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Asymptotic Double Null Coordinates in Kerr Spacetime

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1 Introduction

The theory of general relativity was published by Albert Einstein in 1915 and has been a cornerstone of our understanding of the universe since then. The principal idea is that gravity, classically being thought of as a force, is the result of spacetime being curved and not flat. Even nowadays, more than 100 years later, there are still a lot of unresolved questions regarding general relativity. There is ongoing research, both from the theoretical and the experimental perspective, in order to understand the implications and limits of the theory.

The main equations in general relativity are the Einstein field equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R the Ricci scalar, $g_{\mu\nu}$ the spacetime metric and $T_{\mu\nu}$ the stress-energy tensor. In the case of a vacuum spacetime (i.e. $T_{\mu\nu} = 0$) these equations reduce to the Einstein vacuum equations

$$R_{\mu\nu} = 0. \tag{1.2}$$

While these equations look simple at first glance, they constitute a system of highly non-linear partial differential equations (PDEs). In particular, due to the non-linearities, there is not a unique vacuum solution, as we would expect for the gravitational field in vacuum in Newtonian gravity. Finding explicit solutions, i.e. a spacetime metric $g_{\mu\nu}$, is very hard and only a handful are known.

It is thus remarkable that not even a year after the theory was published, an exact solution to the Einstein vacuum equations, now called the Schwarzschild solution, was found by Karl Schwarzschild. His solution describes the exterior spacetime of a spherically symmetric, non-rotating matter distribution. However, his solution was initially met with scepticism due to its singular behaviour at certain coordinate values. It was not until 1958 that the significance of this coordinate singularity was clarified, now being understood as an event horizon — a boundary in spacetime representing the limit beyond which no timelike or lightlike signal can escape. The region beyond the event horizon was coined as a *black hole*. In 1963, Roy Kerr found a more general solution to the Einstein equations, which describes the spacetime of an uncharged, rotating black hole. This metric, now called the Kerr metric, depends on two parameters: The mass M and the angular momentum a. It is of special interest from both the observational and theoretical perspective because it is thought of as being a good toy model for understanding realistic black holes as observed within our cosmos [3].

While studying phenomena such as gravitational waves or black hole stability, it is beneficial to use a coordinate system which is adapted to the causal structure of spacetime. In the so-called double null coordinate system, the metric is expressed in terms of two null coordinates (also called optical functions) u and v, which are defined as solutions to the Eikonal equation $g(\nabla u, \nabla u) = 0$. This means that the hypersurfaces of constant u, v are null hypersurfaces, making this coordinate system particularly useful for dealing with the propagation of light or gravitational waves, i.e. studying the wave equation in general, globally hyperbolic spacetimes. The double null coordinate system often simplifies calculations and offers a more comprehensible physical interpretation of the results. That is because in this coordinate system the vacuum Einstein equations can be decomposed into several transport equations along the null hypersurfaces of constant u or v. We will see a short glimpse of this in section 2.4. Moreover, double null coordinates are used in the construction of Penrose diagrams, which are helpful for intuitively understanding the causal structure of a given spacetime. These diagrams compactify infinite regions of spacetime onto finite diagrams, while leaving the causal structure intact. Thus, making it easier to understand the causal relationships between different events, especially in the case of black hole spacetimes.

For the aforementioned advantages, having an expression for the Kerr metric in double null coordinates would be very useful. In Minkowski and Schwarzschild spacetime, the optical functions can be obtained explicitly and globally (apart from $r = 0$) and are thus well understood. Due to the absence of spherical symmetry and the twisting of null geodesics, the situation is much more elusive in the case of Kerr spacetime. While Pretorius and Israel have already studied double null coordinates in Kerr spacetime [5], a formal proof of their global existence is not given by them. Dafermos and Luk proved the existence of double null coordinates in the interior of sub-extremal ($a < M$) Kerr spacetimes [4]. However, their proof does not include the extremal case $a = M$ or the existence in the exterior region $r > r_+$ of the black hole.

The goal of this thesis is to prove the existence of double null coordinates in the exterior of Kerr spacetime, in particular in the asymptotic (far away from the event horizon) region where the radial distance r is large. This is equivalent to saying that the reduced mass $\mu = M/r$ is small (< 1) and hence close to the value of Minkowski spacetime (in which it is 0). The construction of general solutions to the Eikonal equation in Kerr spacetime relies on finding solution to a constraint equation $F = 0$. Hence, in order to show that a double null foliation indeed exists, we want to show the existence of solutions to this constraint equation in Kerr spacetime. For this we want to make use of the fact that we can solve the Eikonal equation in Minkowski spacetime explicitly and therefore have a solution to this constraint equation. By the implicit function theorem (under suitable conditions), we are assured that a solution exists locally around a point for small enough μ. The idea is to show that we can incrementally expand this region, eventually arriving at the whole exterior. For this purpose, we will be using the so-called bootstrap method, which can be thought of as a generalized induction principle over $\mathbb{R}$ [6].

This thesis is structured as follows: In the beginning we define and study double null coordinates in general spacetimes. After this we review the work of Pretorius and Israel [5] on null hypersurfaces in Kerr spacetimes and study the simplified case of Minkowski spacetime based on the methods used by Pretorius and Israel. In the end, building up on this work, we consider the general case of Kerr spacetime. We proof of existence of a solution to the constraint equation in (a patch of) the exterior region using the bootstrap method.

2 Double Null Coordinates

In this chapter we want to study the null geometry of a given spacetime. For this we will introduce and give a local construction of the double null coordinate system and have a short look on its properties, in particular the form of the metric in double null coordinates. This introduction is based on the instructive lecture notes of Stefanos Aretakis on general relativity [1].

2.1 Null Hypersurfaces

In the following let (M, g) be a four-dimensional Lorentzian manifold. For the moment we do not need to assume that g is a solution to the Einstein equations. The geometric idea behind the double null coordinate system is to find a foliation of M via null hypersurfaces. Thus, it is important to have a good understanding of the geometric properties of null hypersurfaces.

A hypersurface is called null (or lightlike) if its normal vector fields are null vector fields. In other words: Let $H \subset M$ be a hypersurface with unit normal vector field $L \in TH$. Then H is a null hypersurface if the following holds:

$$g(L, L) = 0, \quad g(L, X) = 0 \quad \forall X \in TH. \tag{2.1}$$

In particular this means that H is degenerate: The normal vector field is also tangent to H. This is a very peculiar situation which does not occur in Riemannian geometry at all. Due to this property we have that null hypersurfaces H are rules by null geodesics. To show this, we note that by definition there exists a global null vector field L on H. Hence, the integral curves defined by $L(\gamma(t)) = \gamma'(t)$ are null and ruled H. To show that they are indeed geodesics, we must show that L satisfies the geodesic equation $\nabla_L L = 0$, where ∇ is the covariant derivative. Let $X \in TH$ be arbitrary. By using the product rule twice we find that

$$g(\nabla_L L, X) = L(\underbrace{g(L, X)}_{0}) - g(L, \nabla_L X) = -g(L, \nabla_X L + \underbrace{[L, X]}_{0}) = -\frac{1}{2}X(\underbrace{g(L, L)}_{0}) = 0. \tag{2.2}$$

Since X was arbitrary, we must have $\nabla_L L = \alpha L$, where α is a scalar function. This is not quite the usual geodesic equation, but still defines a non-affinely parametrized geodesic. The geodesics obtained this way are called the null generators of H.

Later we want to use null hypersurfaces to introduce the double null coordinate system. This means that we need a scalar function which is somehow associated to the null hypersurface. A way to find such a function is the following: We consider H as the level set of a differentiable function $u : M \to \mathbb{R}$:

$$H = \{p \in M \,|\, u(p) = u_0 = \text{const.}\}. \tag{2.3}$$

Since we want H to be a null hypersurface, the function u cannot be arbitrary. The condition is that the gradient vector ∇u is null:

$$g(\nabla u, \nabla u) = 0. \tag{2.4}$$

This equation is known as the Eikonal equation and a solution is called an optical function.[1]

[1]This terminology stems from the so called Eikonal (from Greek εἰκών for image) from geometrical optics, which describes the path travelled by a light ray according to Fermat's principle.

A simple example of a family of null hypersurfaces are the outgoing (or ingoing) light cones in Minkowski spacetime. In spherical coordinates (t, r, θ, φ), they are defined as the level set $H = \{t - r = \text{const}\}$. In this case the associated optical function is given by $u = t - r$ and the normal null vector field is given by $L = \partial_t - \partial_r$. If we consider the slices $t = \text{const}$ of these null hypersurfaces, then r must be constant as well. This leaves only the angular coordinates (θ, φ) free to vary. From this we can infer that H is foliated by spacelike two-surfaces, called sections, homeomorphic to the unit two-sphere $\mathbb{S}^2$. This situation is depicted in the following figure:

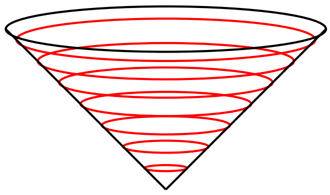


Figure 1: A foliation of the outgoing light cone in Minkowski spacetime via sections of constant t and r. In this picture, one angular direction is suppressed.

This way we can also obtain a foliation of a null hypersurface in a general spacetime. A spacelike two-surface $S \subset H$ is called a section of H if every null generator of H intersects it orthogonally. The section S is usually assumed to be homeomorphic to $\mathbb{S}^2$.

2.2 Pairs of Null Hypersurfaces

Let us start by considering a spacelike two-surface $S \subset M$, homeomorphic to $\mathbb{S}^2$, which is a priori not embedded in any null hypersurface. Given S, we wish to construct a pair of null hypersurfaces from it. First, since S is two-dimensional and spacelike, we have that its tangent bundle TS is isometric to two-dimensional euclidean space at every $p \in S$. In view of this we know that its normal bundle must be isometric to two-dimensional Minkowski spacetime at every $p \in S$. Thus, we find two linearly independent null vectors $L_p, \underline{L}_p$, which correspond to the two null directions $t \pm r = 0$ in Minkowski spacetime. These null vectors can be extended to null vector fields $L, \underline{L}$ on the normal bundle of S. Since the metric length of a null vector vanishes, they can't be normalized in the usual sense. Instead, we can choose them such that

$$g(L, \underline{L}) = -1. \tag{2.5}$$

However, this condition does not uniquely determine L and $\underline{L}$, because the equation is left invariant under the transformation $L \to \alpha L$ and $\underline{L} \to \frac{1}{\alpha}\underline{L}$, where α is a continuous, positive[2] function on S.

Given a choice of L and $\underline{L}$ we can construct two null hypersurfaces in the following way: We extend both vector fields as geodesic vector fields via the geodesic equation

$$\nabla_L L = 0, \quad \nabla_{\underline{L}} \underline{L} = 0. \tag{2.6}$$

This gives rise to two families of null geodesics, namely the integral curves of L and $\underline{L}$, which we call G_p and $\underline{G}_p$ respectively. Here G_p denotes the geodesic starting at p with initial tangent vector L_p and $\underline{G}_p$ denotes the geodesic starting at p with initial tangent vector $\underline{L}$. These geodesics

[2] More generally, α could also be negative, but usually $L, \underline{L}$ are taken to be future-directed vectors and multiplying by a negative function would reverse their orientation.

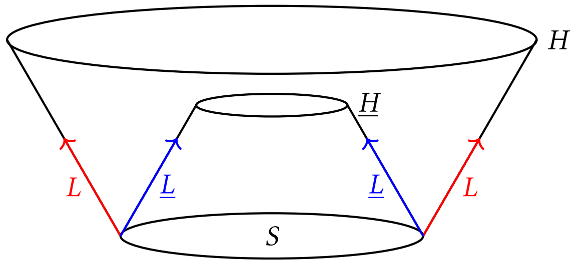


Figure 2: Two null hypersurfaces generated by null vector fields $L, \underline{L}$. In this figure, one dimension of S and the null hypersurfaces is suppressed.

are the null generators of two null hypersurfaces[3] $H, \underline{H}$, which are defined as the unions over all $p \in S$ of the geodesic families:

$$H = \bigcup_{p \in S} G_p, \quad \underline{H} = \bigcup_{p \in S} \underline{G}_p. \tag{2.7}$$

This construction is illustrated in figure 2. Generally these null hypersurfaces will not be globally smooth. This can happen for multiple reasons: (a) The null generators intersect, forming a vertex on which the null hypersurface is singular, or (b) The spacetime is geodesically incomplete, which means that the null generators cannot be extended indefinitely. Here we will assume that both H and $\underline{H}$ are smooth, so these problems do not concern us.

We can find a foliation of H (and similarly for $\underline{H}$) in the following way: Let us consider an affine parameter τ of the null generator G_p such that $L(\tau) = 1$ and $\tau|_S = 0$. Then the level set

$$S_\tau = \{G_p(\tau) \mid p \in S\} \tag{2.8}$$

is a section of H. We can foliate H by the sections S_τ:

$$H = \bigcup_{\tau \in [0,\infty)} S_\tau. \tag{2.9}$$

Thus, we can understand the geometry of H by studying the evolution and properties of the sections S_τ. Since H is ruled by the null generators G_p, if $q \in H$, then there exists $p \in S, \tau > 0$ such that $q = G_p(\tau)$. We can find a basis for T_qH by first considering a basis (E_1, E_2) of T_pS and then propagating this frame along G_p according to the evolution equation

$$[L, E_a] = 0, \tag{2.10}$$

which is called Lie propagation. This means that the Lie derivative of E_a along L vanishes. This ensures that the frame (E_1, E_2) stays orthogonal to L. Hence, we find that $(L_q, (E_1)_q, (E_2)_q)$ is a basis for T_qH, and we can prove the following proposition.

Proposition 2.1. *The sets H and $\underline{H}$ as defined in equation 2.7 are null hypersurfaces.*

Proof. We know that $TH = \text{span}(L, E_1, E_2)$ and thus $\dim(H) = 3$. We need to show that L is orthogonal to any $X \in TH$, i.e. $g(L, X) = 0$ for $X = L, E_1, E_2$. At $p \in S$ we have $g_p(L_p, X_p) = 0$, furthermore we find that

$$L(g(L, X)) = 0,$$

so $g(L, X) = 0$ on all of H. The proof for $\underline{H}$ is identical. ■

[3] Proved in proposition 2.1

We can define coordinates on $H \cup \underline{H}$ in the following way: Let u, v be two functions on $H \cup \underline{H}$ which satisfy

$$\begin{aligned} L(v) &= 1, \quad \underline{L}(v) = 0, \quad v|_S = 0 \\ L(u) &= 0, \quad \underline{L}(u) = 1, \quad u|_S = 0. \end{aligned} \tag{2.11}$$

Then u, v can be thought of as the affine parameters of $G_p, \underline{G}_p$ subject to the conditions above. In particular, since $L(u) = 0$ and $u|_S = 0$, we see that H is the level set $\{u = 0\}$ and similarly $\underline{H} = \{v = 0\}$. Hence, we see that u, v are also optical functions. The sections of H are the level sets $S_\tau = \{v = \tau\}$ and equivalently for $\underline{H}$. Now we consider coordinates θ^1, θ^2 on S and extend them onto $H \cup \underline{H}$ according to

$$L(\theta^a) = 0, \quad \underline{L}(\theta^a) = 0. \tag{2.12}$$

The extended functions θ^1, θ^2 are now also coordinates on the sections S_τ and $\underline{S}_\tau$. We can use the affine parameters u and v to obtain coordinates on $\underline{H}$ and H respectively. We find that (u, θ^1, θ^2) are coordinates for $\underline{H}$, while (v, θ^1, θ^2) are coordinates for H.[4]

2.3 Defining the Null Coordinates

Now we want to construct a (local) coordinate system a spacetime M. For this we want to obtain a foliation of our spacetime by families of null hypersurfaces and then use the coordinates established on the null hypersurfaces to obtain a new coordinate system for the spacetime. This coordinate system is called the double null coordinate system and the foliation of M by two families of null hypersurfaces is known as the double null foliation. Because the construction is a local one, the resulting coordinate system will generally also be a local one.

Again we consider a spacelike two-sphere $S_0 \subset M$ and the out and ingoing null congruences $H_0, \underline{H}_0$ generated by the null generators. Let $\Omega : H_0 \cup \underline{H}_0 \to \mathbb{R}$ be a smooth, positive function and $L', \underline{L}'$ be two null vector fields normal to S_0. We normalize them such that

$$g(L', \underline{L}') = -\Omega^{-2}. \tag{2.13}$$

The function Ω, called the null lapse, can be regarded as a gauge choice related to the foliation density of the sections. In the previous sections we had that $\Omega = 1$, but since we will have families of null hypersurfaces, we need a more general approach. Now we extend L' and $\underline{L}'$ as geodesic vector fields onto H_0 and respectively $\underline{H}_0$ as before:

$$\nabla_{L'} L' = 0, \quad \nabla_{\underline{L}'} \underline{L}' = 0. \tag{2.14}$$

This again means that the integral curves of those vector fields are null geodesics. Additionally, we consider the conformally rescaled vector fields

$$L = \Omega^2 L', \quad \underline{L} = \Omega^2 \underline{L}', \tag{2.15}$$

which satisfy $g(L, \underline{L}) = -\Omega^2$. Using these vector fields, we now define two functions u, v on $H_0 \cup \underline{H}_0$ according to equation 2.11:

$$\begin{aligned} L(v) &= 1, \quad \underline{L}(v) = 0, \quad v|_{S_0} = 0 \\ L(u) &= 0, \quad \underline{L}(u) = 1, \quad u|_{S_0} = 0, \end{aligned} \tag{2.16}$$

[4] More precisely both of them are only local coordinate systems, because $S \simeq \mathbb{S}^2$ possesses no global coordinate system. To cover the complete hypersurfaces, we would have to choose suitable coordinates $\tilde{\theta}^1, \tilde{\theta}^2$, which cover the remaining part of S.

which means that v and u are affine parameters of the null geodesics defined by L and $\underline{L}$. Using these functions, we can define sections of H_0 and $\underline{H}_0$: Let $S_{0,\tau} \subset H_0$ be the embedded two-sphere such that $v = \tau$ and let $S_{\tau,0} \subset \underline{H}_0$ be the embedded two-sphere such that $u = \tau$. Then $S_{\tau,0}$ (and similarly $S_{0,\tau}$) is a section of $H_0 = \bigcup_\tau S_\tau$.

As of now, the vector fields L' and $\underline{L}'$ are only defined on H_0 and $\underline{H}_0$ respectively. We define L' on $\underline{H}_0$ and $\underline{L}'$ on H_0 such that both L' and $\underline{L}'$ are null and satisfy $g(L', \underline{L}') = -\Omega^{-2}$. As we did in equation 2.15, we define $L = \Omega^2 L'$ on $\underline{H}_0$ and $\underline{L} = \Omega^2 \underline{L}'$ on H_0.

Now we can start constructing new null congruences from the sections of H_0 and $\underline{H}_0$. For example, starting at the section $S_{0,v} \subset H_0$, we extend the vector fields $L', \underline{L}'$ as geodesic vector fields onto M. In view of proposition 2.1, we obtain two new null hypersurfaces, which we call $H_{0,v}$ and $\underline{H}_{0,v}$. Next we can also extend the vector fields $L, \underline{L}$ onto these hypersurfaces by defining

$$L = \Omega^2 L', \quad \underline{L} = \Omega^2 \underline{L}', \quad \text{where } \Omega^2 = -g(L', \underline{L}'). \tag{2.17}$$

Finally, we extend u, v onto $H_{0,v} \cup \underline{H}_{0,v}$ according to equation 2.11 again. And we get that

$$H_{0,\tau} = \{u = 0 \wedge v \geq \tau\}, \quad \underline{H}_{0,\tau} = \{u \geq 0 \wedge v = \tau\}. \tag{2.18}$$

Similarly, starting from the sphere $S_{\tau,0}$, we obtain two null hypersurfaces

$$H_{\tau,0} = \{u = \tau \wedge v \geq 0\}, \quad \underline{H}_{\tau,0} = \{u \geq \tau \wedge v = 0\}, \tag{2.19}$$

which means that both u, v are optical functions, i.e. they satisfy equation 2.4. Generally, these null hypersurfaces can intersect other null hypersurfaces. For example the intersection

$$S_{\tau,\tilde{\tau}} = H_{\tau,0} \cap \underline{H}_{0,\tilde{\tau}} \tag{2.20}$$

is again a spacelike two-sphere, from which we could construct two new null hypersurfaces in the same fashion as before. Repeating this process, we obtain a double foliation of M by null hypersurfaces. This construction is illustrated in figure 3.

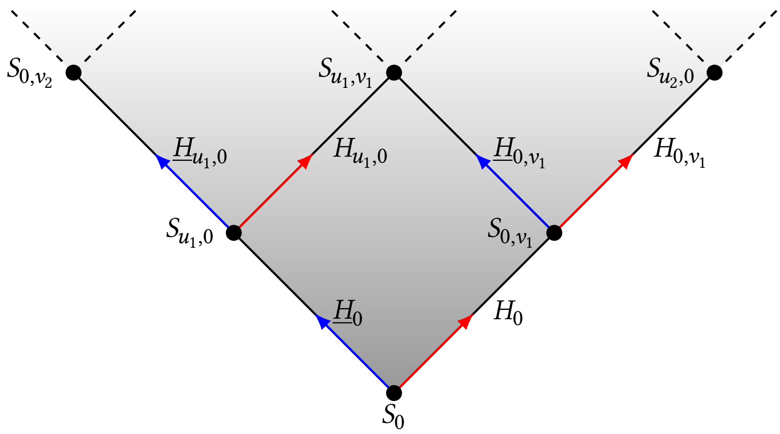


Figure 3: A schematic representation of the double null foliation construction. In this picture every point corresponds to a sphere $\mathbb{S}^2$.

To establish coordinates on the sphere $S_{u,v}$, we pick a local coordinate system (θ^1, θ^2) on S_0. We consider a point $q \in S_{u,v}$ and follow backwards the unique null generator of $\underline{H}_{0,v}$, which passes through a point in $S_{0,v}$. From there we follow the null generator passing through this point to S_0. The point of intersection, call it $p \in S_0$, has coordinates $\theta^1(p), \theta^2(p)$, which we assign to q. In this way we also uniquely assign coordinates to any other $\tilde{q} \in S_{u,v}$ and hence find a coordinate system for the sections $S_{u,v}$. This method will work as long the null generators do not intersect each other, in other words in the absence of focal points. We obtain, using the previously established coordinate systems for the null hypersurfaces, the (local) coordinate system $(u, v, \theta^1, \theta^2)$ on M. This coordinate system is called the double null coordinate system.

In our construction of the coordinates (now viewed with S_0 as the starting point), we first transported θ^a along the L-direction and then the $\underline{L}$-direction. It is important to note that switching the order would result in a different coordinate system. We have that $\underline{L} = \partial_u$ on M, but $L = \partial_v$ only holds on H_0. More generally we will have that $L = \partial_v + b^a \partial_{\theta^a}$, for some vector $b = b^a \partial_{\theta^a}$ and hence

$$[L, \underline{L}] = [\partial_v, \partial_u] + [b^a \partial_{\theta^a}, \partial_u] = -(\partial_u b^a)\partial_{\theta_a}, \tag{2.21}$$

which is generally non-zero. From this we see that ∂_v is only null on H_0, but spacelike elsewhere. The vector $b = b^a \partial_{\theta^a}$ is called the shift vector and b^a indicates how much θ^a changes along L. The (u-derivative of the) components of b can be obtained by

$$d\theta^a([L, \underline{L}]) = -\partial_u(b^a). \tag{2.22}$$

To finish this discussion we find the expression of the metric in double null coordinates.

Proposition 2.2. *The metric g in double null coordinates $(u, v, \theta^1, \theta^2)$ is given by*

$$g = -2\Omega^2 du dv + \not{g}_{ab}(d\theta^a - b^a dv)(d\theta^b - b^b dv), \tag{2.23}$$

where Ω is the null lapse, b the shift vector and $\not{g}_{ab} = g(\partial_{\theta^a}, \partial_{\theta^b})$ the induced metric on $S_{u,v}$.

Proof. In order to derive this expression we compute the components $g_{\alpha\beta} = g(\partial_{x^\alpha}, \partial_{x^\beta})$ (for $x^\alpha = u, v, \theta^1, \theta^2$) of the metric. First we have that

$$g(L, \partial_{\theta^a}) = 0, \quad g(\underline{L}, \partial_{\theta^b}) = 0, \tag{2.24}$$

since $L, \underline{L}$ are normal and ∂_{θ^a} is tangent to the sections $S_{u,v}$. Also, we remember that L and $\underline{L}$ are null vector fields which are normalized with respect to each other by $g(L, \underline{L}) = -\Omega^2$. Hence, we obtain

$$\begin{aligned} g_{uu} &= g(\underline{L}, \underline{L}) = 0 \\ g_{vv} &= g(L, L) - 2g(L, \partial_{\theta^a})\, b^a + g(\partial_{\theta^a}, \partial_{\theta^b})\, b^a b^b = \not{g}_{ab} b^a b^b \\ g_{uv} &= g(\underline{L}, L) - g(\underline{L}, \partial_{\theta^a})\, b^a = -\Omega^2 \\ g_{ua} &= g(\underline{L}, \partial_{\theta^a})\, b^a = 0 \\ g_{va} &= g(L, \partial_{\theta^a}) - g(\partial_{\theta^b}, \partial_{\theta^a})\, b^b = -\not{g}_{ab} b^b \\ g_{ab} &= g(\partial_{\theta^a}, \partial_{\theta^b}) = \not{g}_{ab}, \end{aligned} \tag{2.25}$$

which gives us

$$\begin{aligned} g = g_{\alpha\beta} dx^\alpha dx^\beta &= 2g_{uv}\, du\, dv + g_{vv}\, dv^2 + 2g_{va}\, dv\, d\theta^a + g_{ab}\, d\theta^a d\theta^b \\ &= -2\Omega^2\, du\, dv + \not{g}_{ab}\left(b^a b^b\, dv^2 - 2b^b\, dv\, d\theta^a + d\theta^a d\theta^b\right) \\ &= -2\Omega^2\, du\, dv + \not{g}_{ab}(d\theta^a - b^a\, dv)(d\theta^b - b^b\, dv). \end{aligned} \tag{2.26}$$

■

2.4 Application: The Wave Equation in Minkowski Spacetime

Before we start working on the problem of double null coordinates in Kerr, let us consider a short application which illustrates the usefulness of double null coordinates. They are particularly handy to study conservation laws associated with the scalar wave equation. Here we will consider the wave equation in Minkowski spacetime, which is given by

$$0 = \eta^{ab}\partial_a\partial_b\psi = -\partial_t^2\psi + \frac{1}{r}\partial_r^2(r\psi) + \frac{1}{r^2}\not{\Delta}\psi \tag{2.27}$$

in spherical coordinates (t, r, θ, φ). Here $\not{\Delta}$ denotes the induced Laplacian on the two-spheres of constant t, r. Let us define the functions

$$u = \frac{1}{2}(t-r), \quad v = \frac{1}{2}(t+r), \tag{2.28}$$

which are the optical functions from the usual light cones. We adapt u, v as coordinates instead of (t, r). By the chain rule we obtain

$$\partial_u = \partial_t - \partial_r, \quad \partial_v = \partial_t + \partial_r, \quad \text{and thus } -\partial_v\partial_u = -\partial_t^2 + \partial_r^2. \tag{2.29}$$

To study the wave equation in double null coordinates, we multiply the equation with $r > 0$ and replace the t, r-derivatives with the derivatives with respect to the null coordinates u, v. This gives us a new PDE for $r\psi$:

$$\begin{aligned} 0 &= -\partial_t^2(r\psi) + \partial_r^2(r\psi) + \frac{1}{r^2}\not{\Delta}(r\psi) \\ &= -\partial_v\partial_u(r\psi) + \frac{1}{r^2}\not{\Delta}(r\psi). \end{aligned} \tag{2.30}$$

We can integrate this equation over the sphere $S_{u,v} = (v-u)^2 \cdot \mathbb{S}^2$ of constant u, v. Further, applying Stokes theorem on the term with the Laplacian, we find that this integral vanishes since a sphere has no boundary:

$$\begin{aligned} \int_{S_{u,v}} \not{\Delta}(r\psi)\, dS &= \int_{S_{u,v}} \not{\mathrm{div}}\,\not{\mathrm{grad}}(r\psi)\, dS \\ &= \int_{\partial S_{u,v}} \not{\mathrm{grad}}(r\psi) \cdot N\, ds = 0, \end{aligned} \tag{2.31}$$

where N is the outwards-pointing unit normal vector field on $S_{u,v}$. We are left with

$$\partial_v\Big(\underbrace{\int_{S_{u,v}} \partial_u(r\psi)\, dS}_{Q}\Big) = 0. \tag{2.32}$$

Hence, we find that the quantity Q is conserved along the null generators of ∂_v. In other words specifying the value of Q on a sphere S_{u_0,v_0} fixes the value of Q on any sphere $S_{u_0,v}$, where $v > v_0$. In other coordinate systems such a conservation law would be hard to identify. The reason that the double null coordinates system is very useful for studying the wave equation is that the characteristics of the wave equation are in fact the in and outgoing light cones, i.e. the level sets of u and v.

By Noether's theorem a conservation law gives rise to a continuous symmetry. Therefore, it is an interesting question whether there are such conservation laws along null hypersurfaces in more complicated spacetimes; for example Schwarzschild or (extremal) Kerr spacetime. Stefanos Aretakis showed that sub-extremal Kerr black holes do not admit such conservation laws, but extremal ones do admit exactly one such conservation law along their event horizon [2].

3 Kerr Spacetime and Null Hypersurfaces

From now on we are mainly interested in the specific case of Kerr spacetime. In this chapter we will introduce the Kerr metric and study its null hypersurfaces following and filling in between the steps of Pretorius and Israel [5].

3.1 Properties of Kerr Spacetime

The Kerr metric was first discovered by Roy Kerr in 1963. It is a solution to the vacuum Einstein equations

$$\mathrm{Ric}(g) = 0 \tag{3.1}$$

and describes the spacetime of an uncharged, rotating black hole. The Kerr metric characterized by two parameters: The mass M and the reduced angular momentum a, also called the Kerr parameter. In the case that $a = 0$, the Kerr metric reduces to the non-rotating Schwarzschild metric. Thereby, it generalizes the non-rotating Schwarzschild solution. We will not consider the case of over-extremal Kerr black holes, nor the case Schwarzschild, so $0 < a \leq M$. Originally the Kerr metric was derived in cartesian-like coordinates (called Kerr-Schild coordinates), however nowadays, the metric is often expressed in the latter derived spherical-like Boyer-Lindquist coordinates (t, r, θ, φ) in which it is given by

$$g = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2 + R^2 \sin^2(\theta)\, d\varphi^2 - \frac{4Mar\sin^2(\theta)}{\Sigma}\, d\varphi dt, \tag{3.2}$$

where we defined the three quantities

$$\Sigma = r^2 + a^2\cos^2(\theta), \quad \Delta = r^2 + a^2 - 2Mr, \quad R^2 = \frac{(r^2+a^2)^2 - a^2\Delta\sin^2(\theta)}{\Sigma}. \tag{3.3}$$

Since these coordinates resemble spherical coordinates, it seems natural that the range of the coordinates is

$$t \in (-\infty, \infty), \quad r \in (0, \infty), \quad \theta \in (0, \pi), \quad \varphi \in (0, 2\pi). \tag{3.4}$$

However, the metric expression becomes singular if $\Delta = 0$ or $\Sigma = 0$. Solving for the roots of the polynomial Δ we find two solutions $r_\pm = M \pm \sqrt{M^2 - a^2}$, which define an inner and outer event horizon. However, this singularity is only due to the choice of the coordinates and not a curvature singularity, which can be seen by evaluating the Kretschmann scalar

$$R^{abcd}R_{abcd} = \frac{6M^2\left(r^6 - 15a^2r^4\cos^2(\theta) + 15a^4r^2\cos^4(\theta) - a^6\cos^6(\theta)\right)}{\Sigma^6} \tag{3.5}$$

at $r_\pm$. On the other hand, Σ vanishes when $r = 0$ and also $\theta = \pi/2$. For $\Sigma = 0$ the Kretschmann scalar diverges, and hence we see that $\Sigma = 0$ indicates a curvature singularity. Since this singularity appears for any value of φ it has the same topological structure as a circle and is thus called ring singularity. In this thesis we are mainly interested in the exterior region of the black hole, so we consider the region where $r \in (r_+, \infty)$, which means that $\Delta > 0$ and also that the aforementioned singularities do not occur.

The Kerr metric exhibits a frame-dragging effect, which causes radially inwards falling observers, not subjected to any force, to start rotating with the black hole. This becomes apparent through the $d\varphi\, dt$ cross term in the metric. This also holds true for light rays, which means that the radial null geodesics twist around the axis of rotation.

In the case of a non-rotating Schwarzschild black hole, the spacetime is spherically symmetric. Due to the rotation of the Kerr black hole, the spacetime is no longer spherically symmetric, but rather axisymmetric around the axis of rotation. The surfaces defined by $r = \text{const}$ are oblate spheroids instead of spheres. For this reason, when studying null hypersurfaces in Kerr, we will assume that they are axisymmetric instead of being spherically symmetric.

3.2 Eikonal Equation in Kerr

We want to find axisymmetric null hypersurfaces in Kerr spacetime. We want to consider these null hypersurfaces as level sets of functions $u = u_-, v = u_+$ parametrized as

$$u_\pm(t,r,\theta) = t \pm r_*(r,\theta) = \text{const}\,. \tag{3.6}$$

The function r_* is called the tortoise coordinate, and is to be determined such that u, v are optical functions. This means that they satisfy the Eikonal equation 2.4 and r_*4 satisfies

Proposition 3.1. *Let $u = t - r_*$ be a solution to the Eikonal equation in Kerr spacetime. Then the function r_* satisfies the equation*

$$\Delta(\partial_r r_*)^2 + (\partial_\theta r_*)^2 = \frac{\Sigma R^2}{\Delta} = \frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2(\theta), \tag{3.7}$$

which is a first-order, non-linear PDE for r_*.

Proof. Inserting u into the Eikonal equation we obtain that

$$\begin{aligned} g(\nabla u, \nabla u) &= (g^{-1})^{ab}\partial_a u\, \partial_b u \\ &= (g^{-1})^{tt} + (g^{-1})^{rr}(\partial_r r_*)^2 + (g^{-1})^{\theta\theta}(\partial_\theta r_*)^2 \\ &= 0. \end{aligned} \tag{3.8}$$

To obtain an explicit expression for the Eikonal equation we therefore need to find the components of the inverse metric. Since the metric is block-diagonal, the r- and θ-components are easily found to be

$$(g^{-1})^{rr} = \frac{\Delta}{\Sigma}, \quad (g^{-1})^{\theta\theta} = \frac{1}{\Sigma}. \tag{3.9}$$

In order to obtain $(g^{-1})^{tt}$ we need to invert the t, φ block of the metric:

$$\begin{pmatrix} g_{tt} & g_{\varphi t} \\ g_{\varphi t} & g_{\varphi\varphi} \end{pmatrix}^{-1} = \frac{1}{\det g_{(t,\varphi)}} \begin{pmatrix} g_{\varphi\varphi} & -g_{\varphi t} \\ -g_{\varphi t} & g_{tt} \end{pmatrix} \tag{3.10}$$

The determinant is given by

$$\begin{aligned} \det g_{(t,\varphi)} &= g_{tt}g_{\varphi\varphi} - g_{\varphi t}^2 \\ &= -\left(1 - \frac{2Mr}{\Sigma}\right) R^2\sin^2(\theta) - \left(\frac{2Mar\sin^2(\theta)}{\Sigma}\right)^2 \\ &= \left(-R^2 + \frac{2MrR^2}{\Sigma} - \frac{4M^2a^2r^2\sin^2(\theta)}{\Sigma^2}\right)\sin^2(\theta) \end{aligned}$$

$$
\begin{aligned}
&= \left(-R^2 + \frac{2Mr}{\Sigma^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2(\theta) - 2Mra^2\sin^2(\theta)\right)\right)\sin^2(\theta) \\
&= \left(-R^2 + \frac{2Mr}{\Sigma^2}(r^2+a^2)\left(r^2+a^2\cos^2(\theta)\right)\right)\sin^2(\theta) \\
&= \frac{-(r^2+a^2)^2 - a^2\Delta\sin^2(\theta) + 2Mr(r^2+a^2)}{\Sigma}\sin^2(\theta) \\
&= \frac{-(r^2+a^2-2Mr)\left(r^2+a^2\cos^2(\theta)\right)}{\Sigma}\sin^2(\theta) \\
&= -\Delta\sin^2(\theta).
\end{aligned}
$$

Hence, we find

$$
(g^{-1})^{tt} = \frac{g_{\varphi\varphi}}{\det g_{(t,\varphi)}} = -\frac{R^2}{\Delta}. \tag{3.11}
$$

After multiplying this by Σ we can separate this expression into a r- and θ-dependent part:

$$
\begin{aligned}
\frac{\Sigma R^2}{\Delta} &= \frac{1}{\Delta}\left((r^2+a^2)^2 - a^2\Delta\sin^2(\theta)\right) \\
&= \frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2(\theta).
\end{aligned} \tag{3.12}
$$

Inserting these components of g^{-1} into the Eikonal equation gives the claimed equation. ■

Constructing the General Solution

We now want to proceed solving this equation by finding a so-called complete integral and then constructing the general solution from it. A complete integral is a family $r_*(r,\theta,\lambda,\mu)$ of solutions to equation 3.7 depending on as many parameters (here λ,μ) as independent variables. Here we can take μ to be a function of λ since this function can be chosen arbitrarily: $\mu = f(\lambda)$.

The left-hand side of the Eikonal equation consists of two positive summands while the right-hand side consists of a difference. We can introduce a separation variable $a^2\lambda$ such that the right-hand side also becomes a sum of two (positive) summands:

$$
\frac{\Sigma R^2}{\Delta} = \frac{(r^2+a^2)^2 - a^2\lambda\Delta}{\Delta} + a^2(\lambda - \sin^2(\theta)) = \frac{Q(r,\lambda)^2}{\Delta} + P(\theta,\lambda)^2, \tag{3.13}
$$

where we defined

$$
Q(r,\lambda) = \sqrt{\left(r^2+a^2\right)^2 - a^2\lambda\Delta}, \tag{3.14}
$$

$$
P(\theta,\lambda) = a\sqrt{\lambda - \sin^2(\theta)}. \tag{3.15}
$$

The definition of P suggests that we must have $\lambda \geq \sin^2(\theta)$. We now consider the exact differential

$$
d\rho = \partial_r\rho\, dr + \partial_\theta\rho\, d\theta = \frac{Q}{\Delta}\, dr + P\, d\theta. \tag{3.16}
$$

If we plug the values for $\partial_r\rho$ and $\partial_\theta\rho$ into equation 3.7, we see that $d\rho$ is in fact the differential of a solution ρ to Eikonal equation:

$$
\Delta\frac{Q^2}{\Delta^2} + P^2 = \Delta(\partial_r\rho)^2 + (\partial_\theta\rho)^2 = \frac{Q^2}{\Delta} + P^2. \tag{3.17}
$$

In principle, since Q and P are known, we can obtain a particular solution to equation 3.7 by integrating $d\rho$ along a path. Since the differential is exact, the result is solely dependent on the endpoints and not the specific path. Thus, we can integrate $d\rho$ separately in r and θ[5]:

$$\rho(r,\theta,\lambda) = \int_{\infty}^{r} \frac{Q(r',\lambda)}{\Delta(r')}\,dr' + \int_{0}^{\theta} P(\theta',\lambda)\,d\theta' + C(\lambda), \tag{3.19}$$

which is the complete integral we were looking for. However, this is only a particular solution and not the general solution in which we are interested in because we want to be able to assert general boundary conditions.

We can construct the general solution using envelopes. An envelope is a function $r_*(r,\theta)$ which at each point is tangent to some member of the family $\rho(r,\theta,\lambda)$. To find an envelope we promote the parameter λ to be an independent variable. Since we consider λ as an independent parameter now, the differential 3.16 obtains an extra term:

$$d\rho = \frac{Q(r,\lambda)}{\Delta(r)}\,dr + P(\theta,\lambda)\,d\theta + \frac{a^2}{2}F(r,\theta,\lambda)\,d\lambda \quad \text{with} \quad \frac{a^2}{2}F = \partial_\lambda \rho, \tag{3.20}$$

For r_* to be an envelope we must have that the differential in equations 3.16 and 3.20 are the same, which requires that $F(r,\theta,\lambda) = 0$. This means that we cannot choose λ arbitrarily, but rather that we have an implicit relationship between r,θ and λ. We come back to this relationship in a moment.

By taking λ-derivative of ρ we find an expression for F. First we compute the λ-derivatives of P and Q:

$$\frac{\partial Q(r,\lambda)}{\partial \lambda} = -\frac{a^2\Delta(r)}{2Q(r,\lambda)}, \qquad \frac{\partial P(\theta,\lambda)}{\partial \lambda} = \frac{a^2}{2P(\theta,\lambda)}. \tag{3.21}$$

Thus, we obtain[6] (and by replacing C by $a^2/2\, f$):

$$F(r,\theta,\lambda) = \frac{2}{a^2}\partial_\lambda \rho = \int_{r}^{\infty} \frac{1}{Q(r',\lambda)}dr' + \int_{0}^{\theta} \frac{1}{P(\theta',\lambda)}\,d\theta' + f'(\lambda). \tag{3.22}$$

The relation between r,θ and λ is as follows: Since we must impose $F = 0$, λ is fixed by specifying r,θ. We want to consider λ to be function $\lambda(r,\theta)$ such that $F(r,\theta,\lambda(r,\theta)) = 0$. Then

$$r_*(r,\theta) = \rho(r,\theta,\lambda(r,\theta)) \tag{3.23}$$

is an envelope to the complete integral which we found and thus also is the general solution to the Eikonal equation for axisymmetric null hypersurfaces. In principle, if $\lambda(r,\theta)$ were to be known, we could find r_* explicitly in terms of r,θ and also the integration constant $f(\lambda(r,\theta))$.

[5] Here we made the choice of integrating in the r-direction starting from infinity. We need to be very careful about this since the integral does not convergence when integrating starting from infinity:

$$\int_{r}^{\infty} \frac{Q(r',\lambda)}{\Delta(r')}\,dr' = \int_{r}^{\infty} \frac{\sqrt{(r'^2+a^2)^2 - a^2\lambda\Delta}}{r'^2+a^2-2Mr'}dr' \sim \int_{r}^{\infty} \frac{r'^2}{r'^2}dr' = \infty. \tag{3.18}$$

Thus, we actually need to start integrating from some fixed $r_\infty > r_+$ in order to ensure that the integral converges. As will become clear later, the limit $r_\infty \to \infty$ can be still be taken by choosing appropriate boundary conditions. Hence, we agree to still write ∞, while keeping in mind this caveat.

[6] In this case the r-integral converges even when integrating to infinity.

The Implicit Function Theorem

The construction of envelopes relies on the fact that $F(r, \theta, \lambda(r, \theta)) = 0$. The question is whether there are any solutions to this equation and whether such a function even exists locally in the neighbourhood of a point. This question is answered by the implicit function theorem (IFT): Suppose that F has a zero at a point $(r_0, \theta_0, \lambda_0)$ and also $\partial_\lambda F \neq 0$ at that point. Then there exists a (small) neighbourhood around this point for such a function $\lambda(r, \theta)$ exists. However, this function is not given explicitly, but only implicitly in terms of $F(r, \theta, \lambda(r, \theta)) = 0$. We will concern ourselves with this question in the last chapter.

For the remainder of this chapter we will assume that a global solution $\lambda(r, \theta)$ to the equation $F(r, \theta, \lambda(r, \theta)) = 0$ exists and study the general form of the Kerr metric in double null coordinates.

3.3 Kerr Metric in Double Null Coordinates

In order to express the Kerr metric in double null coordinates we want to replace the Boyer-Lindquist r-coordinate with r_* as defined in equation 3.23. However, since r_* does not only depend on r, but also on the coordinate θ, just changing coordinates from r to r_* would yield an undesirable $dr_* d\theta$ cross term in the metric. We can overcome this problem by also replacing θ as a coordinate. In fact replacing θ by λ is the coordinate change we are looking for:

Proposition 3.2. *The gradients of the functions $r_*(r, \theta)$ and $\lambda(r, \theta)$ are orthogonal with respect to the Kerr metric:*

$$g(\nabla r_*, \nabla \lambda) = 0. \tag{3.24}$$

Before we can proof this and show that λ is indeed the coordinate we are looking for, we need to know the r- and θ-derivatives of r_* and λ in order to compute the metric product.

Proposition 3.3. *Let $\Lambda = \partial_\lambda F$. The partial derivatives of $r_*(r, \theta)$ and $\lambda(r, \theta)$ are given by*

$$\frac{\partial r_*}{\partial r} = \frac{Q}{\Delta}, \qquad \frac{\partial r_*}{\partial \theta} = P, \tag{3.25}$$

$$\frac{\partial \lambda}{\partial r} = \frac{1}{\Lambda Q}, \qquad \frac{\partial \lambda}{\partial \theta} = -\frac{1}{\Lambda P}. \tag{3.26}$$

Proof. The derivatives of r_* can be read of the definition of dr_* (eq. 3.25). In order to obtain the derivatives of λ we consider the differential of F:

$$dF = \Lambda \, d\lambda - \frac{1}{Q} \, dr + \frac{1}{P} \, d\theta. \tag{3.27}$$

Imposing $F(r, \theta, \lambda(r, \theta)) = 0$ yields that $dF = 0$. Since now λ depends on r and θ, we can also express $d\lambda$ in terms of dr and $d\theta$ to find

$$\begin{aligned} 0 &= \Lambda \left(\frac{\partial \lambda}{\partial r} \, dr + \frac{\partial \lambda}{\partial \theta} \, d\theta \right) - \frac{1}{Q} \, dr + \frac{1}{P} \, d\theta \\ &= \left(\Lambda \frac{\partial \lambda}{\partial r} - \frac{1}{Q} \right) dr + \left(\Lambda \frac{\partial \lambda}{\partial \theta} + \frac{1}{P} \right) d\theta. \end{aligned}$$

Since we assumed a global solution $\lambda(r, \theta)$ to the constraint equation, the implicit function theorem guarantees that $\Lambda \neq 0$. As both functions in front of dr and $d\theta$ must vanish, we get the above results. ■

With this we are now able to prove proposition 3.2:

Proof of proposition 3.2. Since both r_* and λ solely depend on r and θ, they need to be orthogonal with respect to the metric $\sigma = \Sigma\left(dr^2/\Delta + d\theta^2\right)$, which is the induced metric on the two-surfaces of constant t, φ. We have that

$$\begin{aligned}\sigma(\nabla r_*, \nabla\lambda) &= (\sigma^{-1})^{ab}(dr_*)_a(d\lambda)_b \\ &= (\sigma^{-1})^{rr}\,\partial_r r_*\,\partial_r\lambda + (\sigma^{-1})^{\theta\theta}\,\partial_\theta r_*\,\partial_\theta\lambda \\ &= \frac{\Delta}{\Sigma}\left(-\frac{1}{\Lambda Q}\frac{Q}{\Delta}\right) + \frac{1}{\Sigma}\left(\frac{1}{\Lambda P}P\right) \\ &= -\frac{1}{\Lambda\Sigma} + \frac{1}{\Lambda\Sigma} = 0\end{aligned} \tag{3.28}$$

Hence ∇r_* and $\nabla\lambda$ are orthogonal. ■

The orthogonality of the gradients means that the coordinate lines intersect orthogonally, and we do not get a (r_*, λ) cross term in the metric. Thus, we are interested in the coordinate change $(t, r, \theta, \phi) \to (t, r_*, \lambda, \varphi)$. Since t and φ are left invariant under this change, we can focus on the transformation $(r, \theta) \to (r_*, \lambda)$ and look at the induced metric σ. In order to express σ in terms of the new coordinates we need to invert the differential relations (eqs. 3.25 and 3.26) between (r, θ) and (r_*, λ).

Proposition 3.4. *Let $L = \Lambda PQ$. The partial derivatives of $r(r_*, \lambda)$ and $\theta(r_*, \lambda)$ are given by.*

$$\frac{\partial r}{\partial r_*} = \frac{Q\Delta}{\Sigma R^2}, \qquad \frac{\partial r}{\partial\lambda} = \frac{-LP\Delta}{\Sigma R^2}, \tag{3.29}$$

$$\frac{\partial\theta}{\partial r_*} = -\frac{P\Delta}{\Sigma R^2}, \qquad \frac{\partial\theta}{\partial\lambda} = \frac{LQ}{\Sigma R^2}. \tag{3.30}$$

Proof. To obtain these formulas we invert the Jacobian matrix

$$J = \begin{pmatrix} Q/\Delta & P \\ 1/(\Lambda Q) & -1/(\Lambda P) \end{pmatrix}, \tag{3.31}$$

with the row entries given by eqs. 3.25 and 3.26. We have that

$$\begin{pmatrix} \partial r/\partial r_* & \partial r/\partial\lambda \\ \partial\theta/\partial r_* & \partial\theta/\partial\lambda \end{pmatrix} = J^{-1} = \frac{1}{\det J}\begin{pmatrix} -1/(\Lambda P) & -P \\ -1/(\Lambda Q) & Q/\Delta \end{pmatrix} \tag{3.32}$$

The determinant is given by

$$\det J = -\frac{Q}{\Delta\Lambda P} - \frac{P}{Q\Lambda} = -\frac{Q^2 + \Delta P^2}{\Lambda QP\Delta} = \frac{\Sigma R^2}{L\Delta}, \tag{3.33}$$

and therefore

$$J^{-1} = \frac{L\Delta}{\Sigma R^2}\begin{pmatrix} -1/(\Lambda P) & -P \\ -1/(\Lambda Q) & Q/\Delta \end{pmatrix} = \frac{1}{\Sigma R^2}\begin{pmatrix} Q\Delta & -LP\Delta \\ P\Delta & LQ \end{pmatrix}. \tag{3.34}$$

■

With the knowledge of the derivatives can rewrite σ in (r_*, λ)-coordinates. From the previous proposition we know that

$$\Sigma R^2\, dr = Q\Delta\, dr_* - L\Delta P\, d\lambda, \tag{3.35}$$

$$\Sigma R^2\, d\theta = P\Delta\, dr_* + LQ\, d\lambda. \tag{3.36}$$

We can plug this into σ to obtain

$$\begin{aligned}
\sigma &= \frac{\Sigma}{R^2}\, dr^2 + \Sigma\, d\theta^2 \\
&= \frac{1}{\Sigma^2 R^4}\left(\frac{\Sigma}{R^2}\left(Q\Delta\, dr_* - L\Delta P\, d\lambda\right)^2 + \Sigma\left(P\Delta\, dr_* + LQ\, d\lambda\right)^2\right) \\
&= \frac{1}{\Sigma R^4}\left(\frac{1}{\Delta}\left(Q^2\Delta^2\, dr_*^2 + L^2P^2\Delta^2\, d\lambda^2\right) + \left(P^2\Delta^2\, dr_*^2 + L^2Q^2\, d\lambda^2\right)\right) \\
&= \frac{1}{\Sigma R^4}\left(\Delta(Q^2 + \Delta P^2)\, dr_*^2 + L^2(Q^2 + \Delta P^2)\, d\lambda^2\right) \\
&= \frac{1}{R^2}\left(\Delta\, dr_*^2 + L^2\, d\lambda^2\right).
\end{aligned} \tag{3.37}$$

We can also rewrite the (t, φ) part of the Kerr metric in the following way: First we define the so called ZAMO (zero angular momentum observer) angular velocity ω_B as

$$\omega_B = -\frac{g_{\varphi t}}{g_{tt}} = \frac{2Mar}{\Sigma R^2}, \tag{3.38}$$

which describes the frame dragging velocity of the black hole. Using this quantity, we then have

$$\begin{aligned}
& g_{tt}\, dt^2 + g_{\varphi\varphi}\, d\varphi^2 + 2g_{\varphi t}\, d\varphi\, dt \\
&= g_{tt}\, dt^2 - \frac{g_{\varphi t}^2}{g_{\varphi\varphi}}\, dt^2 + \frac{g_{\varphi t}^2}{g_{\varphi\varphi}}\, dt^2 + g_{\varphi\varphi}\, d\varphi^2 + 2g_{\varphi t}\, d\varphi\, dt \\
&= \frac{1}{g_{\varphi\varphi}}\big(\underbrace{g_{\varphi\varphi}g_{tt} - g_{\varphi t}^2}_{\det g_{(t,\varphi)}}\big)\, dt^2 + g_{\varphi\varphi}\left(\omega_B^2\, dt^2 + d\varphi^2 - 2\omega_B\, d\varphi\, dt\right) \\
&= -\frac{\Delta}{R^2}\, dt^2 + R^2\sin^2(\theta)\left(d\varphi - \omega_B\, dt\right)^2,
\end{aligned} \tag{3.39}$$

which is the metric in the form of a co-rotating reference frame with angular velocity ω_B. Now we add both metrics in order to obtain the full Kerr metric in $(t, r_*, \lambda, \varphi)$-coordinates:

$$g = -\frac{\Delta}{R^2}\left(dt^2 - dr_*^2\right) + \frac{L^2}{R^2}d\lambda^2 + R^2\sin^2(\theta)\left(d\varphi - \omega_B\, dt\right)^2. \tag{3.40}$$

From here we reintroduce the two null coordinates we started with in the Eikonal equation and obtain the Kerr metric in double null coordinates.

Proposition 3.5. *Let $u = (t - r_*)/2$, $v = (t + r_*)/2$. Also define a new angular coordinate $\varphi_* = \varphi - h(u, v)$ such that $\partial_u h = \omega_B$, $\partial_v h = -\omega_B$. The form of the Kerr metric in double null coordinates $(u, v, \lambda, \varphi_*)$ is*

$$g = -4\frac{\Delta}{R^2}\, du\, dv + \frac{L^2}{R^2}\, d\lambda^2 + R^2\sin^2(\theta)\left(d\varphi_* - 2\omega_B\, dv\right)^2, \tag{3.41}$$

Furthermore, the null lapse function Ω, the shift vector b and the induced metric $\not{g}$ on the two-surfaces of constant u, v are

$$
\begin{aligned}
\Omega^2_{\text{Kerr}} &= \frac{2\Delta}{R^2}, & b^{\lambda}_{\text{Kerr}} &= 0, & b^{\varphi_*}_{\text{Kerr}} &= 2\omega_B \\
\not{g}^{\text{Kerr}}_{\lambda\lambda} &= \frac{L^2}{R^2}, & \not{g}^{\text{Kerr}}_{\varphi_*\varphi_*} &= R^2\sin^2(\theta), & \not{g}^{\text{Kerr}}_{\lambda\varphi_*} &= 0.
\end{aligned}
\tag{3.42}
$$

Proof. The differentials of u and v are given by

$$du = \frac{dt - dr_*}{2}, \quad dv = \frac{dt + dr_*}{2}, \tag{3.43}$$

which means that $dt^2 - dr_*^2 = (dt + dr_*)(dt - dr_*) = 4\,du\,dv$. The metric in (u, v, λ, φ)-coordinates becomes

$$g = -4\frac{\Delta}{R^2}\,du\,dv + \frac{L^2}{R^2}\,d\lambda^2 + R^2\sin^2(\theta)\,(d\varphi - \omega_B\,(du + dv))^2\,. \tag{3.44}$$

This expression looks similar to the general expression for the metric in double null coordinates However, due to the appearance of du in the rotational term, it is not quite of the same form as in equation 2.23. We can overcome this obstacle by introducing a new azimuthal coordinate φ_*, which we define as

$$\varphi_*(u, v, \varphi) = \varphi - h(u, v), \tag{3.45}$$

where $h(u, v)$ is a differentiable function subject to the conditions

$$\frac{\partial h}{\partial u} = \omega_B, \quad \frac{\partial h}{\partial v} = -\omega_B. \tag{3.46}$$

These two conditions imply that the shift is coordinate-time independent: $\partial h/\partial t = 0$. We thereby have

$$d\varphi = d\varphi_* + \omega_B(du - dv), \tag{3.47}$$

which we use to rewrite the non-null part of the metric and obtain the Kerr metric in double null coordinates. Inserting equation 3.47 into the metric (ignoring the $du\,dv$ term) gives

$$
\begin{aligned}
&\frac{L^2}{R^2}\,d\lambda^2 + R^2\sin^2(\theta)\,(d\varphi - \omega_B\,(du + dv))^2 \\
&= \frac{L^2}{R^2}\,(d\lambda - 0\;dv)^2 + R^2\sin^2(\theta)\,(d\varphi_* - 2\omega_B\,dv)^2\,.
\end{aligned}
\tag{3.48}
$$

Comparing this to equation 2.23 (with $\theta^1 = \lambda$ and $\theta^2 = \varphi_*$), we obtain the values for b and $\not{g}$. Furthermore, the null lapse can be read off from the $du\,dv$ term: $-2\Omega^2 = -4\Delta/R^2$. ■

The coordinates are well-behaved as long the null geodesics associated to u, v don't intersect, that is to say that no caustics develop along them. The occurrence of a caustic is indicated by the vanishing of the volume form on the two-spheres: $\sqrt{\det \not{g}} = 0$, where the induced determinant is given by

$$\det \not{g} = \frac{L^2}{R^2}R^2\sin^2(\theta) = \Lambda^2 P^2 Q^2 \sin^2(\theta). \tag{3.49}$$

Later we will show that this coordinate system indeed exists by showing the existence $\lambda(r, \theta)$ in the exterior region of the black hole. In order to do this we will first study the simplified case in which the mass M vanishes.

4 Minkowski Space

In this section we will study the case of vanishing mass M. We will see that in this case we obtain the flat Minkowski metric in a special curvilinear coordinate system. We will first explicitly find the relation of these coordinates to spherical coordinates. Then we use the envelope method from the last chapter to obtain the same results in order to verify that this method produces the expected results and to find the correct boundary conditions. At last, we will look at the properties of the quantities $P, Q, \partial_\lambda F$ in Minkowski spacetime. We will need them later in the proof of existence of the angular function λ.

4.1 Oblate Spheroidal Coordinates

If we plug $M = 0$ into the Kerr metric in Boyer-Lindquist coordinates we are left with the following:

$$g_0 = -dt^2 + \frac{r^2 + a^2\cos^2(\theta)}{r^2 + a^2}\, dr^2 + \left(r^2 + a^2\cos^2(\theta)\right) d\theta^2 + \left(r^2 + a^2\right)\sin^2(\theta)\, d\varphi^2. \tag{4.1}$$

In this case the cross-term vanishes and we are left with a diagonal metric. This is expected since for $M = 0$ there is no source for the frame dragging effect and hence ω_B vanishes. However, this metric looks different from the Minkowski metric in spherical coordinates. Hence, we can infer that r is not the same as the spherical radial distance and θ not the usual spherical polar angle. To distinguish between them, we will denote the radial distance as r_* and the polar angle as θ_*. In spherical coordinates $(t, r_*, \theta_*, \varphi)$ the Minkowski metric is given by

$$\eta = -dt^2 + dr_*^2 + r_*^2\left(d\theta_*^2 + \sin^2(\theta_*)\, d\varphi^2\right). \tag{4.2}$$

As we will see, the metric obtained from setting $M = 0$ in the Kerr metric is the Minkowski metric expressed in so-called oblate spheroidal coordinates. We construct this coordinate system by first considering elliptic coordinates in the xz-plane, then rotating this plane around the z-axis.

Elliptical Coordinates

The elliptic coordinate system is a two-dimensional coordinate system in which we use ellipses and hyperbolas to specify a position on the plane. Furthermore, we choose the ellipses such that every ellipse has the same focal points, which we will fix at $(x, z) = (\pm a, 0)$.

The usual way to define an ellipse is via the equation

$$\frac{x^2}{A^2} + \frac{z^2}{B^2} = 1, \tag{4.3}$$

where A and B are the semi-major/minor-axis of the ellipse. An often used parametrization is

$$x = A\sin(\theta), \quad z = B\cos(\theta), \tag{4.4}$$

where $\theta \in (0, 2\pi)$. Since the focal points are fixed at $(\pm a, 0)$, A and B can not be chosen freely. The relation between the axes and the focal points is

$$a^2 = A^2 - B^2. \tag{4.5}$$

We want to use those ellipses to construct a coordinate system in $\mathbb{R}^2$ and obtain a foliation by ellipses. In order to specify a specific leaf of the foliation we define the coordinate function $r(p) = B(p)$, where $B(p)$ is the semi-minor-axis of the unique ellipse going through the point $p = (x, z)$. To specify the position on this ellipse, we use the parameter θ from the parametrization of the ellipse (eq. 4.4). As we already noted in the Kerr metric with vanishing mass, this θ is not the azimuthal angle θ_*. Only in the case when $a = 0$ the ellipses reduce to circles and $\theta = \theta_*$. In the general case we can construct θ via the point construction of de La Hire (depicted in figure 4).

With this we find a parametrization of (x, z) in terms of (r, θ):

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin(\theta), \\ z &= r\cos(\theta). \end{aligned} \tag{4.6}$$

By noting that $x^2 - z^2 = a^2\sin^2(\theta)$, we see that the coordinate lines of constant θ are hyperbolas. The coordinate lines of the coordinate system (r, θ) are sketched in figure 5.

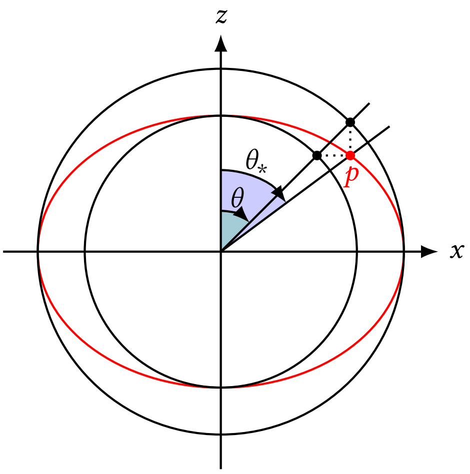


Figure 4: The point construction of an ellipse due to de La Hire.

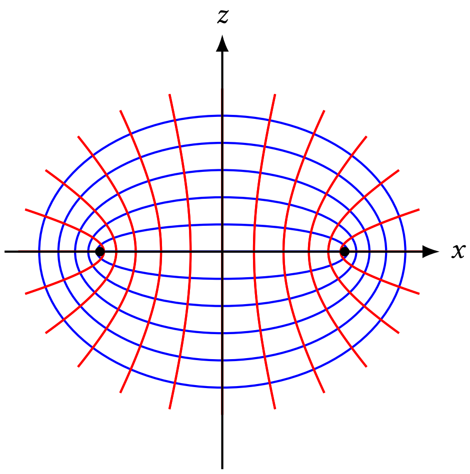


Figure 5: A sketch of the elliptic coordinate system in the xz-plane. The two black dots indicate the focal points of the ellipses.

Oblate Spheroidal Coordinates

We can extend this two-dimensional coordinate system into a three-dimensional one by rotating the plane around the axis separating the two foci, which in our case is the z-axis. We thereby introduce a new coordinate $\varphi \in (0, 2\pi)$ (the polar angle) and obtain a parametrization of (x, y, z) in terms of (r, θ, φ):

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin(\theta)\cos(\varphi) \\ y &= \sqrt{r^2 + a^2}\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta). \end{aligned} \tag{4.7}$$

Since both θ and φ range between 0 to 2π, this parametrization is twofold. Thus, we need to restrict either θ or φ. It is customary to restrict θ to $\theta \in (0, \pi)$. The coordinate system (r, θ, φ) is called the oblate spheroidal coordinate system since the surfaces of constant r are oblate spheres.

Relation to Spherical Coordinates

We now want to find the relation between the oblate and spherical coordinates. In the spherical coordinate system the parametrization of (x, y, z) is

$$\begin{aligned} x &= r_* \sin(\theta_*) \cos(\varphi) \\ y &= r_* \sin(\theta_*) \sin(\varphi) \\ z &= r_* \cos(\theta_*). \end{aligned} \tag{4.8}$$

To find an expression for r_* and θ_* in terms of the oblate coordinates we can compare the two different parametrizations of (x, y, z) and obtain

$$r_*^2 = x^2 + y^2 + z^2 = (r^2 + a^2)\sin^2(\theta) + r^2\cos^2(\theta) = r^2 + a^2\sin^2(\theta) \tag{4.9}$$

and

$$\tan^2(\theta_*) = \frac{x^2 + y^2}{z^2} = \frac{(r^2 + a^2)\sin^2(\theta)}{r^2\cos^2(\theta)} = \frac{r^2 + a^2}{r^2}\tan^2(\theta). \tag{4.10}$$

From the second equation we see that $\theta_* \geq \theta$ with $\theta_* = \theta$ only for $\theta = 0, \pi/2, \pi$. With this we can finally show that the metric g_0 is in fact the Minkowski metric. For this we compute the differentials of r_* and θ:

$$\begin{aligned} dr_* &= \frac{\partial r_*}{\partial r}\,dr + \frac{\partial r_*}{\partial \theta}\,d\theta = \frac{r}{\sqrt{r^2 + a^2\sin^2(\theta)}}\,dr + \frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2 + a^2\sin^2(\theta)}}\,d\theta \\ &= \frac{r}{r_*}\,dr + \frac{a^2\sin(\theta)\cos(\theta)}{r_*}\,d\theta \end{aligned} \tag{4.11}$$

$$\begin{aligned} d\theta_* &= \frac{\partial \theta_*}{\partial r}\,dr + \frac{\partial \theta_*}{\partial \theta}\,d\theta = -\frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2 + a^2}(r^2 + a^2\sin^2(\theta))}\,dr + \frac{r\sqrt{r^2 + a^2}}{r^2 + a^2\sin^2(\theta)}\,d\theta \\ &= -\frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2 + a^2}r_*^2}\,dr + \frac{r\sqrt{r^2 + a^2}}{r_*^2}\,d\theta \end{aligned} \tag{4.12}$$

We can plug these differentials into the expression of the Minkowski metric in spherical coordinates (ignoring the t- and φ-terms) to obtain

$$\begin{aligned} dr_*^2 + r_*^2\,d\theta_*^2 &= \left(\frac{r}{r_*}\,dr + \frac{a^2\sin(\theta)\cos(\theta)}{r_*}\,d\theta\right)^2 + r_*^2\left(-\frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2 + a^2}r_*^2}\,dr + \frac{r\sqrt{r^2 + a^2}}{r_*^2}\,d\theta\right)^2 \\ &= \frac{r^2(r^2 + a^2) + a^2\sin^2(\theta)\,a^2\cos^2(\theta)}{r_*^2(r^2 + a^2)}\,dr^2 + \frac{a^2\sin^2(\theta)\,a^2\cos^2(\theta) + r^2(r^2 + a^2)}{r_*^2}\,d\theta^2 \\ &\quad + \frac{1}{r_*^2}\left(ra^2\sin(\theta)\cos(\theta) - \frac{ra^2\sin(\theta)\cos(\theta)\sqrt{r^2 + a^2}}{\sqrt{r^2 + a^2}}\right)dr\,d\theta \\ &= \frac{1}{r_*^2}\left(\frac{r^4 + r^2a^2(\sin^2(\theta) + \cos^2(\theta)) + a^2\sin^2(\theta)\,a^2\cos^2(\theta)}{r^2 + a^2}\right)dr^2 \\ &\quad + \frac{1}{r_*^2}\left(a^2\sin^2(\theta)\,a^2\cos^2(\theta) + r^4 + r^2a^2(\sin^2(\theta) + \cos^2(\theta))\right)\,d\theta^2 \end{aligned} \tag{4.13}$$

$$= \frac{(r^2 + a^2 \sin^2(\theta))(r^2 + a^2 \cos^2(\theta))}{r_*^2(r^2 + a^2)} \, dr^2 + \frac{(r^2 + a^2 \sin^2(\theta))(r^2 + a^2 \cos^2(\theta))}{r_*^2} \, d\theta^2$$

$$= \frac{r^2 + a^2 \cos^2(\theta)}{r^2 + a^2} \, dr^2 + \left(r^2 + a^2 \cos^2(\theta)\right) \, d\theta^2, \tag{4.14}$$

which is precisely the r, θ-part of g_0. In order to rewrite the $d\varphi^2$-term we need an expression for $\sin^2(\theta_*)$, which we find to be

$$\sin^2(\theta_*) = \frac{\tan^2(\theta_*)}{1 + \tan^2(\theta_*)} = \frac{r^2 + a^2}{r^2 + a^2 \sin^2(\theta)} \sin^2(\theta). \tag{4.15}$$

Hence, we find

$$r_*^2 \sin^2(\theta_*) = \left(r^2 + a^2 \sin^2(\theta)\right) \frac{r^2 + a^2}{r^2 + a^2 \sin^2(\theta)} \sin^2(\theta) = \left(r^2 + a^2\right) \sin^2(\theta). \tag{4.16}$$

With this we have shown that the Kerr metric in Boyer-Lindquist coordinates is indeed the Minkowski metric expressed in oblate spheroidal coordinates.

4.2 Solving the Eikonal Equation

We now proceed to solve the Eikonal equation giving the derivation of Pretorius and Israel in detail. For vanishing mass M, Q simplifies to

$$Q_0(r, \lambda) = \sqrt{(r^2 + a^2)^2 - a^2\lambda(r^2 + a^2)} = \sqrt{(r^2 + a^2)(r^2 + a^2(1 - \lambda))}, \tag{4.17}$$

while P left unchanged because it is independent of M. In the following we will assume that $\sin^2(\theta) < \lambda < 1$. In this case we can find a change of variables from $r \to \psi$ such that the two integrals occurring in the expression of F are of the same form.

By looking at the definition of the constraint function F (eq. 3.22), we can infer that the desired function $\psi(r, \lambda)$ must satisfy

$$\frac{dr}{Q_0(r, \lambda)} = \frac{d\psi}{P(\psi, \lambda)}. \tag{4.18}$$

An implicit solution to this differential equation is given by

$$r(\psi, \lambda)^2 = \frac{a^2(1 - \lambda) \sin^2(\psi)}{\lambda - \sin^2(\psi)}, \tag{4.19}$$

which can be verified by an explicit calculation: First we find that

$$\begin{aligned}
Q_0(r(\psi, \lambda), \lambda) &= \sqrt{\left(\frac{a^2(1 - \lambda) \sin^2(\psi)}{\lambda - \sin^2(\psi)} + a^2\right)\left(\frac{a^2(1 - \lambda) \sin^2(\psi)}{\lambda - \sin^2(\psi)} + a^2(1 - \lambda)\right)} \\
&= a^2 \frac{\sqrt{\left((1 - \lambda) \sin^2(\psi) + (\lambda - \sin^2(\psi))\right)\left((1 - \lambda) \sin^2(\psi) + (1 - \lambda)(\lambda - \sin^2(\psi))\right)}}{\lambda - \sin^2(\psi)} \\
&= \frac{a^2 \lambda \sqrt{1 - \lambda} \cos(\psi)}{\lambda - \sin^2(\psi)}.
\end{aligned} \tag{4.20}$$

Taking the ψ-derivative of r gives

$$\begin{aligned}
\frac{\partial r}{\partial \psi} &= \sqrt{\frac{\lambda - \sin^2(\psi)}{a^2 \sin^2(\psi)(1-\lambda)}} \frac{(1-\lambda)(\lambda - \sin^2(\psi)) + \sin^2(\psi)(1-\lambda)}{\lambda - \sin^2(\psi)} \sin(\psi)\cos(\psi) \\
&= \frac{a^2 \lambda \sqrt{1-\lambda}\cos(\psi)}{a\sqrt{\lambda - \sin^2(\psi)}\,(\lambda - \sin^2(\psi))} \\
&= \frac{Q_0(r(\psi,\lambda),\lambda)}{P(\psi,\lambda)}.
\end{aligned} \tag{4.21}$$

Using the change of variables from r to ψ, we obtain the new expression for F_0:

$$\begin{aligned}
F_0(r,\theta,\lambda) &= \int_{\psi(r,\lambda)}^{\psi(\infty,\lambda)} \frac{d\psi'}{P(\psi',\lambda)} + \int_0^{\theta} \frac{d\theta'}{P(\theta',\lambda)} + f'(\lambda) \\
&= \int_{\psi(r,\lambda)}^{\theta} \frac{d\psi'}{P(\psi',\lambda)} + \int_0^{\psi(\infty,\lambda)} \frac{d\psi'}{P(\psi',\lambda)} + f'(\lambda).
\end{aligned}$$

Solving equation 4.19 for ψ we find

$$\psi(r,\lambda) = \arcsin\left(\sqrt{\frac{\lambda r^2}{r^2 + a^2(1-\lambda)}}\right), \tag{4.22}$$

from which we see that the limit when $r \to \infty$ indeed exists and is given by

$$\lim_{r\to\infty} \psi(r,\lambda) = \lim_{r\to\infty} \arcsin\left(\sqrt{\frac{\lambda}{1 + a^2/r^2(1-\lambda)}}\right) = \arcsin\left(\sqrt{\lambda}\right). \tag{4.23}$$

This is a bit problematic since this implies that

$$\lim_{r\to\infty} P(\psi(r,\lambda),\lambda) = \sqrt{\lambda - \sin^2(\arcsin(\sqrt{\lambda}))} = 0, \tag{4.24}$$

which means, in the limit as $r \to \infty$, that the integrand of the second integral blows up at the upper integration limit for any value of λ. However, by choosing an appropriate boundary condition we can absorb this divergence. Remembering that we actually integrate to a fixed value r_∞ instead of ∞, we can to this by stipulating that

$$\int_0^{\psi(r_\infty,\lambda)} \frac{d\psi'}{P(\psi',\lambda)} + f'(\lambda) = 0. \tag{4.25}$$

Since we can do this for any choice of r_∞, we can define $f'(\lambda)$ as a limit of a family of functions $\tilde{f}'(\lambda, r_\infty)$ such that

$$\lim_{r_\infty\to\infty}\left(\int_0^{\psi(r_\infty,\lambda)} \frac{d\psi'}{P(\psi',\lambda)} + \tilde{f}'(\lambda, r_\infty)\right) = 0 \tag{4.26}$$

holds. Of course the limit itself is not well-defined (it diverges), however in the sum the divergent parts cancel. In this special sense, we can still consider the limit of $r_\infty \to \infty$. By imposing this boundary condition (specifically 4.25), we obtain that

$$F_0(r,\theta,\lambda) = \int_{\psi(r,\lambda)}^{\theta} \frac{d\psi'}{P(\psi',\lambda)}. \tag{4.27}$$

From this form of F_0 we can directly find a solution λ_0:

Proposition 4.1. *Let $F_0(r,\theta,\lambda)$ be defined as in equation 4.27. Then a solution to the constraint equation $F_0(r,\theta,\lambda) = 0$ is given by*

$$\lambda = \lambda_0(r,\theta) = \frac{r^2 + a^2}{r^2 + a^2 \sin^2(\theta)} \sin^2(\theta). \tag{4.28}$$

Proof. We want to find a solution to the equation

$$\int_{\psi(r,\lambda)}^{\theta} \frac{d\psi'}{P(\psi',\lambda)} = 0. \tag{4.29}$$

By our assumptions we know that P is a positive function and hence the only way that this integral vanishes is when the integration domain itself is empty. This is precisely the case if

$$\theta = \psi(r,\lambda) = \psi(r,\lambda_0(r,\theta)), \tag{4.30}$$

which means that the implicit relationship between r, θ, λ is governed by the function ψ. Replacing ψ with θ in equation 4.19 and solving for λ, we obtain the stated result for $\lambda_0(r,\theta)$. ■

With $\lambda_0(r,\theta)$ known explicitly, we can now find an explicit expression for r_* by integrating equation 3.16 to obtain

Proposition 4.2. *In Minkowski spacetime the tortoise coordinate is given by*

$$r_*(r,\theta) = \sqrt{r^2 + a^2 \sin^2(\theta)}. \tag{4.31}$$

Proof. Inserting $\lambda_0(r,\theta)$ into $P(\theta,\lambda)$ gives

$$\begin{aligned} P(\theta,\lambda_0(r,\theta)) &= a\sin(\theta)\sqrt{\frac{r^2 + a^2}{r^2 + a^2\sin^2(\theta)} - 1} = \frac{a^2 \sin(\theta)\cos(\theta)}{\sqrt{r^2 + a^2 \sin^2(\theta)}} \\ &= \frac{\partial}{\partial\theta}\sqrt{r^2 + a^2 \sin^2(\theta)}, \end{aligned} \tag{4.32}$$

from which we can obtain a first expression for r_* by integrating $d\rho$ (eq. 3.16) with respect to θ:

$$r_*(r,\theta) = \int_0^{\theta} P(\theta', \lambda_0(r,\theta'))\, d\theta' = \sqrt{r^2 + a^2 \sin^2(\theta)} + C(r). \tag{4.33}$$

Taking the derivative with respect to r gives

$$\frac{\partial r_*}{\partial r} = \frac{r}{\sqrt{r^2 + a^2 \sin^2(\theta)}} + C'(r). \tag{4.34}$$

From $(d\rho)_r = Q/\Delta$(eq. 3.16) we also know

$$\begin{aligned}
\frac{\partial r_*}{\partial r} = \frac{Q_0(r, \lambda_0(r,\theta))}{\Delta_0(r)} &= \frac{\sqrt{(r^2+a^2)\left(r^2 + a^2(1-\lambda_0(r,\theta))\right)}}{r^2+a^2} \\
&= \left(\frac{r^2\left(r^2 + a^2\sin^2(\theta)\right) + a^2\left(r^2 + a^2\sin^2(\theta) - (r^2+a^2)\sin^2(\theta)\right)}{\left(r^2+a^2\right)\left(r^2 + a^2\sin^2(\theta)\right)}\right)^{1/2} \\
&= \left(\frac{r^2}{r^2 + a^2\sin^2(\theta)} \frac{r^2 + a^2\left(\sin^2(\theta) + \cos^2(\theta)\right)}{r^2+a^2}\right)^{1/2} \\
&= \frac{r}{\sqrt{r^2 + a^2\sin^2(\theta)}}.
\end{aligned} \tag{4.35}$$

Comparing the two expressions for $\partial_r r_*$, we find that $C'(r) = 0$ and hence $C(r) = \text{const}$. We can choose this constant to be zero, which has the effect that $r_* = r$ at $\theta = 0, \pi$. ■

What is left is to relate the polar angle θ_* with newly found λ_0. By defining the polar angle θ_* implicitly via $\lambda_0 = \sin^2(\theta_*)$, we find that

$$\sin^2(\theta_*) = \frac{r^2 + a^2}{r^2 + a^2\sin^2(\theta)} \sin^2(\theta) \tag{4.36}$$

and hence also reproduce the relation between $\tan(\theta)$ and $\tan(\theta_*)$:

$$\tan^2(\theta_*) = \frac{\sin^2(\theta_*)}{1 - \sin^2(\theta_*)} = \frac{(r^2+a^2)\sin^2(\theta)}{r^2 + a^2\sin^2(\theta) - (r^2+a^2)\sin^2(\theta)} = \frac{r^2+a^2}{r^2}\tan^2(\theta). \tag{4.37}$$

It is important to note that the derived coordinate function λ_0 is only defined for $\theta \in (0, \pi/2)$ since $\partial_\theta \lambda_0 = 0$ on the equator $\theta = \pi/2$. Hence, by using λ as a coordinate, we only can establish a local coordinate system on the northern hemisphere of the spheres of constant $t \pm r_*$. To construct a global (ignoring the usual complications which arise from using spherical coordinates) coordinate system, we instead adapt θ_* as a coordinate extend and extend θ_* onto $\theta \in (0, \pi)$ by defining

$$\theta_*(r, \pi/2) = \pi/2 \quad \text{and} \quad \theta_*(r,\theta) = \theta_*(r, \pi - \theta) \quad \text{for} \quad \theta \in (\pi/2, \pi). \tag{4.38}$$

By solving the Eikonal equation we thus confirm the expression for the tortoise coordinate r_* (in the case the spherical distance) and the polar angle θ_*. Thereby we have shown that this method of solving the Eikonal equation, using the boundary conditions above, produces the expected results in Minkowski spacetime.

4.3 Properties of $Q, P, \partial_\lambda F$ in Minkowski

With $\lambda = \lambda_0(r,\theta)$ and $r_*(r,\theta)$ known for $\theta \in (0, \pi/2)$, we can find the form of the quantities P, Q_0 and $\partial_\lambda F_0$ in Minkowski spacetime.

Proposition 4.3. *Let $\lambda_0(r,\theta)$ be the solution to the equation $F_0(r,\theta,\lambda_0(r,\theta)) = 0$ and let $P(\theta,\lambda)$, $Q_0(r,\lambda)$ be defined according to equations 3.14 and 3.15. Then we have that*

$$Q_0(r,\lambda_0(r,\theta)) = \frac{r(r^2+a^2)}{\sqrt{r^2+a^2\sin^2(\theta)}} \tag{4.39}$$

$$P(\theta,\lambda_0(r,\theta)) = \frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2+a^2\sin^2(\theta)}}. \tag{4.40}$$

Proof. Since $r_*(r,\theta) = \rho(r,\theta,\lambda_0(r,\theta))$ is known, we can simply take the r- and θ-derivatives of r_* in order to directly obtain $Q_0(r,\lambda_0(r,\theta))$ and $P(\theta,\lambda_0(r,\theta))$:

$$Q_0(r,\lambda_0(r,\theta)) = \Delta_0\frac{\partial r_*}{\partial r} = (r^2+a^2)\frac{r}{\sqrt{r^2+a^2\sin^2(\theta)}}$$

$$P(\theta,\lambda_0(r,\theta)) = \frac{\partial r_*}{\partial\theta} = \frac{a^2\sin(\theta)\cos(\theta)}{\sqrt{r^2+a^2\sin^2(\theta)}}$$

■

Also, we can find an expression for the λ-derivative of F_0, which is an important quantity in the applicability of the implicit function theorem.

Proposition 4.4. *Let $\lambda_0(r,\theta)$ be the solution to the equation $F_0(r,\theta,\lambda_0(r,\theta)) = 0$. Then the λ-derivative of F_0 is given by*

$$(\partial_\lambda F_0)(r,\theta,\lambda_0(r,\theta)) = -\frac{\left(r^2+a^2\sin^2(\theta)\right)^{5/2}}{2a^2r^2(r^2+a^2)}\frac{1}{\sin^2(\theta)\cos^2(\theta)} < 0, \tag{4.41}$$

Proof. $F_0(r,\theta,\lambda)$ is given by

$$F_0(r,\theta,\lambda) = \int_{\psi(r,\lambda)}^{\theta}\frac{d\psi'}{P(\psi',\lambda)}. \tag{4.42}$$

Taking the λ-derivative gives

$$(\partial_\lambda F_0)(r,\theta,\lambda) = -\frac{a^2}{2}\int_{\psi(r,\lambda)}^{\theta}\frac{d\psi'}{P(\psi',\lambda)^3} - \frac{(\partial_\lambda\psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)}. \tag{4.43}$$

Since we are interested in the $\lambda = \lambda_0(r,\theta)$ case and $\psi(r,\lambda_0(r,\theta)) = \theta$, the integral term vanishes. For the second term we have $\sin^2(\psi) = r^2\lambda/(r^2+a^2(1-\lambda))$ and hence find

$$P(\psi(r,\lambda),\lambda) = a\sqrt{\lambda - \frac{r^2\lambda}{r^2+a^2(1-\lambda)}} = \frac{a^2\sqrt{\lambda(1-\lambda)}}{\sqrt{r^2+a^2(1-\lambda)}}. \tag{4.44}$$

Furthermore, the derivative of ψ is given by

$$\begin{aligned}\partial_\lambda \psi &= \frac{\partial_\lambda \sin^2(\psi)}{2\sin(\psi)\cos(\psi)}\\ &= \frac{1}{2\sin(\psi)\sqrt{1-\sin^2(\psi)}}\,\partial_\lambda\left(\frac{r^2\lambda}{r^2+a^2(1-\lambda)}\right)\\ &= \frac{r^2+a^2(1-\lambda)}{2\sqrt{r^2\lambda}\sqrt{r^2(1-\lambda)+a^2(1-\lambda)}}\frac{r^2(r^2+a^2)}{r^2+a^2(1-\lambda)}\\ &= \frac{r\sqrt{r^2+a^2}}{r^2+a^2(1-\lambda)}\frac{1}{2\sqrt{\lambda(1-\lambda)}}.\end{aligned} \tag{4.45}$$

Putting the pieces together we obtain

$$(\partial_\lambda F_0)(r,\theta,\lambda_0(r,\theta)) = -\left(\frac{r(r^2+a^2)}{r^2+a^2(1-\lambda_0)}\right)^{1/2}\frac{1}{2a^2\lambda_0(1-\lambda_0)} \tag{4.46}$$

To get the stated expression we note that

$$r^2+a^2(1-\lambda_0) = \frac{Q_0(r,\lambda_0)^2}{r^2+a^2} = \frac{r^2(r^2+a^2)}{r^2+a^2\sin^2(\theta)}. \tag{4.47}$$

Thus, inserting $\lambda_0(r,\theta)$ gives us

$$\begin{aligned}(\partial_\lambda F_0)(r,\theta,\lambda_0(r,\theta)) &= -\sqrt{r^2+a^2\sin^2(\theta)}\frac{(r^2+a^2\sin^2(\theta))^2}{2a^2(r^2+a^2)\sin^2(\theta)\,r^2\cos^2(\theta)}\\ &= -\frac{\left(r^2+a^2\sin^2(\theta)\right)^{5/2}}{2a^2r^2(r^2+a^2)\sin^2(\theta)\cos^2(\theta)}.\end{aligned} \tag{4.48}$$

■

From these expressions we see that $\partial_\lambda F_0$ is non-zero for any values of r, θ and hence, in the view of the implicit function theorem, the calculations we did previously were justified. However, we also see that $\partial_\lambda F_0$ blows up if $\theta \to 0, \pi/2$, which underlines the fact that our coordinate system in only defined on the patch for which $\theta \in (0, \pi/2)$. This is because if $\partial_\lambda F_0$ diverges, then the IFT is no longer applicable.

In the general case with non-vanishing mass, $\partial_\lambda F$ also diverges as $\theta \to 0, \pi/2$. We want to show that the divergent behaviour is the same as in this case which would mean (this becomes clear in the next chapter) that we could also find coordinates on the $\theta \in (0, \pi/2)$ patch in the general case.

5 The General Case – Proof of Existence

In this chapter we want to proof that there exist solutions to the constraint equation for non-vanishing mass M. From here on we will instead work with the reduced mass $\mu = M/r$, which has the advantage that it is a dimensionless quantity, and thus we can make assumptions about its smallness. If we assume μ to be small, due to M being fixed, then this assumption means that we are looking at a region far away from the black hole.

We also promote μ to a variable, which means not treating μ merely as a passive parameter, but rather enlarging our variable space from (r, θ) to (r, θ, μ) and looking for solutions $\lambda(r, \theta, \mu)$ to the equation

$$F(r, \theta, \lambda(r, \theta, \mu), \mu) = 0. \tag{5.1}$$

Comparing the new notation to the last chapter we have that $F_0(r, \theta, \lambda) = F(r, \theta, \lambda, 0)$ and also $\lambda_0(r, \theta) = \lambda(r, \theta, 0)$. To ease up the notation later we will often drop the r and θ-dependence of $\lambda(r, \theta, \mu)$ and write λ_μ instead. In contrast, a λ without any subscript will refer to a general λ not dependent on r, θ and μ.

5.1 Boundary Conditions

The function F depends on an arbitrary function $f'(\lambda)$ which we need to fix. In the Minkowski case, after making the coordinate substitution $r \to \psi$, we made the choice of equation 4.25, which had the effect of absorbing the divergent behaviour (in the limit of $r_\infty \to \infty$) of F into the boundary condition. In the limit we have

$$\lim_{r \to \infty} \psi(r, \lambda) = \arcsin(\sqrt{\lambda}) = \theta_*(\lambda). \tag{5.2}$$

As a result, since $\psi(r, \lambda_0) = \theta$, we obtain that $\lim_{r \to \infty} \theta_*(r, \theta) = \theta$. That is to say that asymptotically the angles θ and θ_* become the same. Since Kerr spacetime is asymptotically flat, i.e. is close to the Minkowski metric for large r, we want to have the same property to hold there. This is ensured if we impose the same boundary condition for F, i.e.

$$f'(\lambda) = - \int_0^{\psi(r_\infty, \lambda)} \frac{d\psi'}{P(\psi', \lambda)}, \tag{5.3}$$

where we remember that we actually integrate to r_∞ in order to ensure the convergence of this and also the complete integral ρ. By choosing this condition, F is given by

$$F(r, \theta, \lambda, \mu) = \int_r^{r_\infty} \frac{1}{Q_\mu(r', \lambda)} dr' - \int_\theta^{\psi(r_\infty, \lambda)} \frac{1}{P(\theta', \lambda)} d\theta'. \tag{5.4}$$

In the Minkowski case the boundary condition was precisely chosen such that we could still consider the limit $r_\infty \to \infty$. Here, we would like to consider this limit likewise, since the expressions become more convenient. However, here the transformation $r \to \psi$ does not reduce the expression for F to the same form as before. Thus, we are not able to do this directly.

As we will see in a moment, in the bootstrap argument, we are interested in estimating differences of F's and $\partial_\lambda F$'s. When estimating such differences, we would like to get rid of the auxiliary variable r_∞ in order to obtain simpler estimates. In particular, when estimating the r-integral, the integral still converges when taking the limit $r_\infty \to \infty$. This results in slightly weaker bounds, which are still enough for our argument later. As for the second integral, we will see that we only need to work with differences in λ at $\mu = 0$. This means that in this case we can just use the Minkowski expressions, for which we know the limiting expression.

5.2 Idea of the Proof – Bootstrapping

Let us consider the point $(r_0, \theta_0, \lambda_0, 0)$. Since we know that

$$F(r_0, \theta_0, \lambda_0(r_0, \theta_0), 0) = 0 \quad \text{and} \quad (\partial_\lambda F)(r_0, \theta_0, \lambda_0(r_0, \theta_0), 0) \neq 0, \tag{5.5}$$

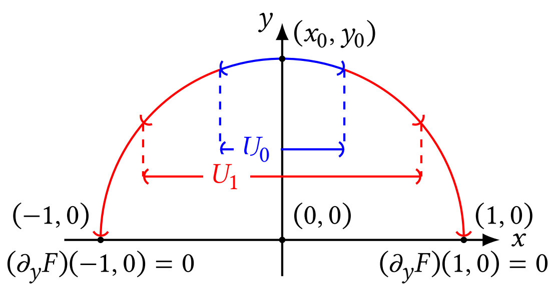
the IFT guarantees that there exists an open domain $U \ni (r_0, \theta_0, 0)$ together with a function $\lambda : U \to \mathbb{R}$ for which $F(r, \theta, \lambda(r, \theta, \mu), \mu) = 0$ is satisfied on U. Notably this already shows that there exist local solutions for non-vanishing μ. However, we want to show that the domain U is in fact the whole $\theta \in (0, \pi/2)$ patch (where μ is suitably small).

A Motivating Example

Before we concern ourselves with the proof of this proposition, let us consider a simpler example to illustrate this problem. Let us consider the function

$$F(x, y) = x^2 + y^2 - 1 = 0, \tag{5.6}$$

which is the locus of the unit circle in $\mathbb{R}^2$. Suppose that we are not able to solve this equation for y directly. If we want to show the existence of a function $y(x)$ such that $F(x, y(x)) = 0$, we first need a starting point (x_0, y_0) which satisfies the constraint equation. In our case we can choose $(x_0, y_0) = (0, 1)$. The y-derivative at this point is given by $(\partial_y F)(x_0, y_0) = 2 \neq 0$. With this the IFT guarantees that there exists a neighbourhood $U_0 \ni x_0$ and a function $y(x)$ on U_0 such that $F(x, y(x)) = 0$ on U_0. However, the IFT does not make any statements about the size of U_0. In principle, we can incrementally increase the size of U_0 to larger neighbourhoods $U_1 \subset \cdots \subset U_n \subset \ldots$ as long as the requirements for the IFT are fulfilled. In particular, we need to check if $\partial_y F \neq 0$ since the IFT also does not give information about the value of the derivative on the neighbourhood. We find the maximal domain to be $(-1, 1)$ because we have that $(\partial_y F)(\pm 1, 0) = 0$ and thus the IFT does not hold at those two points. In our case we can find $y(x)$ explicitly to be $y(x) = \pm\sqrt{1 - x^2}$, from which we see that $y(x)$ only describes the upper (or lower) part of the circle. This example is depicted the following sketch:



Bootstrapping

In our proof in Kerr spacetime we also want to show that we can expand a given domain. We will be using a technique called the bootstrap method to conduct the proof. The assumptions of the bootstrap principle formally captures our idea to incrementally increase the size of the domain for which we have a solution to the constraint equation. The bootstrap method works as follows: Assume we want to prove that an inequality holds on a region X. Then we can show that this inequality is true by assuming it and showing that we can infer a stronger inequality from it. Of course, we are not able to just freely assume the inequality we are trying to prove. We also need to provide a base case for which we independently know the correctness. In this sense the bootstrap principle can be thought of as a continuous version of induction. More generally the bootstrap principle can be formulated in the following way [6]:

Proposition 5.1. *Let $X \subset \mathbb{R}^n$ be connected and consider two statements (hypothesis and conclusion) $H, C\colon X \to \mathbb{Z}_2$ which we call true if they are 1. Assume that*

(a) If $H(x)$ is true then so is $C(x)$

(b) If $C(x)$ is true then there exists a neighbourhood $U \ni x$ such that $H(y)$ is true for all $y \in U$

(c) For any sequence $(x_n) \subset X$ converging to x such that $C(x_n)$ is true then also $C(x)$ is true

(d) There exists $x_0 \in X$ such that $H(x_0)$ is true.

Then $H(x)$ is true for all $x \in X$.

Proof. Combining assumptions (a) and (d) we know that there exists x_0 such that $C(x_0)$ is true. Furthermore, from combining (b) and (a) we know that $C(x)$ is true on an open neighbourhood of x_0. From assumption (c) it follows that this neighbourhood is also closed and hence, since it is non-empty, must be X and thus the hypothesis is true on X. ■

Structure of the Proof

In our case we want to show that the requirements for the IFT are met. In particular, we need to show that $(\partial_\lambda F)(r, \theta, \lambda, \mu) \neq 0$ on U and where $\lambda \in [\lambda_\mu, \lambda_0]$[7]. Because we do not know λ_μ explicitly, we generally do not have a simple expression for $\partial_\lambda F$ as in the Minkowski case from which we could easily infer its properties. Hence, we want to assume that $(\partial_\lambda F)(r, \theta, \lambda, \mu)$ is close enough to $(\partial_\lambda F)(r, \theta, \lambda_0, 0) \neq 0$ such that it doesn't vanish. Since we can split $\partial_\lambda F$ as

$$(\partial_\lambda F)(r, \theta, \lambda, \mu) = (\partial_\lambda F)(r, \theta, \lambda_0, 0) - ((\partial_\lambda F)(r, \theta, \lambda_0, 0) - (\partial_\lambda F)(r, \theta, \lambda, \mu)), \tag{5.7}$$

this is ensured if we assume that an inequality (the bootstrap assumption) of the form

$$\sup_\lambda |(\partial_\lambda F)(r, \theta, \lambda, \mu) - (\partial_\lambda F)(r, \theta, \lambda_0, 0)| \leq \varepsilon |(\partial_\lambda F)(r, \theta, \lambda_0, 0)|, \tag{5.8}$$

with $\varepsilon < 1$ (say $\varepsilon = 1/3$) holds. From this assumption we can then infer a bound on the distance between λ_μ and λ_0:

$$|\lambda_\mu - \lambda_0| \lesssim \mu = \text{small}. \tag{5.9}$$

[7] A priori, we do not know whether $\lambda_0 \geq \lambda_\mu$ or $\lambda_\mu \geq \lambda_0$. In lemma 5.4 we show that the former is true. Interestingly, our proof will still work even without this lemma

Using the smallness of $|\lambda - \lambda_0|$ we can in turn improve the bound of equation 5.8 which "closes the bootstrap" to finish the proof.

5.3 The Detailed Proof

We now turn to the details of the bootstrap argument.

Preliminary Observations

Let us start by finding a bound for the difference of the F's evaluated at λ_0 and λ_μ at same μ:

Lemma 5.1.

$$|F(r,\theta,\lambda_\mu,\mu) - F(r,\theta,\lambda_0,\mu)| \leq \frac{a^2}{3r^3}\,\mu. \tag{5.10}$$

Proof. First we note that we can just consider the difference at fixed $\lambda = \lambda_0$ since

$$|F(r,\theta,\lambda_\mu,\mu) - F(r,\theta,\lambda_0,\mu)| = |F(r,\theta,\lambda_0,0) - F(r,\theta,\lambda_0,\mu)|, \tag{5.11}$$

as both $F(r,\theta,\lambda_\mu,\mu) = F(r,\theta,\lambda_0,0) = 0$. We have that

$$|F(r,\theta,\lambda_0,0) - F(r,\theta,\lambda_0,\mu)| \leq \int_r^\infty \left|\frac{1}{Q_0(r',\lambda_0)} - \frac{1}{Q_\mu(r',\lambda_0)}\right| dr', \tag{5.12}$$

with Q_μ given by

$$Q_\mu(r,\lambda_0) = \sqrt{(r^2+a^2)^2 - a^2\lambda_0\Delta_\mu} = \sqrt{Q_0(r,\lambda_0)^2 + 2a^2r^2\lambda_0\mu} \geq Q_0(r,\lambda_0). \tag{5.13}$$

Using the expressions for $Q_0(r,\lambda_0)$ and λ_0, we can bound the integrand in the following way:

$$\begin{aligned}
\left|\frac{1}{Q_0} - \frac{1}{Q_\mu}\right| &= \frac{|Q_\mu - Q_0|}{Q_0 Q_\mu} = \frac{|Q_\mu^2 - Q_0^2|}{Q_0 Q_\mu (Q_0 + Q_\mu)} \\
&= \frac{2a^2r^2\lambda_0\mu}{Q_0 Q_\mu (Q_0 + Q_\mu)} \leq \frac{2a^2r^2\lambda_0\mu}{2Q_0^3} \\
&= \frac{a^2r^2\left(r^2+a^2\right)\sin^2(\theta)\left(r^2 + a^2\sin^2(\theta)\right)^{3/2}}{\left(r^2 + a^2\sin^2(\theta)\right)\left(r(r^2+a^2)\right)^3}\mu \\
&= \frac{a^2\sin^2(\theta)\left(r^2+a^2\sin^2(\theta)\right)^{1/2}}{r\left(r^2+a^2\right)^2}\mu \\
&\leq \frac{a^2\sin^2(\theta)\left(r^2+a^2\right)^{1/2}}{r\left(r^2\right)^{3/2}\left(r^2+a^2\right)^{1/2}}\mu = \frac{a^2}{r^4}\,\mu.
\end{aligned} \tag{5.14}$$

Integrating this expression gives us

$$|F(r,\theta,\lambda_0,0) - F(r,\theta,\lambda_0,\mu)| \leq a^2\,\mu \int_r^\infty \frac{dr'}{r'^4} = \frac{a^2}{3r^3}\,\mu. \tag{5.15}$$

■

Lemma 5.2. *Assume that the bootstrap assumption (eq. 5.8) holds with $\varepsilon = 1/3$. Then we have that*

$$|\lambda_\mu - \lambda_0| \leq \tilde{C}(r)\mu \sin^2(\theta)\cos^2(\theta), \tag{5.16}$$

with $\tilde{C}(r) = \frac{a^4}{r^4}\left(1 + \frac{a^2}{r^2}\right)$.

Proof. We want to prove this Lemma by exploiting the mean value theorem. Since F is differentiable in λ, the mean value theorem states that there exists a $\lambda_* \in [\lambda_0, \lambda_\mu]$ such that

$$\begin{aligned} |F(r,\theta,\lambda_\mu,\mu) - F(r,\theta,\lambda_0,\mu)| &= |(\partial_\lambda F)(r,\theta,\lambda_*,\mu)| \cdot |\lambda_\mu - \lambda_0| \\ &\geq \inf_{\lambda \in [\lambda_0,\lambda_\mu]} |(\partial_\lambda F)(r,\theta,\lambda,\mu)| \cdot |\lambda_\mu - \lambda_0|, \end{aligned} \tag{5.17}$$

which implies that

$$|\lambda_\mu - \lambda_0| \leq \frac{|F(r,\theta,\lambda_\mu,\mu) - F(r,\theta,\lambda_0,\mu)|}{\inf_{\lambda\in[\lambda_0,\lambda_\mu]} |(\partial_\lambda F)(r,\theta,\lambda,\mu)|}. \tag{5.18}$$

We can estimate the infimum of $\partial_\lambda F$ by using the bootstrap assumption:

$$\begin{aligned} |(\partial_\lambda F)(\lambda_\mu,\mu)| &\geq |(\partial_\lambda F)(\lambda_0,0)| - |(\partial_\lambda F)(\lambda_0,0) - (\partial_\lambda F)(\lambda_\mu,\mu)|, \\ \inf_\lambda |(\partial_\lambda F)(\lambda_\mu,\mu)| &\geq |(\partial_\lambda F)(\lambda_0,0)| - \sup_\lambda |(\partial_\lambda F)(\lambda_0,0) - (\partial_\lambda F)(\lambda_\mu,\mu)| \\ &\geq \frac{2}{3}|(\partial_\lambda F)(\lambda_0,0)|. \end{aligned} \tag{5.19}$$

Combing this estimate with the estimate from lemma 5.1 we obtain

$$\begin{aligned} |\lambda_\mu - \lambda_0| &\leq \frac{a^2}{2r^3\,|(\partial_\lambda F)(r,\theta,\lambda_0,0)|}\mu \\ &= \frac{a^4}{r^3}\frac{(r^2+a^2)\sin^2(\theta)\,r^2\cos^2(\theta)}{(r^2+a^2\sin^2(\theta))^{5/2}}\mu \\ &\leq a^4\frac{r^2+a^2}{r^6}\sin^2(\theta)\cos^2(\theta)\,\mu \\ &= \frac{a^4}{r^4}\left(1+\frac{a^2}{r^2}\right)\sin^2(\theta)\cos^2(\theta)\,\mu. \end{aligned} \tag{5.20}$$

■

In the Minkowski case we know that $\lambda_0 \in [0,1]$. A priori we don't know whether λ_μ is bounded similarly. However, from this lemma we see that λ_μ has fixed values for $\theta = 0, \pi/2$:

$$\lambda_\mu(\theta = 0) = \lambda_0(\theta = 0) = 0 \quad \text{and} \quad \lambda_\mu(\theta = \pi/2) = \lambda_0(\theta = \pi/2) = 1. \tag{5.21}$$

Furthermore, we need to verify that λ_μ doesn't blow up for $\theta \in (0, \pi/2)$.

Before we show this, we first we note that

$$\frac{a}{r} \leq \frac{M}{r} = \mu \leq \frac{M}{r_+} = \frac{M}{M+\sqrt{M^2-a^2}} \leq 1. \tag{5.22}$$

This implies that we can bound the constant appearing in lemma 5.2 purely in terms of μ by

$$\tilde{C}(r)\mu \leq \mu^5\left(1+\mu^2\right) = C(\mu). \tag{5.23}$$

Using this estimate we can prove the following lemma:

Lemma 5.3. *Let $\mu < \mu_* \lesssim 0.815$, where μ_* is the real root of the polynomial $(1+\mu^2)C(\mu) - 1$. Then we have $0 \leq \lambda_\mu \leq 1$.*

Proof. We can verify that for $\mu < \mu_*$, we have $(1+\mu^2)C(\mu) < 1$. From lemma 5.2 we have that

$$\begin{aligned} 1 - \lambda_\mu = (1-\lambda_0) - (\lambda_\mu - \lambda_0) &\geq \frac{r^2\cos^2(\theta)}{r^2 + a^2\sin^2(\theta)} - \left|\lambda_\mu - \lambda_0\right| \\ &\geq \left(\frac{1}{1 + a^2/r^2\sin^2(\theta)} - C(\mu)\sin^2(\theta)\right)\cos^2(\theta) \\ &\geq \left(\frac{1}{1+\mu^2} - C(\mu)\right)\cos^2(\theta) \geq 0, \end{aligned} \tag{5.24}$$

which implies that $\lambda_\mu \leq 1$. Similarly, we find

$$\begin{aligned} \lambda_\mu = \lambda_0 - (\lambda_\mu - \lambda_0) &\geq \frac{(r^2+a^2)\sin^2(\theta)}{r^2 + a^2\sin^2(\theta)} - \left|\lambda_\mu - \lambda_0\right| \\ &\geq \left(\frac{r^2+a^2}{r^2 + a^2\sin^2(\theta)} - C(\mu)\cos^2(\theta)\right)\sin^2(\theta) \\ &\geq \left(\frac{1}{1+\mu^2} - C(\mu)\right)\sin^2(\theta) \geq 0. \end{aligned} \tag{5.25}$$

Furthermore, both estimates do not blow up for $\mu < \mu_*$. ■

In order to ensure that $0 \leq \lambda \leq 1$, we will assume $\mu < \mu_*$ from now on. We could further relax the bound on μ by not omitting the angular dependence in the bound, however for our purpose this result is good enough. Furthermore, later in the bootstrap argument, μ will be required to be even smaller anyway.

As of now we only know an upper bound on the distance $|\lambda_\mu - \lambda_0|$ and not whether $\lambda_0 \geq \lambda_\mu$ or vice versa. The following lemma answers this question.

Lemma 5.4. *Let $\mu < \mu_*$. Then we have that $\lambda_\mu \leq \lambda_0$.*

Proof. From the IFT, we know that for fixed (r,θ) (in the domain such that $F = 0$), we have

$$0 = dF = \frac{\partial F}{\partial \mu} d\mu + \frac{\partial F}{\partial \lambda} d\lambda. \tag{5.26}$$

Therefore, we obtain $\partial_\mu \lambda_\mu = -\partial_\mu F / \partial_\lambda F$. The μ-derivative of F is given by

$$(\partial_\mu F)(r,\theta,\lambda_\mu,\mu) = \int_r^\infty \left(-\frac{(\partial_\mu Q_\mu)(r',\lambda_\mu)}{Q_\mu(r',\lambda_\mu)^2}\right) dr' = -a^2\lambda \int_r^\infty \frac{r'^2}{Q_\mu(r',\lambda_\mu)^3} dr' \leq 0. \tag{5.27}$$

For the λ-derivative of $\partial_\mu F$ we have

$$(\partial_\lambda \partial_\mu F)(r,\theta,\lambda_\mu,\mu) = -a^2 \int_r^\infty \frac{r'^2}{Q_\mu(r',\lambda_\mu)^3} dr' - a^2\lambda_\mu \int_r^\infty \frac{3\Delta_\mu(r')r'^2}{Q_\mu(r',\lambda_\mu)^5} dr' < 0, \tag{5.28}$$

since both integrals are positive. We can switch the order of the derivatives and obtain that $\partial_\mu \partial_\lambda F = \partial_\lambda \partial_\mu F \leq 0$ and hence $(\partial_\lambda F)(\cdot,\mu) \leq (\partial_\lambda F)(\cdot,0) < 0$. In turn this means that $\partial_\mu \lambda_\mu \leq 0$ so that λ_μ is decreasing in μ, which implies that $\lambda_\mu \leq \lambda_0$. ■

It is worth mentioning that the below statements (albeit with slightly different constants) would also work if the opposite were true. The important part is the smallness of $|\lambda_\mu - \lambda_0|$. The only instances in where we make use of this lemma are (a) in lemma 5.7 to circumvent a case distinction; (b) to be able to write $[\lambda_\mu, \lambda_0]$ instead of the verbose expression $[\min\{\lambda_0, \lambda_\mu\}, \max\{\lambda_0, \lambda_\mu\}]$.

The Final Result

We now want to proceed to close the bootstrap by improving the bootstrap assumption. Let us first state the final result:

Theorem 1. *Let $\mu \leq 0.17$ and assume that the bootstrap assumption (eq. 5.8) holds with $\varepsilon = 1/3$. Then there exists a function $\lambda(r, \theta, \mu)$ on $U = [M/0.17, \infty) \times (0, \pi/2) \times [0, 0.17]$ such that*

$$F(r, \theta, \lambda(r, \theta, \mu), \mu) = 0 \quad on \quad U. \tag{5.29}$$

In particular, this implies that we can cover the patch for which

$$(t, r, \theta, \varphi) \in \mathbb{R} \times [M/0.17, \infty) \times (0, \pi/2) \times (0, 2\pi) \tag{5.30}$$

using double null coordinates.

Before we will prove this statement, we first prove three seemingly unrelated estimates of integrals and functions, which we then will use in the proof the theorem.

Lemma 5.5. *Let $\lambda \in [\lambda_\mu, \lambda_0]$, $\mu \leq \mu_*$ and assume that the bootstrap assumption (eq. 5.8) holds with $\varepsilon = 1/3$. Then we have that*

$$|(\partial_\lambda F)(r, \theta, \lambda, \mu) - (\partial_\lambda F)(r, \theta, \lambda, 0)| \leq \frac{13\mu^3}{r}. \tag{5.31}$$

Proof. Since we evaluate $\partial_\lambda F$ at the same values of θ and λ, the θ-integrals cancel each other. We are left with

$$|(\partial_\lambda F)(r, \theta, \lambda, \mu) - (\partial_\lambda F)(r, \theta, \lambda, 0)| \leq \frac{a^2}{2} \int_r^\infty \left| \frac{\Delta_\mu(r')}{Q_\mu(r', \lambda)^3} - \frac{\Delta_0(r')}{Q_0(r', \lambda)^3} \right| dr'. \tag{5.32}$$

Now we need to estimate the integrand. We rewrite it as

$$\begin{aligned} \left| \frac{\Delta_\mu}{Q_\mu} - \frac{\Delta_0}{Q_0} \right| &= \left| \Delta_0 \left(\frac{1}{Q_\mu} - \frac{1}{Q_0} \right) + \frac{\Delta_\mu - \Delta_0}{Q_\mu} \right| \\ &= \left| (r^2 + a^2) \frac{Q_0^3 - Q_\mu^3}{Q_0^3 Q_\mu^3} - \frac{2\mu r^2}{Q_\mu^3} \right| \\ &\leq (r^2 + a^2) \left| \frac{Q_0^3 - Q_\mu^3}{Q_0^3 Q_\mu^3} \right| + \left| \frac{2\mu r^2}{Q_\mu^3} \right| \\ &= \frac{1}{Q_\mu^3} \left(\frac{r^2 + a^2}{Q_0^3} \left| Q_0^3 - Q_\mu^3 \right| + 2\mu r^2 \right). \end{aligned} \tag{5.33}$$

Further we note that

$$\begin{aligned} Q_0^3 &= \left((r^2 + a^2)(r^2 + a^2(1 - \lambda)) \right)^{3/2} \geq (r^2 + a^2) r^4 \\ Q_\mu &= \sqrt{Q_0^2 + 2a^2 \lambda r^2 \mu} \geq Q_0 \geq r^2, \end{aligned} \tag{5.34}$$

from which we obtain

$$\left|\frac{\Delta_\mu}{Q_\mu} - \frac{\Delta_0}{Q_0}\right| \leq \frac{1}{r^6}\left(\frac{\left|Q_0^3 - Q_\mu^3\right|}{r^4} + 2\mu r^2\right). \tag{5.35}$$

What is left is to estimate $\left|Q_0^3 - Q_\mu^3\right|$:

$$\left|Q_0^3 - Q_\mu^3\right| = \frac{\left|(Q_0^2)^3 - (Q_\mu^2)^3\right|}{(Q_0^2)^{3/2} + (Q_\mu^2)^{3/2}} \leq \frac{\left|(Q_0^2)^3 - (Q_\mu^2)^3\right|}{2(Q_0^2)^{3/2}} \leq \frac{\left|(Q_0^2)^3 - (Q_\mu^2)^3\right|}{2r^6}. \tag{5.36}$$

The enumerator is a polynomial in Q_0^2 and $y = Q_\mu^2 - Q_0^2$. Since we have that

$$Q_0^2 \leq (r^2 + a^2)^2 = r^4\left(1 + a^2/r^2\right)^2 \leq 4r^4, \tag{5.37}$$

and also $y \leq 2a^2r^2\mu$ (since $\lambda \leq 1$) we get

$$\begin{aligned} \left|(Q_0^2)^3 - (Q_\mu^2)^3\right| &= \left|(Q_0^2)^3 - (Q_0^2 + y)^3\right| \\ &= 3(Q_0^2)^2 y + 3Q_0^2 y^2 + y^3 \\ &\leq 6(r^2 + a^2)^4 r^2 a^2 \mu + 12(r^2 + a^2)^2 r^4 a^4 \mu^2 + 8r^6 a^6 \mu^3 \\ &\leq 96 r^{10} a^2 \mu + 48 r^8 a^4 \mu^2 + 8 r^6 a^6 \mu^3 \\ &= r^{12}\left(96\frac{a^2}{r^2}\mu + 48\frac{a^4}{r^4}\mu^2 + 8\frac{a^6}{r^6}\mu^3\right) \\ &\leq r^{12}(96\mu^3 + 48\mu^6 + 8\mu^9) \\ &\leq 152\, r^{12}\mu. \end{aligned} \tag{5.38}$$

Plugging this into the integrand we obtain the bound

$$\left|\frac{\Delta_\mu}{Q_\mu} - \frac{\Delta_0}{Q_0}\right| \leq \frac{1}{r^6}\left(76\mu r^2 + 2\mu r^2\right) = \frac{78\mu}{r^4}. \tag{5.39}$$

Integrating this expression we find the desired result:

$$|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda,0)| \leq \frac{a^2}{2}\int_r^\infty \frac{78\mu}{r'^4}dr' = \frac{78a^2\mu}{6r^3} \leq \frac{13\mu^3}{r}. \tag{5.40}$$

■

Lemma 5.6. *Let $\lambda \in [\lambda_\mu, \lambda_0]$, $\mu < \mu_*$ and assume that the bootstrap assumption (eq. 5.8) holds with $\varepsilon = 1/3$. Then we have that*

$$\left|\frac{(\partial_\lambda\psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} - \frac{(\partial_\lambda\psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)}\right| \leq B(\mu)\,|(\partial_\lambda F)(r,\theta,\lambda_0,0)|, \tag{5.41}$$

where

$$B(\mu) = \left(1 + \mu^2 + \frac{(1+\mu^2)^3 C(\mu)}{\left(1 - (1+\mu^2)\,C(\mu)\right)^2}\right)^{1/2} - 1. \tag{5.42}$$

Proof. We already calculated $\partial_\lambda \psi$ and $P(\psi(r,\lambda),\lambda)$ in lemma 4.4, eqs. 4.45, 4.44:

$$\frac{(\partial_\lambda \psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} = \sqrt{\frac{r^2(r^2+a^2)}{r^2+a^2(1-\lambda)}}\frac{1}{2a^2\sqrt{\lambda(1-\lambda)}}. \tag{5.43}$$

Additionally, we have that

$$\frac{(\partial_\lambda \psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)} = -(\partial_\lambda F)(r,\theta,\lambda_0,0). \tag{5.44}$$

Thus, we can write

$$\left|\frac{(\partial_\lambda \psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} - \frac{(\partial_\lambda \psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)}\right| = \left|\frac{(\partial_\lambda \psi)(r,\lambda)\,P(\psi(r,\lambda_0),\lambda_0)}{P(\psi(r,\lambda),\lambda)\,(\partial_\lambda \psi)(r,\lambda_0)} - 1\right| \cdot |(\partial_\lambda F)(r,\theta,\lambda_0,0)|\,. \tag{5.45}$$

$$\begin{aligned}\frac{(\partial_\lambda \psi)(r,\lambda)\,P(\psi(r,\lambda_0),\lambda_0)}{P(\psi(r,\lambda),\lambda)\,(\partial_\lambda \psi)(r,\lambda_0)} &= \left(\frac{r^2+a^2(1-\lambda_0)}{r^2+a^2(1-\lambda)}\right)^{1/2}\left(\frac{\lambda_0(1-\lambda_0)}{\lambda(1-\lambda)}\right)^{1/2} \\ &\le \left(\frac{r^2+a^2}{r^2}\right)^{1/2}\left(\frac{\lambda_0(1-\lambda_0)}{\lambda(1-\lambda)}\right)^{1/2} \\ &\le (1+\mu^2)^{1/2}\left(\frac{\lambda_0(1-\lambda_0)}{\lambda(1-\lambda)}\right)^{1/2}.\end{aligned} \tag{5.46}$$

Now we need to show that the ratio of the λ's is small. A priori the limits of $\lambda \to 0, 1$ seem problematic because the expression would blow up for fixed λ_0. However, from lemma 5.2 we know that $\lambda = \lambda_0$ at the endpoints where $\theta = 0, \pi/2$. Using this lemma we rewrite the expression as

$$\begin{aligned}\frac{\lambda_0(1-\lambda_0)}{\lambda(1-\lambda)} &\le \frac{\lambda(1-\lambda)}{\lambda(1-\lambda)} + \left|\frac{\lambda_0(1-\lambda_0)-\lambda(1-\lambda)}{\lambda(1-\lambda)}\right| \\ &= 1 + \frac{\left|(\lambda_0-\lambda)-(\lambda_0^2-\lambda^2)\right|}{\lambda(1-\lambda)} \\ &= 1 + \frac{|\lambda_0-\lambda|\,|1-(\lambda_0+\lambda)|}{\lambda(1-\lambda)} \\ &\le 1 + \frac{C(\mu)\sin^2(\theta)\cos^2(\theta)}{\lambda(1-\lambda)}.\end{aligned} \tag{5.47}$$

We already estimated (with λ_μ instead of λ) the ratios $\sin^2(\theta)/\lambda$ and $\cos^2(\theta)/(1-\lambda)$ in the proof of lemma 5.3, where we obtained expressions equivalent to

$$\frac{\sin^2(\theta)}{\lambda} \le \left(\frac{1}{1+\mu^2} - C(\mu)\right)^{-1} = \frac{1+\mu^2}{1-(1+\mu^2)C(\mu)}, \tag{5.48}$$

$$\frac{\cos^2(\theta)}{1-\lambda} \le \left(\frac{1}{1+\mu^2} - C(\mu)\right)^{-1} = \frac{1+\mu^2}{1-(1+\mu^2)C(\mu)}, \tag{5.49}$$

which can blow up if μ is large enough. However, by considering $\mu \le 0.8$ this does not occur. Plugging these estimates into eq. 5.45 gives

$$\begin{aligned}&\left|\frac{(\partial_\lambda \psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} - \frac{(\partial_\lambda \psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)}\right| \\ &\le \left(\left(1+\mu^2+\frac{(1+\mu^2)^3C(\mu)}{\left(1-(1+\mu^2)\,C(\mu)\right)^2}\right)^{1/2} - 1\right)|(\partial_\lambda F)(r,\theta,\lambda_0,0)|\,.\end{aligned} \tag{5.50}$$

■

Lemma 5.7. *Define* $\mu_\# \lesssim 0.745$ *as the real root of* $1-(1+\mu^2)^2\mu^3$. *Let* $\lambda \in [\lambda_\mu, \lambda_0]$, $\mu < \mu_\#$ *and assume that the bootstrap assumption (eq. 5.8) holds with* $\varepsilon = 1/3$. *Then we have that*

$$\frac{a^2}{2}\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)}\left|\frac{d\psi'}{P(\psi',\lambda)^3}\right| \le A(\mu)\,|(\partial_\lambda F)(r,\theta,\lambda_0,0)|, \tag{5.51}$$

where

$$A(\mu) = \frac{(1+\mu^2)^5\mu^3}{2\big(1-(1+\mu^2)C(\mu)\big)\big(1-(1+\mu^2)^2\mu^3\big)^{3/2}}. \tag{5.52}$$

Proof. We can find an upper bound of the integral by considering the supremum of the integrand:

$$\begin{aligned}\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)}\left|\frac{d\psi'}{P(\psi',\lambda)^3}\right| &\le \sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right|\cdot|\psi(r,\lambda_0)-\psi(r,\lambda)| \\ &= \sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right|\frac{|\psi(r,\lambda_0)-\psi(r,\lambda)|}{|\lambda-\lambda_0|}\,|\lambda-\lambda_0|\,.\end{aligned} \tag{5.53}$$

The term in the middle can be rewritten by invoking the mean value theorem, which states that there exists a $\lambda_* \in [\lambda, \lambda_0]$ such that

$$\frac{|\psi(r,\lambda_0)-\psi(r,\lambda)|}{|\lambda-\lambda_0|} = |(\partial_\lambda\psi)(r,\lambda_*)|. \tag{5.54}$$

By eq. 4.19 we have that

$$\begin{aligned}|(\partial_\lambda\psi)(r,\lambda_*)| &= \frac{r\sqrt{r^2+a^2}}{r^2+a^2(1-\lambda)}\frac{1}{2\sqrt{\lambda_*(1-\lambda_*)}} \le \frac{r^2+a^2}{r^2}\frac{1}{2\sqrt{\lambda_*(1-\lambda_*)}} \\ &\le \frac{1+\mu^2}{2\sqrt{\lambda_*(1-\lambda_*)}}.\end{aligned} \tag{5.55}$$

Also using lemma 5.2 (to estimate $|\lambda-\lambda_0|$) we obtain

$$\begin{aligned}\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)}\left|\frac{d\psi'}{P(\psi',\lambda)^3}\right| &\le \sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right|\frac{1+\mu^2}{2\sqrt{\lambda_*(1-\lambda_*)}}\tilde{C}(r)\mu\sin^2(\theta)\cos^2(\theta) \\ &= \sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right|\frac{(1+\mu^2)\tilde{C}(r)\mu}{2}\left(\frac{\sin^2(\theta)}{\lambda_*}\right)^{1/2}\left(\frac{\cos^2(\theta)}{1-\lambda_*}\right)^{1/2}\sin(\theta)\cos(\theta) \\ &\le \sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right|\underbrace{\frac{\tilde{C}(r)(1+\mu^2)^3\mu}{2(1-(1+\mu^2)C(\mu))}}_{\tilde{C}(r)f(\mu)}\sin(\theta)\cos(\theta),\end{aligned} \tag{5.56}$$

where we used eqs. 5.48, 5.49 from lemma 5.6 to estimate $\sin^2(\theta)/\lambda_*$ and $\cos^2(\theta)/(1-\lambda_*)$. We now need to also estimate the supremum. Since $P(\theta,\lambda_0) \sim \sin(\theta)\cos(\theta)$ vanishes at $\theta = 0, \pi/2$, we expect that the supremum blows up in these limits, but still can be bounded in terms of sin and cos:

$$\sup_{\psi'}\left|\frac{1}{P(\psi',\lambda)^3}\right| \lesssim \frac{1}{\sin^3(\theta)\cos^3(\theta)}. \tag{5.57}$$

To find an actual upper bound we note that $\partial_\lambda \psi > 0$ and hence that $\psi(r,\lambda)$ is strictly increasing in λ. Since $P(\psi', \lambda)$ is strictly decreasing in ψ', we have that the supremum is attained at the larger value of the boundary points. Therefore, if $\lambda_0 \geq \lambda$, we have that

$$\sup_{\psi'} \left| \frac{1}{P(\psi', \lambda)^3} \right| = \frac{1}{P(\psi(r, \lambda_0), \lambda)^3}. \tag{5.58}$$

Furthermore,

$$\begin{aligned}
P(\psi(r,\lambda_0),\lambda)^3 &= a^3 \left(\lambda - \sin^2(\psi(r,\lambda_0))\right)^{3/2} \\
&= a^3 \left(\lambda - \lambda_0 + \lambda_0 - \sin^2(\psi(r,\lambda_0))\right)^{3/2} \\
&= \left(a^2(\lambda - \lambda_0) + P(\psi(r,\lambda_0),\lambda_0)\right)^{3/2} \\
&\geq \left(P(\psi(r,\lambda_0),\lambda_0) - a^2|\lambda - \lambda_0|\right)^{3/2} \\
&\geq \left(\frac{a^4 \sin^2(\theta)\cos^2(\theta)}{r^2 + a^2\sin^2(\theta)} - a^2\tilde{C}(r)\mu \sin^2(\theta)\cos^2(\theta)\right)^{3/2} \\
&\geq \frac{a^6}{(r^2 + a^2\sin^2(\theta))^{3/2}} \left(1 - \frac{r^2 + a^2}{a^2}\tilde{C}(r)\mu\right)^{3/2} \sin^3(\theta)\cos^3(\theta).
\end{aligned}$$

Hence, we obtain

$$\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)} \left| \frac{d\psi'}{P(\psi',\lambda)^3} \right| \leq \frac{\tilde{C}(r) f(\mu) \left(r^2 + a^2 \sin^2(\theta)\right)^{3/2}}{a^6 \left(1 - \frac{r^2 + a^2}{a^2}\tilde{C}(r)\mu\right)^{3/2} \sin^2(\theta)\cos^2(\theta)}, \tag{5.59}$$

which of course diverges as $\theta \to 0, \pi/2$. However, this divergence is not problematic as we want to bound the integral in terms of $|(\partial_\lambda F)(\lambda_0, 0)|$, which is explicitly given by lemma 4.4, eq. 4.41:

$$|(\partial_\lambda F)(r,\theta,\lambda_0,0)| = \frac{(r^2 + a^2\sin^2(\theta))^{5/2}}{2a^2 r^2 (r^2 + a^2)\sin^2(\theta)\cos^2(\theta)} \sim \sin^{-2}(\theta)\cos^{-2}(\theta), \tag{5.60}$$

and has the same divergent behaviour in θ. Multiplying and dividing with it gives

$$\begin{aligned}
\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)} \left| \frac{d\psi'}{P(\psi',\lambda)^3} \right| &\leq \frac{\tilde{C}(r) f(\mu) \left(r^2 + a^2\sin^2(\theta)\right)^{3/2}}{a^6 \left(1 - \frac{r^2 + a^2}{a^2}\tilde{C}(r)\mu\right)^{3/2}} \frac{2a^2 r^2 (r^2 + a^2)}{(r^2 + a^2\sin^2(\theta))^{5/2}} |(\partial_\lambda F)(r,\theta,\lambda_0,0)| \\
&= \frac{\tilde{C}(r) f(\mu)}{a^4 \left(1 - \frac{r^2 + a^2}{a^2}\tilde{C}(r)\mu\right)^{3/2}} \frac{2r^2(r^2 + a^2)}{r^2 + a^2\sin^2(\theta)} |(\partial_\lambda F)(r,\theta,\lambda_0,0)|.
\end{aligned} \tag{5.61}$$

To finish the proof let us rewrite and estimate the term in front of $|(\partial_\lambda F)(\lambda_0, 0)|$ (almost) purely in terms of μ. First we note, using the expression for C (lemma 5.2) in terms of a/r, that

$$\begin{aligned}
1 - \frac{r^2 + a^2}{a^2}\tilde{C}(r)\mu &= 1 - \tilde{C}(r)\mu - \frac{r^2}{a^2}\frac{a^4}{r^4}\left(1 + \frac{a^2}{r^2}\right)\mu \\
&\geq 1 - \mu^5(1 + \mu^2) - \mu^3(1 + \mu^2) \\
&= 1 - \mu^3(1 + \mu^2)^2
\end{aligned} \tag{5.62}$$

and

$$\tilde{C}(r)\frac{2r^2(r^2+a^2)}{a^4\big(r^2+a^2\sin^2(\theta)\big)} \leq \frac{2}{r^2}\tilde{C}(r)\frac{r^4}{a^4}\left(1+\frac{a^2}{r^2}\right)$$
$$= \frac{2}{r^2}\frac{a^4}{r^4}\left(1+\frac{a^2}{r^2}\right)\frac{r^4}{a^4}\left(1+\frac{a^2}{r^2}\right) \tag{5.63}$$
$$\leq \frac{2}{r^2}(1+\mu^2)^2.$$

Hence, by also multiplying with $a^2/2$, we obtain

$$\frac{a^2}{2}\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)}\left|\frac{d\psi'}{P(\psi',\lambda)^3}\right| \tag{5.64}$$
$$\leq \frac{a^2}{r^2}\frac{(1+\mu^2)^3\mu}{2\big(1-(1+\mu^2)C(\mu)\big)}\frac{(1+\mu^2)^2}{\big(1-(1+\mu^2)^2\mu^3\big)^{3/2}}|(\partial_\lambda F)(r,\theta,\lambda_0,0)|$$
$$\leq \frac{(1+\mu^2)^5\mu^3}{2\big(1-(1+\mu^2)C(\mu)\big)\big(1-(1+\mu^2)^2\mu^3\big)^{3/2}}|(\partial_\lambda F)(r,\theta,\lambda_0,0)|. \tag{5.65}$$

The factor in front does not diverge because we assumed $\mu < \mu_\# < \mu_*$. ■

We have that $B(0) = 0$ and that $B(\mu)$ is continuous on $[0, \mu_*)$. Similarly, $A(0) = 0$ and $A(\mu)$ is continuous on $[0, \mu_\#)$. Hence, we can make these constants as small as desired by considering suitable $\mu < \mu_\# < \mu_*$, which will be important in closing the bootstrap argument. Of course, by considering arbitrarily small μ, we have to deal with the fact that these results are only valid in the region where $r \geq M/\mu$, since the mass M is interpreted to be a fixed parameter of Kerr spacetime.

The Proof

By using all the lemmas so far we are now able to prove theorem 1:

Proof of theorem 1. Let $\mu \in [0, 0.17], \theta \in (0, \pi/2)$ and $r \in [M/\mu, \infty)$. First, since $\mu < \mu_*$, we know that $0 \leq \lambda \leq 1$. In view of the bootstrap principle (prop. 5.1), we need to verify the four properties (a) to (d) on this domain in order to proof the existence of λ, which crucially relies on the (to be proven) fact $\partial_\lambda F \neq 0$. Our hypothesis (the bootstrap assumption, eq. 5.8) ensures this. We want to show that for suitable μ this inequality can be made even sharper:

$$|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda_0,0)| \leq \frac{1}{6}\,|(\partial_\lambda F)(r,\theta,\lambda_0,0)|, \tag{5.66}$$

which is our conclusion. Then the bootstrap principle ensures that our assumed hypothesis is indeed true. Using the triangle inequality we can split the difference in the following way:

$$\begin{aligned}&|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda_0,0)|\\ &= |(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda,0) + (\partial_\lambda F)(r,\theta,\lambda,0) - (\partial_\lambda F)(r,\theta,\lambda_0,0)|\\ &\leq \underbrace{|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda,0)|}_{\text{(I)}} + \underbrace{|(\partial_\lambda F)(r,\theta,\lambda,0) - (\partial_\lambda F)(r,\theta,\lambda_0,0)|}_{\text{(II)}}.\end{aligned} \tag{5.67}$$

The main work in estimating (I) and (II) was already done in the previous lemmas. We already estimated (I) in lemma 5.5. For our purpose here we want to bound (I) in terms of $(\partial_\lambda F)(\lambda_0, 0)$ by inserting $\partial_\lambda F$ into the estimate:

$$\begin{aligned}
(\mathrm{I}) &\leq \frac{13\mu^3}{r} = \frac{13\mu^3}{r}\frac{|(\partial_\lambda F)(r,\theta,\lambda_0,0)|}{|(\partial_\lambda F)(r,\theta,\lambda_0,0)|} \\
&= \frac{13\mu\, 2a^2 r^2 (r^2+a^2)\sin^2(\theta)\cos^2(\theta)}{r(r^2+a^2\sin^2(\theta))^{5/2}}|(\partial_\lambda F)(r,\theta,\lambda_0,0)| \\
&\leq 26\frac{\mu a^2 r(r^2+a^2)}{r^5}|(\partial_\lambda F)(r,\theta,\lambda_0,0)| \\
&\leq \underbrace{26\mu^3(1+\mu^2)}_{D(\mu)}|(\partial_\lambda F)(r,\theta,\lambda_0,0)|.
\end{aligned} \tag{5.68}$$

Furthermore, (II) is given by

$$\begin{aligned}
(\mathrm{II}) &= \left| -\frac{a^2}{2}\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)} \frac{d\psi'}{P(\psi',\lambda)^3} - \frac{(\partial_\lambda\psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} + \frac{a^2}{2}\int_{\psi(r,\lambda_0)}^{\psi(r,\lambda_0)} \frac{d\psi'}{P(\psi',\lambda_0)^3} + \frac{(\partial_\lambda\psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)} \right| \\
&\leq \frac{a^2}{2}\underbrace{\left|\int_{\psi(r,\lambda)}^{\psi(r,\lambda_0)} \frac{d\psi'}{P(\psi',\lambda)^3}\right|}_{(\mathrm{IIa})} + \underbrace{\left|\frac{(\partial_\lambda\psi)(r,\lambda)}{P(\psi(r,\lambda),\lambda)} - \frac{(\partial_\lambda\psi)(r,\lambda_0)}{P(\psi(r,\lambda_0),\lambda_0)}\right|}_{(\mathrm{IIb})},
\end{aligned} \tag{5.69}$$

where we used the Minkowski expression for $\partial_\lambda F$. We already estimated (IIa) in lemma 5.7. Moreover, we also already estimated (IIb) in lemma 5.6. Hence, as a whole, we obtain that

$$|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda_0,0)| \leq \big(A(\mu)+B(\mu)+D(\mu)\big)\,|(\partial_\lambda F)(r,\theta,\lambda_0,0)|. \tag{5.70}$$

Since $\mu \leq 0.17$ we have that $A(\mu)+B(\mu)+D(\mu) \leq 1/6$[8] and thus

$$|(\partial_\lambda F)(r,\theta,\lambda,\mu) - (\partial_\lambda F)(r,\theta,\lambda_0,0)| \leq \frac{1}{6}|(\partial_\lambda F)(r,\theta,\lambda_0,0)|, \tag{5.71}$$

which means that we have strengthened our initial ε from $1/3$ to $1/6$. This is to say that we showed property (a). Property (b), i.e. that the conclusion implies the hypothesis, is trivially true because of continuity. Also, (c) is true since limits preserve inequalities. At last, property (d) is also true because plugging in $(\lambda_0, 0)$, i.e. the Minkowski values, for (λ, μ) implies that the bootstrap assumption is trivially fulfilled. ■

5.4 Results

With this we have proven that the angular function λ, defined as the solution to the constraint equation, exists between $0 < \theta < \pi/2$ and for $r > M/0.17$. By our work in chapter 3, we know that the tortoise coordinate r_*, defined as $r_*(r,\theta) = \rho(r,\theta,\lambda(r,\theta))$ exists on this patch and hence we can introduce the optical functions $u = t - r_*$ and $v = t + r_*$ for which the level sets are null hypersurfaces. Adapting u, v as coordinates, the Kerr metric can be expressed in the double null form derived in chapter 3, eq. 3.41.

[8] This can be easily verified numerically.

In Minkowski space we were able to extend this coordinate system for values of $\theta \in (0, \pi)$ by adapting the polar angle θ_* as a coordinate and defining it on $[\pi/2, \pi]$ via

$$\theta_*(r, \theta = \pi/2) = \pi/2, \quad \text{and} \quad \theta_*(r, \theta) = \theta_*(r, \pi - \theta). \tag{5.72}$$

There we do know that the metric in spherical coordinates in regular at $\theta_* = \pi/2$.

In order to show that we also can expand the coordinate patch in Kerr, we need to show that the metric on the two-surfaces of constant u, v is regular at $\theta = \pi/2$ and beyond. Changing from λ to θ_*, we obtain $d\lambda = 2\sin(\theta_*)\cos(\theta_*)\,d\theta_*$ and thus

$$\not{g} = \frac{L^2}{R^2} d\lambda^2 + R^2 \sin^2(\theta)\,d\varphi_*^2 = 4\frac{L^2}{R^2}\sin^2(\theta_*)\cos^2(\theta_*)\,d\theta_*^2 + R^2\sin^2(\theta)\,d\varphi_*^2. \tag{5.73}$$

For the determinant we obtain

$$\sqrt{\det \not{g}} = 2L\sin(\theta)\sin(\theta_*)\cos(\theta_*) = 2(-\partial_\lambda F)PQ_\mu \sin(\theta)\sin(\theta_*)\cos(\theta_*). \tag{5.74}$$

From our work in the bootstrap argument we know that

$$(-\partial_\lambda F)P\sin(\theta_*)\cos(\theta_*) \sim \frac{\sin(\theta_*)\cos(\theta_*)}{\sin(\theta)\cos(\theta)} \sim 1. \tag{5.75}$$

This implies that L, and by that also $\sqrt{\det \not{g}}$ is well-defined in the limit of $\theta \to \pi/2$. Furthermore, since we defined θ_* for $\theta \in (\pi/2, \pi)$ via mirroring along $\theta = \pi/2$, the in this way extended quantities $P, Q, \partial_\lambda F$ are also well-defined there. Hence, we have shown (or rather sketched a proof) that we in fact can cover Kerr spacetime using double null coordinates in the full range of the Boyer-Lindquist coordinate θ.

6 Conclusion and Outlook

In this work, we studied the question of the existence of a foliation of the Kerr spacetime exterior by two families of null hypersurfaces, known as the double null coordinate system. We started by locally constructing double null coordinates in general spacetimes. Then, assuming the global existence of such coordinates in Kerr, we found an expression for the Kerr metric in these coordinates. The drawback is that the expression is not a closed-form expression in terms of the new coordinates, but rather in terms of the Boyer-Lindquist coordinates. In Minkowski spacetime, we found that the Eikonal equation is globally solvable. Thus, in this special case, we can obtain, not surprisingly, a closed-form expression of the Minkowski metric in double null coordinates. Furthermore, we found that the boundary conditions imposed by Pretorius and Israel in [5] were not well-defined. In this work made sense of their definitions and properly defined them.

The goal of this thesis was to show that we can cover the exterior region Kerr spacetime with double null coordinates, which would thereby justify the expressions derived in chapter 3. The main idea was to make use of the fact that for large values of the Boyer-Lindquist coordinate r, Kerr spacetime asymptotically settles down to Minkowski spacetime, for which we know a global solution exist. Making use of this and employing the so-called bootstrap method in theorem 1, we showed that for $r > M/0.17$ a solution to the Eikonal equation exists and by that also the aforementioned null hypersurfaces.

By optimizing the bootstrap assumption and the established estimates, the bound on r could be potentially relaxed further. In particular, if we were able to show that our proof also holds for $r > M/0.5$, then we would know that for small enough a the region is already the complete exterior. For general a we would need to show that the estimates hold even for $r > M$, for which all our lemmas break down. Thus, an interesting follow-up question is whether we can extend the domain up onto the outer event horizon. This question was already studied, albeit in not much detail, by in [5]. The general idea is to look whether caustics develop along the ingoing null generators of constant λ. The occurrence of a caustic is indicated by the vanishing of the induced volume element $\sqrt{\det \not{g}} = -\partial_\lambda F P Q \sin(\theta)$ on the two-surfaces of constant u, v.

Double null coordinates have already been constructed in the Kerr interior region by Dafermos and Luk in [4]. Thus, another question worth asking is how these two potentially different families of null hypersurface relate to each other in the region around the outer event horizon. And whether it is possible to extend their coordinates over the horizon using this method.

In view of the Einstein equations, the C^2-regularity of the metric is important. Hence, it would be beneficial to know whether this property holds for the metric expressions found.

In the local construction of the double null coordinate system, we assumed the starting point to be a sphere. By the construction of the null hypersurfaces, we found that the intersections of the null hypersurfaces, i.e. the surfaces of constant u, v are spheres as well. A priori, in our coordinate system for Kerr, we do not whether this also holds true. A potential way to show this is to consider the induced metric $\not{g}$ (see eq. 5.73) and look at its asymptotic properties as $r \to \infty$. If this metric reduces to the standard metric on $\mathbb{S}^2$, then, by following the null generators inwards, we could infer that the surfaces are two-spheres. This idea again relies on the claim the the null generators are free of caustics in order to obtain a diffeomorphism between the surfaces.

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