it follows from Proposition 5.15 (iv) that $M_{\text{Bondi}}(u) \geq 0$, so in particular $M_i \geq 0$. If $M_i = 0$, then $M_{\text{Bondi}}(u) = 0$, so by $\partial_v m \geq 0$ in $\mathcal{R} \cup \mathcal{A}$, we must have that $m \equiv 0$ in $\mathcal{R} \cup \mathcal{A}$. Suppose that $\mathcal{T} \cup \mathcal{A} \neq \emptyset$. Then $\mathcal{A} \neq \emptyset$, so along $\mathcal{A}$, $0 = m = \frac{r}{2}$, which is a contradiction, since $r > 0$ in $\mathcal{Q}^+$. Therefore $\mathcal{Q}^+ = \mathcal{R}$.

By (5.11) and (5.12), together with the fact that $\partial_u r < 0$, $\partial_v r \geq 0$ in $\mathcal{R}$ and $\mathbb{T}_{uu}, \mathbb{T}_{uv}, \mathbb{T}_{vv} \geq 0$, we must have that $\mathbb{T}_{uu} = \mathbb{T}_{uv} = \mathbb{T}_{vv} = 0$. Now it follows from Birkhoff's theorem (Theorem 5.9) that $(\mathcal{M}, g)$ must be isometric to a region of the Minkowski spacetime. □

[Exercise: 1) Drop the assumption that $\partial_u r < 0$ along H_0. Assume that $\Gamma = \emptyset$ and $m|_{H_0} < 0$ and *derive* $\partial_u r|_{H_0} < 0$. 2) Explain why $\underline{H}_0$ must be future-null-geodesically incomplete.]

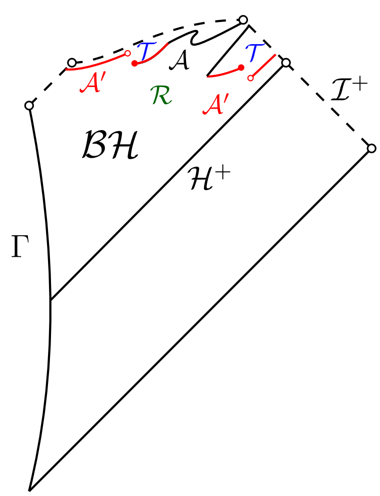


Figure 27: Example of a spherically symmetric spacetime with a complicated apparent horizon.

Definition 5.6. *The outermost apparent horizon $\mathcal{A}'$ is the following subset of $\mathcal{A}$:*

$$\mathcal{A}' := \{(u,v) \in \mathcal{A} \,|\, (u',v) \in \mathcal{R} \text{ for all } u' < u \text{ such that } (u',v) \in \mathcal{Q}^+\}.$$

Proposition 5.17 (Penrose inequality in spherical symmetry)**.** *$r \leq 2M_f$ along $\mathcal{A}'$.*

Proof. EXERCISE. Hint: Use that $r(u,v) = 2m(u,v)$ for $(u,v) \in \mathcal{A}'$ and connect (u,v) to $\mathcal{I}^+$ via ingoing and outgoing null segments. □

Note finally that the apparent horizon $\mathcal{A}$ can have a very complicated structure (even in spherical symmetry!). Unlike the future event horizon $\mathcal{H}^+$, it does not need to be connected, for example.

5.7 Cold static stars and collapsing dust clouds

We consider spherically symmetric spacetimes, with metrics that are solutions to the Einstein–Euler equations with the perfect fluid stress-energy tensor:

$$\mathbb{T}^{\mu\nu} := (\rho + p)U^\mu U^\nu + p(g^{-1})^{\mu\nu}.$$

We moreover assume that U is spherically symmetric, which we take to mean that U is independent of θ, φ and $U^A \equiv 0$, where for $A \in \{\theta, \varphi\}$.[22]

Therefore,

$$-1 = g(U,U) = -\Omega^2 U^u U^v$$

Furthermore $U_u = -\frac{1}{2}\Omega^2 U^v$ and $U_v = -\frac{1}{2}\Omega^2 U^u$.

Let X be another spherically symmetric vector field such that $g(X,U) = 0$ and $X(r) = 1$ (which fixes X uniquely). Then

$$X = \frac{1}{U^v \partial_v r - U^u \partial_u r}\left(U^v \partial_v - U^u \partial_u\right).$$

[EXERCISE: Check that $g(X,U) = 0$ and $X(r) = 1$.]

We therefore obtain:

$$\begin{aligned}
\mathbb{T}_{uu} &= (\rho + p)(U_u)^2 = \frac{1}{4}(\rho + p)\Omega^4 (U^v)^2, \\
\mathbb{T}_{vv} &= (\rho + p)(U_v)^2 = \frac{1}{4}(\rho + p)\Omega^4 (U^u)^2 \\
\mathbb{T}_{uv} &= (\rho + p)U_u U_v - \frac{1}{2}p\Omega^2 = \frac{1}{4}(\rho - p)\Omega^2.
\end{aligned}$$

Recall that the Hawking mass satisfies:

$$\begin{aligned}
\partial_u m &= -8\pi r^2 \Omega^{-2}(\partial_v r \mathbb{T}_{uu} - \partial_u r \mathbb{T}_{uv}) \\
\partial_v m &= 8\pi r^2 \Omega^{-2}(-\partial_u r \mathbb{T}_{vv} + \partial_v r \mathbb{T}_{uv}).
\end{aligned}$$

Note that therefore

$$\begin{aligned}
X(m) = 8\pi r^2 \Omega^{-2} \mathbb{T}_{uv} - \frac{8\pi r^2 \Omega^{-2}}{U^v \partial_v r - U^u \partial_u r}&(U^v \partial_u r \mathbb{T}_{vv} - U^u \partial_v r \mathbb{T}_{uu}) \\
&= 2\pi r^2(\rho - p) + 2\pi r^2(\rho + p) = 4\pi r^2 \rho. \quad (5.21)
\end{aligned}$$

[22]This is equivalent to: $\mathcal{L}_{L_i} U \equiv 0$, where L_i, $i \in \{1,2,3\}$ are the angular momentum vector fields.

5.7.1 Cold static stars

We will first investigate static solutions describing "stars" of some fixed radius $r = R$ at zero temperature. That is to say. We make the following assumptions on our spherically symmetric spacetime solutions to the Euler equations:

- $\mathcal{M} \cong \mathbb{R} \times \mathbb{R}^3$,
- $(\mathcal{M}, g)$ is static. This means that there exists a timelike Killing vector field K, such that we can decompose:
$$g = -e^{\Phi} dt^2 + \underline{g},$$
with $K = \frac{\partial}{\partial t}$, $\underline{g}$ is a Riemannian metric on $\Sigma = \{t = 0\}$ and $\Phi : \Sigma \to \mathbb{R}$. Together with the spherical symmetry assumption, this means that we can write:
$$g = -e^{\Phi} dt^2 + e^{-\Psi} dr^2 + r^2 \mathring{\not{g}},$$
with $\Phi, \Psi : [0, \infty)_r \to \mathbb{R}$. We will assume that K is future-directed.
- The fluid is at rest in (t, r, θ, φ) coordinates, so U points in the ∂_t direction and ρ, p do not depend on t. By the condition $g(U, U) = -1$, this means that $U = e^{-\frac{1}{2}\Phi} \partial_t$.
- $\rho, p \geq 0$ and in the region $\{r > R\}$, $p(r) = \rho(r) = 0$.
- We will assume that p is a function of ρ. This is called a *barotropic equation of state*. For general equations of state, p would be a function of ρ and the temperature T. The barotropic assumption may be motivated by the assumption that the temperature in zero; the star is "cold" and no longer radiating.
- We assume that $\frac{dp}{d\rho} \geq 0$. If $\frac{dp}{d\rho} < 0$ at some point $x \in \mathcal{M}$, then the static solution $(u, p, \rho) = (0, p, \rho)$ to the Euler equations would be unstable in the following heuristic sense: a small increase in ρ at x would lead to a decrease in pressure, which would cause more fluid to flow to x, which will cause ρ to increase further, etc. .

By Birkhoff's theorem, the region $r > R$ must be isometric to a region in a Schwarzschild spacetime with mass M, so $\Phi(r) = \Psi(r) = \log\left(1 - \frac{2M}{r}\right)$ for $r > R$. Furthermore, by (5.21), we obtain

$$M = m(R) = 4\pi \int_0^R \rho(r)\, r^2 dr > 0.$$

Furthermore, using that $g(U, U) = -1$, we must have that $U = e^{-\frac{1}{2}\Phi} \partial_t$.

We can write the metric in double null coordinates by introducing $u = t - r_*$ and $v = t + r_*$ ($t = \frac{1}{2}(u + v)$ and $r_* = \frac{1}{2}(v - u)$), with

$$\frac{dr_*}{dr} = e^{-\frac{1}{2}(\Phi + \Psi)}.$$

Indeed, then

$$g = e^{\Phi}(-dt^2 + dr_*^2) + r^2 \mathring{\not{g}} = -e^{\Phi} du dv + r^2 \mathring{\not{g}}.$$

We have the following identities:

$$\Omega^2 = e^{\Phi},$$
$$\partial_v = \frac{1}{2}(K + e^{\frac{1}{2}(\Phi + \Psi)} X),$$

$$\partial_u = \frac{1}{2}(K - e^{\frac{1}{2}(\Phi+\Psi)}X),$$
$$U = e^{-\frac{1}{2}\Phi}(\partial_u + \partial_v),$$
$$X = e^{-\frac{1}{2}(\Phi+\Psi)}(\partial_v - \partial_u),$$
$$\partial_v r = -\partial_u r = \frac{1}{2}e^{\frac{1}{2}(\Phi+\Psi)}.$$

By definition of m, we therefore have that

$$m = \frac{r}{2}(1 + 4\Omega^{-2}(\partial_u r)(\partial_v r)) = \frac{r}{2}\left(1 - e^{\Psi}\right).$$

Rearranging the above, we have that

$$e^{\Psi} = 1 - \frac{2m}{r}.$$

This implies in particular that $m(r) < \frac{r}{2}$. This means that $R > 2M$. There can therefore exist no static stars with radius $R \leq 2M$. We will see below that this lower bound can be sharpened.

[Exercise: We define E, the total energy of the fluid by integrating the matter energy density $\mathbb{T}_{00}$, i.e. $E = \int_{\{t=0\}} \mathbb{T}_{00}\sqrt{-\det h}\, d\theta d\varphi dr$, with h the induced metric on $\{t = 0\}$. Show that $M < E$. Hence, the total energy of the star, M, is smaller than the energy of the matter, E. One can interpret the difference $E - M$ as the gravitational binding energy of the star.]

The Raychaudhuri equations (5.9) and (5.10), together with the above equations then give (Exercise):

$$\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} e^{\frac{\Phi}{2}} \frac{d}{dr}\left(\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} e^{-\frac{\Phi}{2}}\right) = -4\pi r(\rho + p).$$

We apply the Leibniz rule for differentiation to obtain:

$$-\frac{d}{dr}\left(\frac{m}{r}\right) - \frac{1}{2}\left(1 - \frac{2m}{r}\right)\frac{d\Phi}{dr} = -4\pi r(\rho + p).$$

Using that $\frac{d(r^{-1}m)}{dr} = 4\pi\rho r - r^{-2}m$, we obtain:

$$2mr^{-2} - \left(1 - \frac{2m}{r}\right)\frac{d\Phi}{dr} = -8\pi r p.$$

so

$$\frac{d\Phi}{dr} = 2\frac{m + 4\pi r^3 p}{r(r - 2m)}.$$

The Euler equations give:

$$(\rho + p)U^{\alpha}\nabla_{\alpha}U^{\mu} = -((g^{-1})^{\mu\nu} + U^{\mu}U^{\nu})\nabla_{\nu}p.$$

Using that $U = e^{-\frac{1}{2}\Phi}K$, with K a Killing vector field and $(\mathcal{L}_K g)_{\alpha\mu} = \nabla_{\alpha}K_{\mu} + \nabla_{\mu}K_{\alpha} = 0$, so

$$\begin{aligned}
X(p) = (g^{-1})^{\mu\nu}X_{\mu}\nabla_{\nu}p = ((g^{-1})^{\mu\nu} + U^{\mu}U^{\nu})X_{\mu}\nabla_{\nu}p = -(\rho+p)U^{\alpha}X^{\mu}\nabla_{\alpha}U_{\mu} = -(\rho+p)e^{-\frac{\Phi}{2}}K^{\alpha}X^{\mu}\nabla_{\alpha}(e^{-\frac{1}{2}\Phi}K)_{\mu} \\
= -(\rho+p)e^{-\Phi}K^{\alpha}X^{\mu}\nabla_{\alpha}K_{\mu} = (\rho+p)e^{-\Phi}K^{\alpha}X^{\mu}\nabla_{\mu}K_{\alpha} \\
= \frac{1}{2}(\rho+p)e^{-\Phi}X(g(K,K)) = -\frac{1}{2}(\rho+p)e^{-\Phi}\frac{d}{dr}(e^{\Phi}) \\
= -\frac{1}{2}(\rho+p)\frac{d\Phi}{dr} = -(\rho+p)\frac{m + 4\pi r^3 p}{r(r-2m)}.
\end{aligned}$$

The equations:

$$\begin{aligned}\frac{dm}{dr} &= 4\pi r^2\rho,\\ \frac{d\Phi}{dr} &= 2\frac{m+4\pi r^3 p}{r(r-2m)},\\ \frac{dp}{dr} &= -(\rho+p)\frac{m+4\pi r^3 p}{r(r-2m)}\end{aligned}$$

are called the *Tolman–Oppenheimer–Volkoff* (TOV) equations of hydrostatic equilibrium.

Suppose now that the star has uniform density, so $\rho(r)=\rho_0$ in $r\le R$. Then $m(r)=\frac{4\pi}{3}\rho_0 r^3$ and $M=\frac{4\pi}{3}\rho_0 R^3$, so

$$\frac{dp}{dr} = -4\pi r^3(\rho_0+p)\frac{\frac{1}{3}\rho_0+p}{r(r-\frac{8\pi}{3}\rho_0 r^3)}. \tag{5.22}$$

We can solve this ODE as follows:

$$\int_0^{p(0)} \frac{1}{(\rho_0+p)\left(\frac{1}{3}\rho_0+p\right)}dp = 4\pi\int_0^R \frac{r}{(1-\frac{8\pi}{3}\rho_0 r^2)}\,dr,$$

which is equivalent to:

$$\frac{3}{2\rho_0}\log\left(\frac{p(0)+\frac{\rho_0}{3}}{p(0)+\rho_0}\right)-\frac{3}{2\rho_0}\log\left(\frac{1}{3}\right) = -\frac{3}{4\rho_0}\log(3-8\pi\rho_0 R^2)+\frac{3}{4\rho_0}\log 3,$$

which we can rearrange to obtain:

$$(3p(0)+\rho_0) = \left(1-\frac{8}{3}\pi\rho_0 R^2\right)^{-\frac{1}{2}}(p(0)+\rho_0) = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}(p(0)+\rho_0)$$

so

$$p(0) = \frac{1}{3-\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}}\left(\left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}+1\right)\rho_0.$$

Note that $p(0)$ blows up if

$$3 = \left(1-\frac{2M}{R}\right)^{-\frac{1}{2}}$$

or

$$\frac{1}{9} = 1-\frac{2M}{R},$$

which is satisfied if $R=\frac{9}{4}M$. Hence, the static solution is only well defined if $R>\frac{9}{4}M$: there can be no static, cold, spherically symmetric stars with radius less or equal to $\frac{9}{4}M$. This is called the *Buchdahl inequality.* There is no such upper bound in the Newtonian setting!

More generally, even if ρ is not constant in $\{r\le R\}$, we can use the assumption that $\frac{dp}{d\rho}\ge 0$, together with the inequality $\frac{dp}{dr}\le 0$ that follows from the TOV equations, to obtain $\frac{d\rho}{dr}\le 0$ and this can then be used to derive the following inequality:

$$\frac{m(r)}{r} \le \frac{2}{9}\left[1-6\pi r^2 p(r)+\sqrt{1+6\pi r^2 p(r)}\right]$$

Evaluating the above inequality at $r=R$, where $p(R)=0$, we similarly obtain $R>\frac{9}{4}M$.

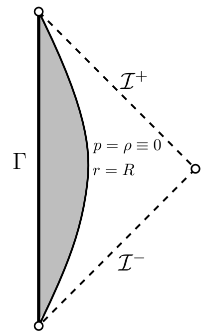


Figure 28: The Penrose diagram of a spherically symmetric static star.

5.7.2 Collapsing dust clouds

Now consider the spherically symmetric Einstein-dust model by taking $p = 0$ in Euler equations. Let $\mathcal{M} \cong \mathbb{R} \times \mathbb{R}^3$. Furthermore, assume that there exists a timelike curve γ in $\mathcal{Q}$, such that $r = r_\gamma(v)$ along γ, with $\rho(r) = \rho_0$ for $r \leq r_\gamma(v)$, $\rho(r) = 0$ for $r > r_\gamma(v)$. We can think of $r_\gamma(v)$ as describing the boundary of a star. The region $r \geq r_\gamma(v)$ must be isometric to a region in Schwarzschild with mass some mass $M > 0$, by Birkhoff's theorem. Since ρ is discontinuous, g will not be a C^2 solution to the Einstein equations at the boundary of the star, but can nevertheless be interpreted as a solution in a "weak sense" (multiplying the Einstein equations with a test function and integrating over a spacetime region).

As our boundary condition, we will assume that γ is a spherically symmetric timelike geodesic in Schwarzschild, since the Euler equations for dust imply that $\nabla_U U = 0$ in the region of non-zero ρ, so the integral curves of U are timelike geodesics. We will assume that $\dot{r}_\gamma(0) = 0$. Then must have that $r_\gamma(\tau) \to 0$ in finite affine time τ.[23] That is to say, there is no pressure to hold off gravitational collapse of the dust cloud. Hence, the spacetime contains a subset of the Schwarzschild black hole interior and therefore the black hole region $\mathcal{BH} = \mathcal{Q} \setminus J^-(\mathcal{I}^+) \neq \emptyset$ is non-empty.

These spacetimes are called the *Oppenheimer–Snyder spacetimes* (Oppenheimer–Snyder 1939).

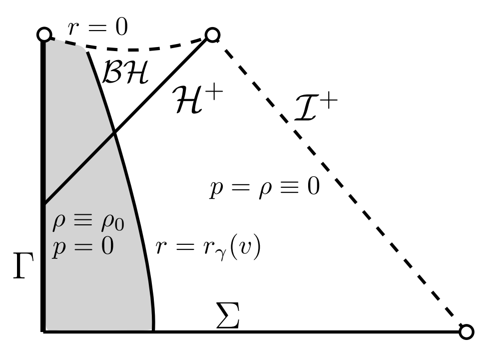


Figure 29: The Penrose diagram of an Oppenheimer–Snyder spacetime with initial hypersurface Σ.

Within the spherically symmetric dust model, the behaviour of spherically symmetric, homogeneous collapsing dust clouds is special. By considering instead a more general class of initial

[23] This follows from the timelike geodesic equation in Schwarzschild and the consideration of the potential $V_1(r)$ with total angular momentum $L = 0$.

data including *inhomogeneous* spherically symmetric dust (of compact support), Christodoulou proved in 1984 that within this model, for an *open* subset of initial data (within spherical symmetry), $\mathcal{T} \cup \mathcal{A} = \emptyset$ and $\mathcal{BH} = \emptyset$. Nevertheless, the boundary $\partial\mathcal{PD}$ has a singular point $\mathcal{O}$ that can be reached in finite affine time, where moreover $\rho \to \infty$. Furthermore, observers moving along $\{r = r_0\}$ exist for only finite time and outgoing null geodesics emanating from the centre Γ get infinitely red-shifted when they are intercepted by the observer at their final time of existence. The singularity can be "seen" from far-away, in contrast with the singularity inside a black hole region, which is hidden behind an event horizon. These types of spacetimes are therefore called "naked singularities".

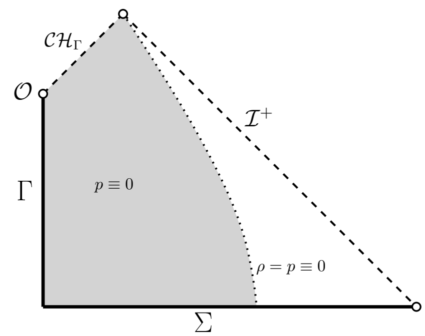


Figure 30: The Christodoulou Einstein-dust naked singularities. The radius r has a non-zero limit on $\mathcal{CH}_\Gamma$.

It is fortuitous that the naked singularities of Christodoulou were only discovered decades after the discovery of the Oppenheimer–Snyder spacetimes, since the later played a fundamental role in developing intuition regarding black hole formation and trapped surfaces.

The existence of naked singularities in the spherically symmetric dust model can be attributed to the highly idealized nature of the model and the singular nature of the Einstein–Euler equations without pressure. Indeed, since $\rho \to \infty$, the assumption of $p = 0$ becomes questionable. One might therefore *hope* that these kind of spacetimes will be entirely excluded by considering less singular matter models. In this next section, we will see that this is, however, not the case.

5.8 The cosmic censorship conjectures

Christodolou's naked singularities can be characterized from future null infinity as follows: consider a sequence of ingoing, future-directed, affinely parametrized radial null geodesics γ_n, such that

$$\gamma_n(0) = (u_0, v_n),$$
$$\dot{\gamma}_n(0) = \partial_u|_{(u_0,v_n)},$$

with $v_n \to \infty$ as $n \to \infty$. Let $s_n^{\sup}$ denote the supremum of the affine parameter s_n along γ_n. In the naked singularity spacetimes, there exists an $S < \infty$ such that $s_n^{\sup} \leq S$ for all n. You can interpreted this as the statement that "$\mathcal{I}^+$ is future-null-geodesically incomplete".

If we think of the corresponding spacetimes as arising from initial data on some Cauchy hypersurface Σ, there is no *global existence in time* of solutions to the Einstein–Euler equations. From the perspective $\mathcal{I}^+$, we cannot use the equations to predict what happens "for all times".

In 1994, Christodoulou proved that naked singularities also exist for the spherically symmetric Einstein–scalar field system, arising from initial data that are not smooth, but still sufficiently regular to have a well-posed initial value problem for the equations.[24] The singular behaviour at $\mathcal{O}$, on the other hand, is too strong to extend the metric across it. In 2019, it was shown by Shlapentokh-Rothman and Rodnianski that (non-spherically symmetric) naked singularities even exist in vacuum.

Sometimes, the $M < 0$ Schwarzschild spacetimes or the $M < e$ super-extremal Reissner–Nordström spacetimes are called "naked singularities". In these cases, however, the singularity is already present at the level of initial data and does not *form* in the evolution of more regular initial data.

(a) A naked singularity solution to the spherically symmetric Einstein-scalar field system with a null boundary $\mathcal{CH}_\Gamma$ emanating from a singular point $\mathcal{O}$ across which the metric can be extended in a suitably regular manner.

(b) A naked singularity solution to the spherically symmetric Einstein-scalar field system with a null boundary $\mathcal{S}_\Gamma$ emanating from a singular point $\mathcal{O}$ at which $r = 0$ and the metric cannot be extended in a suitably regular manner.

Figure 31: Penrose diagrams of naked singularity solutions to the spherically symmetric Einstein-scalar field system.

The *weak cosmic censorship conjecture* asserts that naked singularities are nevertheless "special".

Conjecture 5.18 (The weak cosmic censorship conjecture)**.** *Spacetimes arising from "generic", asymptotically flat, geodesically complete initial data to the Einstein equations coupled to a "reasonable" matter model have a future-complete future null infinity (i.e. there are no naked singularities).*

[Exercise: Find spacetimes arising from geodesically incomplete (characteristic) initial data with a null infinity that is not future-null-geodesically complete. Hint: consider spacetime regions in Minkowski arising from appropriate characteristic initial data to the spherically symmetric Einstein-scalar field equations with a vanishing scalar field.]

We already encountered another failure in the predictability of the Einstein equations, namely the lack of "global uniqueness" in the case of Reissner–Nordström spacetimes, which could be thought of as arising from geodesically complete initial data, but nevertheless had Cauchy horizons across which the spacetime could be extended smoothly in a *highly non-unique* manner. The *strong cosmic censorship conjecture* asserts that there should generically be *global uniqueness* of solutions:

Conjecture 5.19 (The strong cosmic censorship conjecture)**.** *Spacetimes arising from "generic", geodesically complete initial data to the Einstein equations coupled to a "reasonable" matter model are inextendible as "suitably regular" solutions to the equations.*

Despite what the naming suggests, the strong cosmic censorship conjecture does not imply the weak cosmic censorship conjecture. Indeed, this is the case in the Einstein–scalar field singularities of Christodoulou; see the Penrose diagrams in Figure 31.

[24]The existence of naked singularties arising from smooth initial data remains an open problem and the time of writing of these notes.

6 Kerr black hole spacetimes

We will now consider our first non-spherically symmetric black hole spacetimes: the *Kerr black holes spacetimes* (KERR 1963). Just like in the Schwarzschild case, we start by defining the *exterior* of the spacetimes, before extending into the black hole interior. In contrast with Schwarzschild, we will not construct a global double null foliation (this is considerably harder on Kerr spacetimes!). Since the existence of a global double null foliation is directly connected to the construction of a Penrose diagram, we will, strictly speaking, not be able to draw the corresponding Penrose diagram. Nevertheless, we will use "Penrose-like" diagrams to draw boundaries of different spacetime regions schematically.

6.1 Kerr exterior

The Kerr exterior is the spacetime $(\mathcal{M}_{\text{ext}}, g_{M,a})$. Here, $\mathcal{M}_{\text{ext}} = \mathbb{R} \times (r_+, \infty) \times \mathbb{S}^2$. When $a^2 \leq M^2$, $0 < r_- < r_+$ are the roots of of the following polynomial:

$$\Delta(r) := r^2 - 2Mr + a^2,$$

More explicitly,

$$r_\pm = M \pm \sqrt{M^2 - a^2}.$$

When $a^2 > M^2$, we can take $r_+ := 0$.

We take $M > 0$ and interpreted this parameter as the mass or energy of the spacetime and $a = J/M \in \mathbb{R}$, with J the "angular momentum of the spacetime".[25] We will assume for now that $|a| \leq M$. Note that when $a = 0$, we recover the Schwarzschild metric.

The Kerr metric $g_{M,a}$ is defined as follows:[26]

$$\begin{aligned} g_{M,a} &= -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}(adt - (r^2 + a^2)d\varphi)^2, \\ \rho^2(r,\theta) &:= r^2 + a^2\cos^2\theta. \end{aligned}$$

[EXERCISE: Check that the Kerr metric does <u>not</u> fit the definition of a spherically symmetric or static spacetime.]

We say a Kerr spacetime is *extremal* if $|a| = M$, *sub-extremal* if $|a| < M$ and *super-extremal* if $|a| > M$. Note that for $|a| = M$, $r_+ = r_- = M$.

Since $(g_{M,a})_{\mu\nu}$ are independent of t and φ, the vector fields $T := \partial_t$ and $\Phi := \partial_\varphi$ are Killing vector fields. The vector field Φ can be interpreted as the angular momentum vector field in the z-direction. The Kerr metric is therefore said to be *axisymmetric*. The coordinates (t, r, θ, φ) are called *Boyer–Lindquist coordinates*.

Consider T and note that

$$g_{M,a}(T,T) = (g_{M,a})_{tt} = -\frac{\Delta}{\rho^2} + \frac{a^2\sin^2\theta}{\rho^2} = -\frac{r^2 - 2Mr + a^2 - a^2\sin^2\theta}{\rho^2} = -\frac{r^2 - 2Mr + a^2\cos^2\theta}{\rho^2}.$$

The above expression is non-negative when

$$r_+ \leq r \leq M + \sqrt{M^2 - a^2\cos^2\theta}.$$

[25]The notion of mass and angular momentum of a spacetime can be defined precisely in a very general setting at the level of asymptotically flat initial data.

[26]Technically, we also have to complement (θ, φ) with another spherical coordinate chart to extend the metric to the full sphere $\mathbb{S}^2$. It is a nice exercise to convince yourself that this is indeed the case, using the form of the metric on $\{t = \text{const.}\} \cap \{r = \text{const.}\}$.

The region $\{r_+ < r < M + \sqrt{M^2 - a^2\cos^2\theta}\}$ is called the *ergoregion*. Outside the ergoregion T is timelike. The Kerr spacetime is said to be *stationary* since it admits a Killing vector field that is timelike outside a bounded region in r. We will fix the time orientation by demanding T outside the ergoregion to be timelike.

[EXERCISE: 1) Show that the vector field $dt^\sharp$ is timelike everywhere and hence the level sets $\{t = \text{const.}\}$ are spacelike. 2) Show that Φ is everywhere spacelike . 3) Show that the Killing vector fields K and Φ evaluated at any $p \in \mathcal{M}_{\text{ext}}$ span a timelike hyperplane in $T_p\mathcal{M}_{\text{ext}}$ and explain why around every point $p \in \mathcal{M}_{\text{ext}}$ there exists a Killing vector field that is timelike in a neighbourhood of p.]

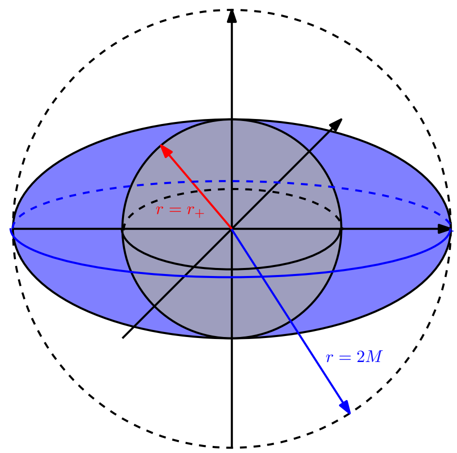


Figure 32: The ergoregion in a Kerr exterior spacetime restricted to a t-level set.

6.2 Kerr black hole interior

In this section, we will restrict to $|a| \leq M$.

Recall that in Schwarzschild, we could extend the spacetime across $r = r_+ = 2M$, into the black hole exterior. One way doing this, was to consider ingoing (or outgoing) Eddington–Finkelstein coordinates (v, r, θ, φ). We then obtained:

$$g_M = -\frac{\Delta}{r^2}dv^2 + 2dvdr + r^2\mathring{\not{g}}.$$

Note that $Y = \partial_r$ is null so the integral curve to $-Y$ is a null curve that points in the direction of decreasing r.[27]

[EXERCISE: Show that with respect to (t, r) coordinates on Schwarzschild, we can express:

$$Y = -\frac{r^2}{\Delta}\partial_t + \partial_r.]$$

We would like to consider similar coordinates to ingoing Eddington–Finkelstein coordinates (v, r) in Kerr with $a \neq 0$. The above choice of Y will no longer be null. However, we can consider the following modification (which agrees with the Schwarzschild choice if $a = 0$):

$$Y = -\frac{r^2 + a^2}{\Delta}\partial_t + \partial_r - \frac{a}{\Delta}\partial_\varphi.$$

[27] And the curve can be reparametrized to obtain an affinely parametrized null geodesic, which reaches $r = r_+$ at finite affine time.

Then

$$\begin{aligned}
\rho^2\Delta^2 g(Y,Y) &= \rho^2[(r^2+a^2)^2 g_{tt} + 2a(r^2+a^2)g_{t\varphi} + a^2 g_{\varphi\varphi}] + \rho^2\Delta^2 g_{rr} \\
&= (r^2+a^2)^2(-\Delta + \cancel{a^2\sin^2\theta}) + 2a(r^2+a^2)(a\sin^2\theta\Delta \cancel{- a\sin^2\theta(r^2+a^2)}) + a^2(-\Delta a^2\sin^4\theta + \cancel{(r^2+a^2)^2\sin^2\theta}) \\
&\quad + (r^2+a^2\cos^2\theta)^2\Delta \\
&= -\Delta[(r^2+a^2)^2 - 2a^2\sin^2\theta(r^2+a^2) + a^4\sin^4\theta - (r^2+a^2\cos^2\theta)^2] \\
&= -\Delta[(r^2+a^2)(r^2+a^2\cos^2\theta) - a^2\sin^2\theta(r^2+a^2(1-\sin^2\theta)) - (r^2+a^2\cos^2\theta)^2] \\
&= -\Delta\rho^2[r^2+a^2-a^2\sin^2\theta - (r^2+a^2\cos^2\theta)] = 0.
\end{aligned}$$

Since Y has a φ-component, we need to change (t,r,φ) to (v,r,φ_*) for an appropriate choice of v and φ_*. To ensure $Y=\partial_r$ in these new coordinates, we moreover need $Y(v)=Y(\varphi_*)=0$.

Define the tortoise coordinate r_* as follows: $r_*:(r_+,\infty)\to\mathbb{R}$, with $\frac{dr_*}{dr}=\frac{r^2+a^2}{\Delta}$ and define $v=t+r_*$.[28] Then clearly $Y(v)=0$.

Similarly, let

$$\varphi_* = \varphi - \int_r^\infty \frac{a}{\Delta(r')}\,dr' \mod 2\pi.$$

Then also $Y(\varphi_*)=0$. We refer to the coordinates (v,r,θ,φ_*) as *Kerr-star coordinates* or *ingoing Kerr coordinates*. With respect to these coordinates, the Kerr metric takes the following form (Exercise):

$$\begin{aligned}
g_{M,a} = -\rho^{-2}\left(\Delta - a^2\sin^2\theta\right)dv^2 + 2dvdr - 4Mar\rho^{-2}\sin^2\theta dvd\varphi_* - 2a\sin^2\theta drd\varphi_* + \rho^2 d\theta^2 \\
+ \rho^{-2}((r^2+a^2)^2 - a^2\Delta\sin^2\theta)\sin^2\theta d\varphi_*^2.
\end{aligned}$$

[Exercise: Show that the inverse metric is given by:

$$\begin{aligned}
g^{-1}_{M,a} = a^2\rho^{-2}\sin^2\theta\partial_v\otimes\partial_v + \rho^{-2}(r^2+a^2)[\partial_v\otimes\partial_r + \partial_r\otimes\partial_v] + \Delta\rho^{-2}\partial_r\otimes\partial_r \\
+ a\rho^{-2}[(\partial_v+\partial_r)\otimes\partial_{\varphi_*} + \partial_{\varphi_*}\otimes(\partial_v+\partial_r)] + \rho^{-2}[\partial_\theta\otimes\partial_\theta + (\sin^2\theta)^{-1}\partial_{\varphi_*}\otimes\partial_{\varphi_*}].
\end{aligned}$$

Hint: Put the matrix corresponding to $g_{M,a}$ in (t,r,θ,φ) coordinates in block-diagonal form by grouping (r,θ) and (t,φ) together.]

From the above expression, it follows in particular that the level sets $\{v=\text{const.}\}$ are <u>not</u> null hypersurfaces when $a\neq 0$.

Since the metric is not singular when $0<r\leq r_+$, we can extend the manifold $\mathcal{M}_{\text{ext}} = \mathbb{R}_v\times(r_+,\infty)_r\times\mathbb{S}^2_{\theta,\varphi_*}$ to obtain $\mathcal{M}=\mathbb{R}_v\times(0,\infty)_r\times\mathbb{S}^2_{\theta,\varphi_*}$ and extend $g_{M,a}$ analytically to $\mathcal{M}$.

We denote

$$\begin{aligned}
\mathcal{H}^+_R &:= \{r=r_+\}, \\
\mathcal{CH}^+_L &:= \{r=r_-\}.
\end{aligned}$$

These level sets are null hypersurfaces. We will refer to $\mathcal{H}^+_R$ as the *right future event horizon* and $\mathcal{CH}^+_L$ as the *left future inner horizon.*

[28] It can be shown that r_* takes the following form:

$$\begin{aligned}
r_* &= r - \frac{M^2}{\sqrt{M^2-a^2}}\log\left(\frac{r-r_-}{r-r_+}\right) + M\log\Delta + c_0 \quad |a|<M, \\
r_* &= r - \frac{2M^2}{r-M} + M\log\Delta + c_0 \quad |a|=M
\end{aligned}$$

with $c_0\in\mathbb{R}$ a constant.

In the region $\{r_- < r < r_+\}$, we moreover can revert "back" to Boyer–Lindquist-like coordinates, by defining $t = v - r_*$ and $\varphi = \varphi_* + \int_r^\infty \frac{a}{\Delta(r')}\,dr' \mod 2\pi$.

Since the Kerr metric is invariant under the reflection map $(t,\varphi) \mapsto (-t,-\varphi)$, we can carry out similar procedure in the region $\{r_- < r < r_+\}$ with the following null vector field:

$$\widetilde{Y} = \frac{r^2+a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\varphi,$$

which motivates the coordinate changes $u = t - r_*$ and $\tilde{\varphi}_* = \varphi + \int_r^\infty \frac{a^2}{\Delta(r')}\,dr' \mod 2\pi$. This allows us to further extend the manifold to obtain $\widetilde{\mathcal{M}} = \mathbb{R}_u \times (0,\infty)_r \times \mathbb{S}^2_{\theta,\tilde{\varphi}_*}$. We define $\widetilde{\mathcal{M}}_{\text{ext}} := \widetilde{\mathcal{M}} \cap \{r > r_+\}$.

In analogy with before, the following level sets are null hypersurfaces:

$$\mathcal{H}_L^+ = \{r = r_+\},$$
$$\mathcal{CH}_R^+ = \{r = r_-\},$$

which we will call the *left future event horizon* and *right inner horizon*, respectively. When $|a| = M$, $r_+ = r_-$, so $\mathcal{H}_L^+$ and $\mathcal{CH}_R^+$ coincide.

Note that we can also introduce $(u, r, \theta, \tilde{\varphi}_*)$ coordinates in $\mathcal{M}_{\text{ext}}$ to define the left/right past event horizons $\mathcal{H}^-_{L,R}$ and the right/left past inner horizons $\mathcal{CH}^-_{R,L}$.

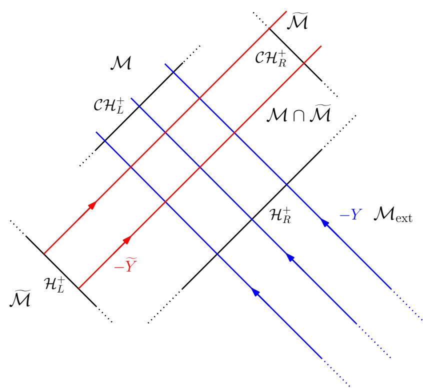


Figure 33: The integral curves of $-Y$ and $-\widetilde{Y}$ and the extensions of $\mathcal{M}_{\text{ext}}$.

We denote

$$\mathcal{BH} = \mathcal{M} \setminus \mathcal{M}_{\text{ext}}.$$

and refer to it as *the black hole region.* Since we do not have a Penrose diagram and we have not defined $\mathcal{I}^+$, it is not immediate that that $\mathcal{BH} \cap J^-(\mathcal{M}_{\text{ext}}) = \emptyset$.

Our choice of time-orientation implies that $-Y$ is future directed. Indeed, this guarantees that $-g(T,Y) = -2 < 0$, which is consistent with T being a future-directed, timelike vector field outside the ergoregion, as we imposed.

Proposition 6.1. *$\mathcal{H}_R^+$ is a null hypersurface, with the null tangent vector field $K = \partial_v + \omega_+ \partial_{\varphi_*}$, where $\omega_+ = \frac{a}{r_+^2+a^2}$. The vector field K is called the* Hawking vector field *and the constant ω_+ is called the angular velocity of the black hole.*

Proof. First since K does not have an r-component, it is tangent to $\mathcal{H}_R^+$. Note furthermore that

$$g(K,K)|_{\mathcal{H}_R^+} = g_{vv} + 2\omega_+ g_{v\varphi_*} + \omega_+^2 g_{\varphi_*\varphi_*}\Big|_{\mathcal{H}_R^+} = \rho^{-2}\sin^2\theta\left[a^2 - 4Mar_+\omega_+ + (r_+^2+a^2)^2\omega_+^2\right].$$

So, using that $2Mr_+ = r_+^2 + a^2$, we have that $g(K,K) = 0$ if and only if

$$0 = \omega_+^2 - 2a(r_+^2+a^2)^{-1}\omega_+ + a^2(r_+^2+a^2)^{-2} = \left(\omega_+ - a(r_+^2+a^2)^{-1}\right)^2.$$

Hence $g(K,K) = 0$ if we take $\omega_+ = a(r_+^2+a^2)^{-1}$.

To conclude that $\mathcal{H}_R^+$ is a null hypersurface, we need to show that $g(K,X)$ for every vector field X tangent to $\mathcal{H}_R^+$. We immediately have that $g(K,\partial_\theta) = 0$. Furthermore,

$$g(K,\partial_{\varphi_*})|_{\mathcal{H}_R^+} = g_{v\varphi_*} + a(r_+^2+a^2)^{-1}g_{\varphi_*\varphi_*}|_{\mathcal{H}_R^+} = \rho^{-2}\sin^2\theta[-2Mar_+ + a(r_+^2+a^2)]|_{\mathcal{H}_R^+} = 0.$$

□

[Exercise: Show that $\mathcal{CH}_L^+$ is a null hypersurface and find a null vector field that is tangent to $\mathcal{CH}_L^+$.]

[Exercise: Show that $\varphi_+ := \varphi - \omega_+ t$ is constant along the integral curves of K and is well-defined on $\mathcal{H}_R^+$.]

[Exercise: Show that $\nabla_K K|_{\mathcal{H}_R^+} = \kappa_+ K|_{\mathcal{H}_R^+}$, with $\kappa_+ = \frac{r_+ - r_-}{4Mr_+}$]

Proposition 6.2. $\mathcal{BH} \cap J^-(\mathcal{M}_{\text{ext}}) = \emptyset$.

Proof. Suppose that $\mathcal{BH} \cap J^-(\mathcal{M}_{\text{ext}}) \neq \emptyset$. Then there exist a future-directed causal curve $\gamma : [0,2] \to \mathcal{M}$, such that $\gamma(0) \in \mathcal{BH} \setminus \mathcal{H}_R^+$, $\gamma(2) \in \mathcal{M}_{\text{ext}}$ and $p = \gamma(1) \in \mathcal{H}_R^+$.

Denote $w = \dot\gamma(1)$. It will be useful to expand:

$$w = w^Y Y_p + w^K K_p + w^\theta \partial_\theta|_p + w^{\varphi_*}\partial_{\varphi_*}|_p.$$

Since γ is future-directed, $0 \leq g_p(Y_p, w)$. By causality of γ we moreover have that:

$$\begin{aligned} &0 \geq g(w,w) = g_p(w^Y Y_p + (w - w^Y Y_p), w^Y Y_+ + (w - w^Y Y_p)) \\ &= \cancel{(w^Y)^2 g_p(Y_p,Y_p)} + 2w^Y g_p(Y_p,w) - \cancel{(w^Y) g_p(Y_p,Y_p)} + (w^\theta)^2 g_{\theta\theta}(p) + (w^{\varphi_*})^2 g_{\varphi_*\varphi_*}(p) \geq 2\underbrace{g_p(Y_p,w)}_{\geq 0} w^Y. \end{aligned}$$

Hence, we need $w^Y \leq 0$, which implies that $\dot r_\gamma(1) \leq 0$. However, since γ is entering $\mathcal{M}_{\text{ext}}$ from $\mathcal{BH}$, we need to have $\dot r(1) > 0$, which is a contradiction.

We conclude that $\mathcal{BH} \cap J^-(\mathcal{M}_{\text{ext}}) = \emptyset$. □

An observer in the black hole region $\mathcal{BH}$ can therefore not travel to $\mathcal{M}_{\text{ext}}$, or send signals to $\mathcal{M}_{\text{ext}}$.

[Exercise: Show that for $|a| < M$, there exists an $\epsilon > 0$ such that K is timelike in the region $\{r_+ < r \leq (1+\epsilon)r_+\}$. Is this also true for $|a| = M$? Show that integral curves of K in $\{r \leq (1+\epsilon)r_+\}$ rotate at an angular velocity ω_+ with respect to observers represented by integral curves of T outside the ergoregion. Since K is tangent to $\mathcal{H}_R^+$, the boundary of the black hole region, we say that the black hole itself rotates at an angular velocity ω_+.]

6.3 Kruskal-like coordinates and global hyperbolicity

We will show that null hypersurfaces $\mathcal{H}_R^+$ and $\mathcal{H}_L^+$ actually intersect. This will result in a global understanding of the Kerr geometry in terms of Kruskal-like coordinates (U, V). In contrast with Schwarzschild and Reissner–Nordström, U- and V-level sets are not going to be null hypersurfaces.

Recall, $v = t + r_*$ and $u = t - r_*$. We restrict to sub-extremal Kerr ($|a| < M$) and we introduce Kruskal-like rescaled coordinates:

$$U := -\, e^{-\kappa_+ u},$$
$$V := e^{\kappa_+ v},$$

with κ_+ the surface gravity of $\mathcal{H}_R^+$, in analogy with the Kruskal-like coordinates on Reissner–Nordström.

We recall moreover the modified azimuthal angular coordinate φ_+:

$$\varphi_+ = \varphi - \omega_+ t \mod 2\pi = \varphi - \frac{a}{r_+^2 + a^2} t \mod 2\pi,$$

which is conserved along the integral curves of the Hawking vector field K.

Proposition 6.3. *Let $|a| < M$. The coordinates $(U, V, \theta, \varphi_+)$ cover*[29] $(\mathcal{M} \cap \{r > r_-\}) \cup (\widetilde{\mathcal{M}} \cap \{r > r_-\})$ *and the components of $g_{M,a}$ are analytic in U, V, θ, φ. Furthermore:*

$$\begin{aligned} \mathcal{M}_{\text{ext}} &= \{U < 0, V > 0\}, \\ \mathcal{M} \cap \{r > r_-\} &= \{U \geq 0, V > 0\}, \\ \widetilde{\mathcal{M}}_{\text{ext}} &= \{U > 0, V < 0\}, \\ \widetilde{\mathcal{M}} \cap \{r > r_-\} &= \{U \leq 0, V < 0\}, \end{aligned}$$

We can extend $g_{M,a}$ further to $\{U \leq 0, V \leq 0\}$, including the bifurcation sphere $\{U = 0,\ V = 0\}$ and we denote the extension by $\mathcal{M}_{\text{Krus}} = \{(U, V, \theta, \varphi_+) \mid (U, V) \in \mathbb{R}^2,\ (\theta, \varphi) \in \mathbb{S}^2\}$.

Proof. (*Sketch*) The proof relies on two main observations.

1. With respect to $(u, v, \theta, \varphi_+)$:

$$\begin{aligned} (g_{M,a})_{uu} &= (g_{M,a})_{vv} = a^2 \Delta^2 h_1(r, \sin^2\theta), \\ (g_{M,a})_{v\varphi_+} &= (g_{M,a})_{u\varphi_+} = a^2 \Delta h_2(r, \sin^2\theta), \\ (g_{M,a})_{uv} &= \Delta h_3(r, \sin^2\theta), \end{aligned}$$

   with h_1, h_2, h_3 analytic functions that are well-defined for $r \in (r_-, \infty)$ and h_3 is non-vanishing.

2. We can express:

$$-UV = e^{2\kappa_+(v-u)} = e^{\kappa_+ r_*} = \Delta k(r),$$

   with k a non-vanishing, analytic function on (r_-, ∞).

Then we can analytically extend the function $r(U, V)$ and the components of $g_{M,a}$ from the region $\mathcal{M}_{\text{ext}} = \{U < 0, V > 0\}$ to the extensions of $\mathcal{M}_{\text{ext}}$. Since analytic extensions are unique, $g_{M,a}$ in $\mathcal{M}_{\text{Krus}} \setminus \{U = 0,\ V = 0\}$ must agree with the analytically extended g_M obtained by switching to Kerr-star coordinates.

[29] At least away from the usual breakdown of spherical coordinates at a meridian connecting the north and south poles of the sphere $\mathbb{S}^2$.

To extend across $\mathcal{H}^{\pm}_{L,R}$ after changing coordinates, the factor Δ in g_{uv}, $g_{u\varphi_+}$ and $g_{v\varphi_+}$ is important. Similarly, the factor Δ^2 in g_{uu} and g_{vv} is important.

To determine the range of U and V, we observe that we can express in the region $r_- < r < r_+$:

$$U = -e^{-\kappa_+(v-2r_*)}$$

and $r_* \in \mathbb{R}$, so as we approach $\mathcal{CH}^+_L$, $U \to \infty$. Similarly, as we approach $\mathcal{CH}^+_R$, $V \to \infty$. We similarly have $U, V \to -\infty$ as we approach the time inverses $\mathcal{CH}^-_R$ and $\mathcal{CH}^-_L$, respectively. □

Proposition 6.4. *The hypersurface $\Sigma = \{U + V = 0\}$ is spacelike and is a Cauchy hypersurface for $(\mathcal{M}_{\text{Krus}}, g_{M,a})$.*

Proof. EXERCISE. Hint: To prove the Cauchy surface property, you could consider a past-directed and past-inextendible causal curve $\gamma(s)$, with s the proper time, and investigate the sign of $\frac{d}{ds}(U + V)(\gamma(s))$. □

6.4 Dynamics of null geodesics

We will investigate the behaviour of future-directed null geodesics on Kerr black hole exteriors $(\mathcal{M}_{\text{ext}}, g_{M,a})$. We proceed as in Schwarzschild by writing out the equation $g(\dot\gamma, \dot\gamma) = 0$:

$$0 = g_{M,a}(\dot\gamma, \dot\gamma) = -\Delta\rho^{-2}(\dot t - a\sin^2\theta\dot\varphi)^2 + \sin^2\theta\rho^{-2}(a\dot t - (r^2 + a^2)\dot\varphi)^2 + \frac{\rho^2}{\Delta}\dot r^2 + \rho^2\dot\theta^2. \qquad (6.1)$$

Using the Killing property of $T = \partial_t$ and $\Phi = \partial_\varphi$, we identify the following conserved quantities along γ:

$$\begin{aligned}
E =& - g_{M,a}(\dot\gamma, T) = -g_{tt}\dot t - g_{t\varphi}\dot\varphi = \rho^{-2}\left[\Delta - a^2\sin^2\theta\right]\dot t + a\sin^2\theta\rho^{-2}\left[r^2 + a^2 - \Delta\right]\dot\varphi,\\
L =& g_{M,a}(\dot\gamma, \Phi) = g_{\varphi\varphi}\dot\varphi + g_{t\varphi}\dot t = \sin^2\theta\rho^{-2}\left[(r^2 + a^2)^2 - a^2\sin^2\theta\Delta\right]\dot\varphi - a\sin^2\theta\rho^{-2}\left[r^2 + a^2 - \Delta\right]\dot t.
\end{aligned}$$

We interpret E as the energy of γ according to integral curves of T and L as the z-component of the angular momentum of γ. Note that E can in principle be non-positive if γ is restricted to the ergoregion (and integral curves of T are not timelike).

We can write these equations as the following single matrix equation:

$$\begin{pmatrix} E \\ L \end{pmatrix} = A \begin{pmatrix} \rho^{-2}\dot t \\ \rho^{-2}\sin^2\theta\dot\varphi, \end{pmatrix}$$

with A a $2x2$ matrix that is defined as follows:

$$A = \begin{pmatrix} \Delta - a^2\sin^2\theta & a(r^2 + a^2 - \Delta) \\ -a\sin^2\theta(r^2 + a^2 - \Delta) & (r^2 + a^2)^2 - a^2\sin^2\theta\Delta \end{pmatrix} = \begin{pmatrix} \rho^2 - 2Mr & 2Mra \\ -2Mar\sin^2\theta & (r^2 + a^2)\rho^2 + 2Mra^2\sin^2\theta \end{pmatrix}.$$

Then

$$\det A = \rho^4(r^2 + a^2) + 2Mra^2\sin^2\theta\rho^2 - 2Mr(r^2 + a^2)\rho^2 = \rho^4(r^2 + a^2 - 2Mr) = \rho^4\Delta.$$

Therefore

$$A^{-1} = \begin{pmatrix} \rho^{-4}((r^2 + a^2)^2\Delta^{-1} - a^2\sin^2\theta) & -\rho^{-4}a((r^2 + a^2)\Delta^{-1} - 1) \\ \rho^{-4}a\sin^2\theta((r^2 + a^2)\Delta^{-1} - 1) & \rho^{-4}(1 - a^2\sin^2\theta\Delta^{-1}) \end{pmatrix}$$

and

$$\begin{aligned}
\rho^2\dot t =& a(L - Ea\sin^2\theta) + \Delta^{-1}(r^2 + a^2)(E(r^2 + a^2) - aL),\\
\rho^2\sin^2\theta\dot\varphi =& - Ea\sin^2\theta + L + \Delta^{-1}a\sin^2\theta(E(r^2 + a^2) - aL).
\end{aligned}$$

In contrast with the Schwarzschild case, we cannot simply assume without loss of generality that $\theta \equiv \frac{\pi}{2}$. Instead, we will make use of the existence of an additional, hidden, conserved quantity:

Lemma 6.5. *Let γ be an affinely parametrized null geodesic and define*

$$Q := \rho^4\dot{\theta}^2 + \frac{L^2}{\sin^2\theta} - a^2E^2\cos^2\theta.$$

Then

$$\frac{d}{ds}Q(s) = 0.$$

Furthermore,

$$Q + a^2E^2 \geq L^2.$$

The quantity Q is called Carter's constant.

The existence of the conserved quantity Q is related to the existence of a Killing tensor $\mathcal{K} \in \mathcal{T}^{(0,2)}(\mathcal{M})$, which is a symmetric tensor field that satisfies the following generalization of the equation satisfied by a Killing vector field: $\nabla_{(\mu}\mathcal{K}_{\nu\rho)} = 0$.

[Exercise: Show that for γ an affinely parametrized geodesic, $\frac{d}{ds}(\mathcal{K}_{\nu\rho}(\gamma(s))\dot{\gamma}^\nu(s)\dot{\gamma}^\rho(s)) = 0$.]

Using the above lemma, we obtain the following additional equation:

$$\rho^4\dot{\theta}^2 = Q - \frac{L^2}{\sin^2\theta} + a^2E^2\cos^2\theta = Q + a^2E^2 - 2aEL - \frac{(L - aE\sin^2\theta)^2}{\sin^2\theta}$$

We plug the equations for $\dot{t}$, $\dot{\varphi}$ and $\rho^4\dot{\theta}^2$ in terms of E, L, Q, into (6.1) to obtain:

$$\begin{aligned}
0 = \rho^4\dot{r}^2 &- \Delta^2(\dot{t} - a\sin^2\theta\dot{\varphi})^2 + \sin^2\theta\Delta(a\dot{t} - (r^2+a^2)\dot{\varphi})^2 + \rho^4\Delta\dot{\theta}^2 \\
&= \rho^4\dot{r}^2 - \rho^{-4}((E(r^2+a^2) - aL)\rho^2)^2 \\
&+ \sin^2\theta\Delta\rho^{-4}(a^2(L - Ea\sin^2\theta) - Ea(r^2+a^2) + L(r^2+a^2)(\sin^2\theta)^{-1})^2 \\
&+ \Delta\left(Q + a^2E^2 - 2aEL - \frac{(L - aE\sin^2\theta)^2}{\sin^2\theta}\right) \\
&= \rho^4\dot{r}^2 - (E(r^2+a^2) - aL)^2 + \Delta\rho^{-4}\sin^2\theta((L - Ea\sin^2\theta)(r^2+a^2)\rho^2(\sin^2\theta)^{-1})^2 \\
&+ \Delta\left(Q + a^2E^2 - 2aEL - \frac{(L - aE\sin^2\theta)^2}{\sin^2\theta}\right) \\
&= \rho^4\dot{r}^2 - (E(r^2+a^2) - aL)^2 + \Delta\left(Q + a^2E^2 - 2aEL\right) \\
&= \rho^4\dot{r}^2 - E^2(r^2+a^2)^2 + 4MaELr - a^2L^2 + \Delta(Q + a^2E^2).
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{\rho^4}{(r^2+a^2)^2}\dot{r}^2 &= E^2 - V_0(r;E,L,Q), \\
V_0(r;E,L,Q) &:= \frac{1}{(r^2+a^2)^2}\left[4MaELr - a^2L^2 + \Delta(Q + a^2E^2)\right]
\end{aligned}$$

Now we can analyze the dynamics of null geodesics by determining the existence and location of maxima/minima of the potential V_0, just like in Schwarzschild. Recall that in Schwarzschild, for $\theta \equiv \frac{\pi}{2}$, we obtain $Q = L^2$ and we can write $V_0(r;L) = r^{-4}L^2\Delta$.

Proposition 6.6. *The potential function $V_0 : (r_+, \infty) \to \mathbb{R}$ either:*

- *is non-increasing,*
- *has a unique critical point $r_+ \leq r_{\max}$ that corresponds to a global maximum, or*

- *has exactly two critical points $r_+ \leq r_{\min} < r_{\max}$ that correspond to a local minimum at $r_{\min}$ and a local maximum at $r_{\max}$.*

Proof. To determine the critical points of V_0, we compute:

$$\begin{aligned}\frac{d}{dr}V_0(r) &= \frac{1}{(r^2+a^2)^3}\left[-4r(4MarEL - a^2L^2 + \Delta(Q+a^2E^2)) + (r^2+a^2)(4MaEL + 2(r-M)(Q+a^2E^2))\right] \\ &= \frac{1}{(r^2+a^2)^3}\left[4MaEL(-3r^2+a^2) + 4ra^2L^2 - 2(Q+a^2E^2)(r^3 - 3Mr^2 + a^2r + Ma^2)\right].\end{aligned}$$

We will investigate first the critical points of $(r^2+a^2)^3\frac{dV_0}{dr}(r)$. We have that:

$$\begin{aligned}\frac{d}{dr}((r^2+a^2)^3\frac{dV_0}{dr}(r)) &= -24MaELr + 4a^2L^2 - 2(Q+a^2E^2)(3r^2 - 6Mr + a^2) \\ &= -6(Q+a^2E^2)\left[r^2 - 2M(1-2\xi)r + \frac{a^2}{3}(1-2\mu)\right],\end{aligned}$$

where we introduced the variables $\xi = EL(Q+a^2E^2)^{-1}$ and $\mu = L^2(Q+a^2E^2)^{-1}$. The critical points of $(r^2+a^2)^3\frac{dV_0}{dr}(r)$ correspond to the roots of the above quadratic polynomial:

$$r_{1,2} = M(1-2\xi) \mp \sqrt{M^2(1-2\xi)^2 - \frac{a^2}{3}(1-2\mu)}$$

We now distinguish two cases:

Case I: $\xi \geq 0$ Since $r_1 \leq M(1-2\xi) \leq M \leq r_+$ (with equality iff $|a| = M$), we have that the only possible critical point in (r_+, ∞) is r_2.

Case II: $\xi < 0$ In this case, we can express:

$$r_1 = M(1-2\xi)\left[1 - \sqrt{1 - \frac{a^2}{3M^2}\frac{1-2\mu}{(1-2\xi)^2}}\right]$$

Note that

$$\frac{a^2}{3M^2}\frac{1-2\mu}{(1-2\xi)^2} < \frac{a^2}{3M^2} \leq \frac{1}{3},$$

where we used that $\xi < 0$, $\mu \geq 0$ and $|a| \leq M$.

Let $0 \leq x \leq \frac{1}{3}$. Then we use that

$$1 - x \geq 1 - x - \frac{x}{3}\left(1 - \frac{4}{3}x\right) = \left(1 - \frac{2}{3}x\right)^2,$$

to obtain $\sqrt{1-x} \geq 1 - \frac{2}{3}x$ and therefore

$$r_1 \leq M(1-2\xi)\frac{2}{3}\frac{a^2}{3M^2}\frac{1-2\mu}{(1-2\xi)^2} = \frac{2a^2}{9M}\frac{1-2\mu}{1-2\xi} \leq \frac{2a^2}{9M} \leq \frac{2}{9}r_+.$$

So in Case II, we also conclude that $(r^2+a^2)^3\frac{dV_0}{dr}(r)$ has at most one critical point, r_2, in (r_+, ∞).

Since $Q + a^2E^2 \geq 0$, we have that

$$\lim_{r\to\infty}(r^2+a^2)^3\frac{dV_0}{dr}(r) = -\infty.$$

This implies the following properties.

- $V_0(r)$ has at most two critical points. Indeed, suppose there are ≥ 3 critical points. Then $(r^2+a^2)^3\frac{dV_0}{dr}(r)$ must have at least 3 zeroes and hence, at least two critical point, which is in contradiction with the existence of at most one critical point, r_2.

- Suppose that $V_0(r)$ has no critical points. Then $(r^2+a^2)^3\frac{dV_0}{dr}(r)$ must have a constant sign. By $\lim_{r\to\infty}(r^2+a^2)^3\frac{dV_0}{dr}(r)=-\infty$, this sign must be negative.

- Suppose that $V_0(r)$ has exactly one critical point. By the limiting property of $(r^2+a^2)^3\frac{dV_0}{dr}(r)$, it must be a maximum.

- Suppose that $V_0(r)$ has exactly two critical points. By the limiting property of $(r^2+a^2)^3\frac{dV_0}{dr}(r)$, the critical point with the largest r-value must be a local maximum. The other critical point cannot be a maximum, since then there would have to be a third critical point in between, which would contradict the existence of exactly two critical points. Similarly, the other critical point cannot be a saddle point, because then it would be a critical point and a zero of $(r^2+a^2)^3\frac{dV_0}{dr}$. However, since $(r^2+a^2)^3\frac{dV_0}{dr}$ must also be zero at the maximum of V_0, it would have another critical point between the maximum and saddle points of V_0, which contradicts the existence of at most one critical point for $(r^2+a^2)^3\frac{dV_0}{dr}$. Hence, V_0 in this case has a local minimum and a local maximum. □

The possible existence of a minimum for $V_0(r)$ might suggest the existence of trapped null geodesics that are stable under small perturbations of their initial data in phase space. The proposition below demonstrates that this is not the case: trapped null geodesics are also unstable in Kerr.

The following conserved quantity will play an important role:

$$E_K := -g_{M,a}(\dot\gamma,K) = -g_{M,a}\left(\dot\gamma, T+\frac{a}{r_+^2+a^2}\Phi\right) = E-\frac{a}{r_+^2+a^2}L = E-\frac{a}{2Mr_+}L = E-\omega_+L.$$

At $\mathcal{H}^+$, where K is causal and future-directed, we must have that $E_K\geq 0$ and we can interpret E_K as the energy of $\dot\gamma$ as measured by observers rotating with the black hole.

Proposition 6.7. *We have that*

$$E^2\geq V_0(r_{\min})$$

with equality if and only if $E_K=0$ and $r_{\min}=r_+$, which can only occur when the Kerr black hole is extremal. In particular, if $E_K\neq 0$ or $r_{\min}\neq r_+$, then there are no trapped null geodesics at $r=r_{\min}$.

If $aE_KL<0$, then $\frac{dV_0}{dr}(r_+)>0$ and $V_0(r)$ has exactly one critical point: a maximum at $r_{\max}$.

Proof. We first compute, using that $r_+^2+a^2-2Mr_+=0$:

$$\begin{aligned}E^2-V_0(r_+) &= E^2-\frac{1}{(r_+^2+a^2)^2}(4MaELr_+-a^2L^2)\\ &= \frac{E^2(r_+^2+a^2)^2-4MaELr_++a^2L^2}{(r_+^2+a^2)^2} = \left(\frac{2MEr_+-aL}{r_+^2+a^2}\right)^2 = E_K^2.\end{aligned}$$

Hence, if $E_K\neq 0$, then

$$E^2>V_0(r_+)\geq V_0(r_{\min}).$$

If $r_{\min}\neq r_+$, then $E^2\geq V_0(r_+)>V_0(r_{\min})$.

Note moreover that

$$
\begin{aligned}
(r_+^2+a^2)^3\frac{dV_0}{dr}(r_+) &= -4r_+(r_+^2+a^2)^2V_0(r_+)+(r_+^2+a^2)(4MaEL+\frac{d\Delta}{dr}(Q+a^2E^2))\\
&= -4r_+(4Mar_+EL-a^2L^2)+8MaELr_++2(Q+a^2E^2)(r_+^2+a^2)(r_+-M)\\
&= -8Mr_+^2aE_KL+2(Q+a^2E^2)(r_+^2+a^2)(r_+-M).
\end{aligned}
$$

Hence, we can only have $r_+ = r_{\min}$ and $E^2 - V_0(r_+) = 0$ if $Q + a^2E^2 = 0$ or $r_+ = M$, the latter which is equivalent to $|a| = M$. Note however that since $Q^2 + a^2E^2 \geq L^2$, the first case implies that $L^2 = Q^2 + a^2E^2 = 0$, so $V_0 \equiv 0$ and there does not exist a $r_{\min}$

The above computation implies moreover that in the case $aE_KL < 0$:

$$
\frac{dV_0}{dr}(r_+) > 2(Q+a^2E^2)(r_+^2+a^2)^{-2}(r_+-M) \geq 0.
$$

We cannot have the existence of $r_{\min}$, since that would require $\frac{dV_0}{dr}(r_+) \leq 0$. □

Remark 6.1. *The above argument also excludes the existence of trapped null geodesics with non-constant (oscillating) r-values (r bounded below and above).* [EXERCISE*: Convince yourself of this.*]

[EXERCISE: Suppose that $E < 0$. Show that a future-directed affinely parametrized null geodesic γ in $\mathcal{M}_{\text{ext}}$ must intersect the future event horizon $\mathcal{H}_R^+$. Hint: Use that γ must remain in the ergoregion. Then suppose there exists a future-directed null geodesic with $E < 0$ approaching $r = r_{\max}$ asymptotically. Use that the geodesic can be perturbed by slightly changing (E, L, Q) to obtain $E < 0$ and $E^2 > V_0(r_{\max})$ or $E^2 > V_0(r_{\max})$ and use this to reach a contradiction.]

The presence of an ergoregion allows one to "extract energy" from a black hole. We will first explore this in a heuristic setting. Let γ be a future-directed causal geodesic with energy E and angular momentum L that enters a Kerr black hole with mass M_i and angular momentum $J_i = a_iM_i$. We will assume that this dynamical process leads to a black hole that settles down to another Kerr black hole with mass M_f and angular momentum J_f, where $M_f = M_i + E$ and $J_f = J_i + L$. Since E can be negative, the energy or mass of the black hole can decrease. Extracting this energy difference is known as the *Penrose process*.

Since γ intersects $\mathcal{H}^+$, we must have that $E_K = E - \omega_+L \geq 0$. This implies that:

$$
\begin{aligned}
J_f - J_i = L = \frac{1}{\omega_+}(E-E_K) = \frac{1}{\omega_+}(M_f-M_i) - \frac{1}{\omega_+}E_K &= \frac{2M_ir_+}{a_i}(M_f-M_i) - \frac{1}{\omega_+}E_K\\
&= \frac{2M_i(M_i^2+\sqrt{M_i^4-J_i^2})}{J_i}(M_f-M_i) - \frac{1}{\omega_+}E_K,
\end{aligned}
$$

where we plugged in the expression for ω_+ and r_+ with respect to M_i and J_i, assuming that $E \ll M_i$ and $L \ll J_i$. We can use this inequality to derive properties of the so-called *irreducible mass* M_{irr}, which is defined as follows:

$$
M_{\text{irr}}(M,J) := \sqrt{\frac{1}{2}(M^2+\sqrt{M^4-J^2})}.
$$

By applying Taylor's theorem, we can obtain

$$
\begin{aligned}
(M_{\text{irr}}^2)_f - (M_{\text{irr}}^2)_i &= \frac{\partial M_{\text{irr}}^2}{\partial M}|_{M=M_i,J=J_i}(M_f-M_i) + \frac{\partial M_{\text{irr}}^2}{\partial J}|_{M=M_i,J=J_i}(M_f-M_i)(J_f-J_i)+O(E^2)+O(L^2)\\
&= \frac{J_i}{2\sqrt{M_i^4-J_i^4}}\left[\frac{2M_i(M_i^2+\sqrt{M_i^4-J_i^2})}{J_i}(M_f-M_i)-(J_f-J_i)\right]+O(E^2)+O(M_i^{-2}L^2)
\end{aligned}
$$

$$\geq \frac{J_i}{2\sqrt{M_i^4 - J_i^4}\omega_+} E_K + O(E^2) + O(M_i^{-2}L^2) = \frac{M_i^2 r_+}{2\sqrt{M_i^4 - J_i^4}} E_K + O(E^2) + O(M_i^{-2}L^2).$$

Since $EM_i^{-1} \ll 1$ and $LM_i^{-2} \ll 1$, the right hand side above is positive if $M_i^{-1}E_K \gg M_i^{-2}E^2 + M_i^{-4}L^2$. In this case, M_{irr} is increasing as consequence of this dynamical process.

The name "irreducible mass" comes from the following inequality:

$$M_{\text{irr}}^2 = \frac{1}{2}(M^2 + \sqrt{M^4 - J^2}) \leq M^2,$$

so the mass M must be bounded below by M_{irr}. The Penrose process cannot lead to a decrease in black hole mass below $M_{\text{irr}}(M_i, J_i)$.

The concept of irreducible mass was introduced by Christodoulou in his PhD thesis and it played an important role in the development of black hole thermodynamics, where M_{irr}^2 can be interpreted as a non-decreasing *entropy of the Kerr black hole.*

[EXERCISE: Show that the area A of the spheres of intersection of $\mathcal{H}^+$ and $\{v = \text{const.}\}$ satisfies $A = 16\pi M_{\text{irr}}^2$.]

The phenomenon of *superradiance* is also related to the ability to extract energy from a black hole. Consider solutions ϕ to the wave equation on a (sub)-extremal Kerr background $\Box_{g_{M,a}}\phi = 0$, such that that ϕ vanishes at the past event horizon $\mathcal{H}_L^-$ and has energy equal to 1 at $\mathcal{I}_R^-$. The energy at $\mathcal{I}_R^+$ (that does not escape to the black hole) can then be bigger than 1: there can be an energy amplification caused by the presence of a black hole. If we study this process in frequency space by taking a Fourier transform of the solution to the wave equation and decomposing $\Phi = \sum_{m\in\mathbb{Z}} \phi_m$, where $\Phi(\phi_m) = im\phi_m$, we will see that only the following frequency range can lead to an energy increase:

$$0 < am\omega < a\omega_+ m^2.$$

These are called *superradiant frequencies.*

The phenomenon of superradiance is intimately connected to the behaviour of null geodesics. At the level of null geodesics, the superradiance frequency regime corresponds to the condition:

$$0 < aL \cdot E < \frac{a^2L^2}{2Mr_+} = a\omega_+ L^2.$$

We will refer to null geodesics satisfying this condition as "superradiant null geodesics". For null geodesics with $E > 0$, this amounts to the condition that the geodesic is rotating in the same direction as the black hole and $E_K < 0$. This implies in particular that the geodesic cannot cross $\mathcal{H}_R^+$, where necessarily $E_K > 0$, since K and $\dot{\gamma}$ are causal and future-directed at $\mathcal{H}_R^+$.

The following proposition shows that superadiant null geodesics cannot be (unstably) trapped in sub-extremal black hole exteriors. At the level of the wave equation, this is crucial for establishing decay of waves on Kerr black hole backgrounds and, ultimately, the conjectured (as of the writing of these notes) stability of all sub-extremal Kerr black holes under suitably small initial data perturbations.

Proposition 6.8. *Let γ be a superradiant null geodesic on a sub-extremal Kerr black hole. Then γ cannot be trapped, i.e. $V_0(r_{\max}) - E^2 > 0$.*

Proof. We may assume without loss of generality that $a > 0$, so $a < M$. We have that $aE_K L < 0$ so by the previous proposition, V_0 has exactly one critical point, a maximum $r_{\max}$. Denote $\Lambda := Q + a^2E^2$. We will consider three cases: I) $E_K \leq \epsilon\sqrt{\Lambda}$, II) $E^2 \leq \epsilon\Lambda$ and III) $E_K > \epsilon\sqrt{\Lambda}$ and $E^2 > \epsilon\Lambda$, for some suitably small $\epsilon > 0$.

Case I: $E_K \leq \epsilon\sqrt{\Lambda}$ By the proof of the above proposition

$$E^2 - V_0(r_+) = E_K^2 \leq \epsilon^2\Lambda.$$

We also showed that

$$\frac{dV_0}{dr}(r_+) \geq 2\Lambda(r_+^2 + a^2)^{-2}(r_+ - M) \geq 4r_+^{-1}\Lambda$$

Hence, there exists a $0 < \epsilon^2 < \delta$ such that

$$V_0(r_+(1+\delta)) - E^2 \geq (2\delta - \epsilon^2)\Lambda.$$

so in particular, $V_0(r_{\max}) - E^2 \geq (2\delta - \epsilon^2)\Lambda$.

Case II: $E^2 \leq \epsilon\Lambda$ As $r \to \infty$, we have in this case (using that $L^2 \leq \Lambda$ and $E^2 \leq \epsilon\Lambda$):

$$V_0(r) - E^2 = \Lambda r^{-2} - E^2 + \Lambda O(r^{-3}).$$

Hence, there exists a $R > r_+$ suitably large and an $\epsilon > 0$ suitably small such that

$$V_0(R) - E^2 \geq \frac{1}{2}\Lambda R^{-2} - E^2 \geq (\frac{1}{2} - \epsilon)\Lambda R^{-2}.$$

We conclude also in this case that $V_0(r_{\max}) - E^2 \geq V_0(R) - E^2 \geq \left(\frac{1}{2} - \epsilon\right)\Lambda R^{-2}$.

Case III: $E_K > \epsilon\sqrt{\Lambda}$ **and** $E^2 > \epsilon\Lambda$ Let $r_0 = \frac{aL^2}{2MLE}$. Since $0 < EL \leq \frac{a}{2Mr_+}L^2(1 - \epsilon^2|L|^{-1}\sqrt{\Lambda})$, we can estimate

$$r_0 \geq r_+\frac{1}{1 - \epsilon^2|L|^{-1}\sqrt{\Lambda}} \geq (1-\epsilon^2)^{-1}r_+ > r_+.$$

Furthermore, it can be shown that (Exercise):

$$V_0(r_0) - E^2 = \frac{\Delta(r_0)}{(r_0^2 + a^2)^2}\left(\Lambda - \frac{a^2L^2}{4M^2}(1 + 2Mr_0^{-1} + a^2r_0^{-2})\right) \geq \frac{\Delta(r_0)}{4M(r_0^2 + a^2)^2}(M-a)\Lambda$$

where we used that $L^2 \leq \Lambda$ and $1 + 2Mr^{-1} + a^2r^{-2} \leq 1 + 2Ma^{-1} + a^2a^{-2} = 2(1 + \frac{a}{M}) = 4 - 2\frac{M-a}{M}$.

Hence, $V_0(r_{\max}) - E^2 \geq V_0(r_0) - E^2 \geq b\Lambda$, for some $b > 0$. □

Exercises:

1. Show that trapped null geodesics that are orthogonal to the integral curves of Φ must lie outside the ergoregion. [Hint: Show that $\frac{dV_0}{dr} > 0$ in the ergoregion.]

2. Show that in the extremal case, these null geodesics are located at $r = (1+\sqrt{2})M$. The hypersurface $\{r = (1+\sqrt{2})M\}$ therefore serves as an *effective photon sphere* in this case.

3. ($\star$) Let $b = \frac{|L|}{E}$ be the impact parameter associated to an untrapped null geodesic that does not cross the event horizon. Show that $b_c = \inf_{Q,E}\frac{|L|}{E}$ is larger when $aL < 0$ than when $aL > 0$. [Hint: Show and use that $E^2 < V_0(r_{\max})$, with V_0 attaining its maximum at $r_{\max} \geq r_+$.] Explain why a far-away observer on the equatorial plane ($\theta = \frac{\pi}{2}$) will see a shadow that "bulges outwards" in one direction when looking at a rotating black hole with a uniformly distributed shell of stars surrounding both the observer and black hole, when compared to a round spherical shadow. On which side of the black hole will the shadow bulge outwards? You may use that, as in the Schwarzschild case, b_c can be related to the size of the shadow in the equatorial plane .

We will not discuss in detail the dynamics of timelike geodesics, but it can be shown that, just like in Schwarzschild, there exist stable orbits around Kerr black holes. In contrast with Schwarzschild, these can occur at r-values arbitrarily close to $r = r_+$ for $1 - \frac{|a|}{M}$ suitably small.

7 Asymptotic flatness

Throughout the lectures, we have used the term "asymptotically flat" in a rather vague sense. In particular, we referred to the Kerr and Reissner–Nordström spacetime families as being asymptotically flat. We will now give a precise definition of asymptotic flatness at the level of spacelike hypersurfaces that we can also interpret as "initial data hypersurfaces". We will then use this to state some important propositions and theorems without proof.

First, we need to define the notion of *extrinsic curvature*:

Definition 7.1. *Let $(\mathcal{M}, g)$ be a spacetime with a spacelike hypersurface Σ. Let $N \in \mathcal{T}(M)$ be (an extension) of the future-directed normal vector field to Σ. The* extrinsic curvature *or the* second fundamental form *of Σ is the tensor field $k \in \mathcal{T}^{(0,2)}(\Sigma)$ defined as follows: let $X, Y \in \mathcal{T}(\Sigma)$, then*

$$k(X,Y) := g(\nabla_X N, Y)|_{\Sigma}.$$

[Exercise: 1) Show that k is symmetric. 2) Show that we can express $k(X,Y) = \frac{1}{2}(\mathcal{L}_N g)(X,Y)$ for $X, Y \in \mathcal{T}(\Sigma)$.]

Definition 7.2. *Let $(\mathcal{M}, g)$ be an 4-dimensional spacetime with a spacelike hypersurface Σ and corresponding induced metric $\overline{g}$. The hypersurface Σ is said to have an* asymptotically flat end *if there exists an open subset $\Sigma_{\text{ext}} \subset \Sigma$, such that:*

1. *Σ_{ext} is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_R(0)}$, with $\overline{B_R(0)}$ the closed ball of radius R centred around $0 \in \mathbb{R}^3$.*
2. *There exists a coordinate chart (x^1, x^2, x^3) on Σ_{ext}, such that with respect to this coordinate chart:*

$$\overline{g}_{ij} = \delta_{ij} + o_2(r^{-\alpha}),$$
$$k_{ij} = o_1(r^{-\alpha-1}),$$

for $\mathbb{R} \ni \alpha > \frac{1}{2}$, with δ_{ij} the Kronecker delta. Here the "Little-o" notation $o_k(r^{-\alpha})$ groups functions $f : \Sigma_{\text{ext}} \to \mathbb{R}$ that satisfy for $I \in \mathbb{N}^3$, $|I| = k$:

$$\lim_{r\to\infty} \sup_{\partial B_r(0)} |r^{\alpha+k}\partial_{x^1}^{I_1}\partial_{x^2}^{I_2}\partial_{x^3}^{I_3} f|_{\partial B_r(0)}| = 0,$$

for $r := \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

We say Σ is a one-ended asymptotically flat hypersurface*(-with-boundary) if $\Sigma \setminus \Sigma_{\text{ext}}$ is compact.*

7.1 Conservation of the total energy-momentum

With the notion of asymptotic flatness, we can define the following notions of total *energy*, *linear momentum* and *angular momentum* associated to a one-ended asymptotically flat hypersurface:

Proposition 7.1. *Let Σ be one-ended asymptotically flat hypersurface, with induced metric $\overline{g}$ and second fundamental form k. Let (x^1, x^2, x^3) be a global coordinate chart with respect tot which asymptotic flatness is defined.*

(i) *Then the* ADM energy[30] *$E \in \mathbb{R}$ and* ADM linear momentum *$\mathbf{P} \in \mathbb{R}^3$ are defined as follows:*

$$E := \frac{1}{16\pi} \lim_{r\to\infty} \int_{\partial B_r(0)} \sum_{j,m=1}^{3} (\partial_j \overline{g}_{jm} - \partial_m \overline{g}_{jj}) n^m \, r^2 d\sigma,$$

[30] The acronym "ADM" stands for Arnowitt–Deser–Misner.

$$P_i := -\frac{1}{8\pi}\lim_{r\to\infty}\int_{\partial B_r(0)}\sum_{j=1}^{3}(k_{ij}-\overline{g}_{ij}\,\mathrm{tr}\,k)n^j\,r^2 d\sigma,$$

with $r^2 = x^2+y^2+z^2$, $d\sigma = \sin\theta d\theta d\varphi$ the standard volume form on the unit round sphere in $\mathbb{R}^3$, $n = (x^2, x^2, x^3)^T$ the outward unit normal to $\partial B_r(0)$ in $\mathbb{R}^3$.

The quantities E and P_i are finite and independent of the choice of coordinate chart (x^1, x^2, x^3) corresponding to asymptotic flatness.

(ii) *If we make the stronger assumptions:*

$$\begin{aligned}\overline{g}_{ij} &= \left(1+\frac{2M}{r}\right)\delta_{ij} + O_2(r^{-1-\epsilon}),\\ k_{ij} &= O_1(r^{-2-\epsilon}),\end{aligned}$$

for some $\epsilon > 0$,[31] *then $\mathbf{P} = 0$, $E = M$ and the* ADM angular momentum $\mathbf{J}\in\mathbb{R}^3$ *associated to Σ is determined by the following well-defined expressions:*

$$J_i := -\frac{1}{8\pi}\lim_{r\to\infty}\int_{\partial B_r(0)}\sum_{j,m,n=1}^{3}\epsilon_{ijm}(k_{mn}-\overline{g}_{mn}\,\mathrm{tr}\,k)x^j n^n\,r^2 d\sigma,$$

with ϵ_{ijk} the Levi-Civita symbol. The quantity $\mathbf{J}$ is independent of the choice of coordinate chart (x^1, x^2, x^3) corresponding to asymptotic flatness

By considering appropriate coordinates in the Kerr black hole exterior, it can be shown that $E = M$ and $\mathbf{J} = aM(0,0,1)^T$, with M, a the Kerr parameters. Furthermore, by making the following global assumption: there exists a coordinate chart (t, x^1, x^2, x^3) such that

$$g_{\mu\nu} = m_{\mu\nu} + o_2(r^{-\alpha}),$$

with $\alpha > \frac{1}{2}$ and $m_{\mu\nu}$ the Minkowski metric components with respect to Cartesian coordinates, it can be shown that the quantities E and $\mathbf{P}$ are also independent of the precise choice of hypersurface Σ: they are conserved quantities. With respect to a suitable choice of time function, the same can be shown for $\mathbf{J}$.

7.2 Black hole uniqueness theorems

In this section, we will state the *black hole uniqueness theorems*, which roughly state that asymptotically flat, stationary black hole spacetimes must be isometric to members of the Kerr family. To state these theorems more precisely, we first need a suitable notion of stationarity.

Definition 7.3. *A spacetime $(\mathcal{M}, g)$ with a one-ended asymptotically flat hypersurface Σ is* stationary *if there exists a Killing vector field T, which is timelike on Σ_{ext} and complete (i.e. the domain of its integral curves is $\mathbb{R}$).*

Let $\mathbb{R}\times\Sigma_{\mathrm{ext}} \subseteq \mathcal{M}$ be the spacetime region obtained by flowing along the integral curves of T, starting from Σ_{ext}. We will use this definition (in place of future null infinity), to define the domain of outer commutations and the black hole region.

[31] Here we applied the "Big-O notation" $O_k(r^{-\alpha})$, to group functions $f : \Sigma_{\mathrm{ext}} \to \mathbb{R}$ satisfying $r^{\alpha+k}\sup_{\partial B_r(0)}|\partial^{I_1}_{x^1}\partial^{I_2}_{x^2}\partial^{I_3}_{x^3} f|_{\partial B_r(0)}| \leq C_k$ for some constant $C_k > 0$ and $I\in\mathbb{N}^3$, $|I| = k$.

Definition 7.4. *We define the* domain of outer communications $\mathcal{M}_{\text{ext}}$ *as follows:*

$$\mathcal{M}_{\text{ext}} := I^+(\mathbb{R} \times \Sigma_{\text{ext}}) \cap I^-(\mathbb{R} \times \Sigma_{\text{ext}}).$$

Then the black hole region $\mathcal{BH}$,[32] *the* future event horizon $\mathcal{H}^+$ *and the* past event horizon $\mathcal{H}^-$ *as defined as follows:*

$$\begin{aligned}\mathcal{BH} &:= I^+(\mathbb{R} \times \Sigma_{\text{ext}}) \setminus \mathcal{M}_{\text{ext}},\\ \mathcal{H}^{\pm} &:= \partial\mathcal{M}_{\text{ext}} \cap I^{\pm}(\mathbb{R} \times \Sigma_{\text{ext}}).\end{aligned}$$

It can be shown that $\mathcal{H}^{\pm}$ are null hypersurfaces, so they are foliated by non-intersecting null geodesics, the "null generators" of $\mathcal{H}^+$.

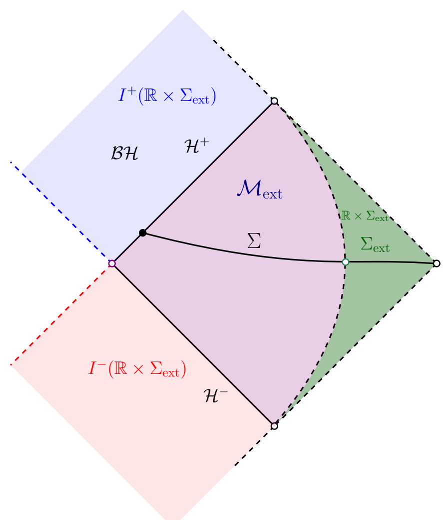


Figure 34: An picture of the objects introduced in Definition 7.4 in a Penrose diagram of sub-extremal Reissner–Nordström.

Finally, we define the notion of *axisymmetry* of a spacetime.

Definition 7.5. *A spacetime* $(\mathcal{M}, g)$ *is* axisymmetric *if there exists a Killing vector field* Φ *with periodic integral curves and an* axis of rotation*, which is a two-dimensional, totally geodesic submanifold*[33] *of* $\mathcal{M}$ *at which* Φ *vanishes.*

Now we obtain the following *rigidity theorem* for stationary, axisymmetric vacuum spacetimes:

Theorem 7.2 (Carter 1971, Chruściel–Costa 2008, Chruściel–Nguyen 2010)**.** *Let* $(\mathcal{M}, g)$ *be a stationary and axisymmetric spacetime solution to* $\text{Ric}[g] = 0$ *with a one-ended asymptotically flat hypersurface* Σ*. Assume additionally that*

- $\mathcal{M}_{\text{ext}}$ *is globally hyperbolic,*
- $\mathcal{H}^+$ *is connected,*
- *If* $\mathcal{H}^+ \neq \emptyset$*, then* $\partial\Sigma \subset \mathcal{H}^+$ *and* $\partial\Sigma$ *intersects each null generator of* $\mathcal{H}^+$ *exactly once,*

[32] The white hole region can be defined analogously as $I^-(\mathbb{R} \times \Sigma_{\text{ext}}) \setminus \mathcal{M}_{\text{ext}}$.
[33] This means: every geodesic that is initially tangent to the submanifold, stays tangent to the submanifold.

Then $\mathcal{M}_{\text{ext}}$ *is isometric to the domain of outer communications of a Kerr spacetime with* $|a| \leq M$ *(if* $\mathcal{H}^+ \neq \emptyset$*) or to the Minkowski spacetime (if* $\mathcal{H}^+ = \emptyset$*).*

One can remove the axisymmetry assumption, but this comes at a high cost: the restriction to *analytic* spacetimes.

Theorem 7.3 (Hawking 1972, Chruściel–Costa 2008, Chruściel–Nguyen 2010)**.** *Let* $(\mathcal{M}, g)$ *be a stationary and* ***analytic*** *spacetime solution to* $\mathrm{Ric}[g] = 0$ *with a one-ended asymptotically flat hypersurface* Σ*. Assume additionally that*

- $\mathcal{M}_{\text{ext}}$ *is globally hyperbolic,*
- $\mathcal{H}^+$ *is connected,*
- *If* $\mathcal{H}^+ \neq \emptyset$*, then* $\partial\Sigma \subset \mathcal{H}^+$ *and* $\partial\Sigma$ *intersects each null generator of* $\mathcal{H}^+$ *exactly once,*

Then $\mathcal{M}_{\text{ext}}$ *is isometric to the domain of outer communications of a Kerr spacetime with* $|a| \leq M$ *(if* $\mathcal{H}^+ \neq \emptyset$*) or to the Minkowski spacetime (if* $\mathcal{H}^+ = \emptyset$*).*

It remains an open problem whether the analyticity assumption can be removed in the above problem!