On the invariance of sequential equilibrium and quasi-perfect equilibrium under isomorphism of extensive games

André Casajus[†‡]

(March 2007, this version: October 20, 2008, 16:47)

Abstract

Sequential equilibrium and quasi-perfect equilibrium are not invariant under isomorphism of the standard form. We introduce two relaxations of super weak isomorphism which reflect those features of extensive games which determine sequential equilibrium or quasi-perfect equilibrium, respectively. Though these concepts rely on details of the extensive form, they are essentially weaker than other such concepts as strong, weak, and super weak isomorphism.

Journal of Economic Literature Classification Number: C72.

Key Words: Equivalence, invariance, genericity, sequential equilibrium, quasi-perfect equilibrium.

1. Introduction

It is well known that sequential rationality considerations in games already can be made within the normal form: Proper equilibria (Myerson, 1978) of the normal form induce sequential equilibria (Kreps and Wilson, 1982, henceforth SEQ) and quasi-perfect equilibria (van Damme, 1984, henceforth QPE) in every extensive

[†]Chair of Economics and Information Systems, HHL Leipzig Graduate School of Management, Jahnallee 59, D-04109 Leipzig, Germany.

[‡]Professur für Mikroökonomik, Wirtschaftswissenschaftliche Fakultät, Universität Leipzig, PF 100920, D-04009 Leipzig, Germany. e-mail: casajus@wifa.uni-leipzig.de

I thank nobody for valuable comments on earlier versions of this note. Of course, the usual disclaimer applies.

form having this normal form. Weakening the reduced normal form invariance requirement of Kohlberg and Mertens (1986), however, Govindan and Wilson (2004) "accept the relevance of extensive form analysis" and employ QPE in their axiomatic justification of stable equilibria (Kohlberg and Mertens, 1986). Moreover, Mertens (1995) argues that QPE seems to be the right combination of admissibility and backward induction. Hence, one might be interested in concepts of isomorphism for extensive games under which QPE is invariant.

Consider the games in Figure 1 due to van Damme (1984) which both have the same standard form G (Harsanyi and Selten, 1988) below.

G

	ℓ	r
L	$1,1$	$1,1$
R	$1,1$	$0,0$

Hence, the identity mapping on the action set establishes an isomorphism of the standard form (Harsanyi and Selten, 1988, henceforth SFI). Since this mapping switches the order of information sets, QPE is not SFI invariant. While (R, ℓ) is a QPE in Γ, it is not so in $\bar{\Gamma}$.

Despite its drawbacks, SEQ frequently is employed in applications because it is easier to compute than perfect equilibrium (Selten, 1975). As QPE, however, SEQ is not SFI invariant (see e.g. Kreps and Wilson, 1982, Figures 2 and 13). In contrast, perfect equilibrium can be defined in terms of the agent normal form (Selten, 1975, henceforth ANF) and therefore is invariant under isomorphism of the ANF (ANFI). Yet, it shares a major disadvantage with SEQ: It may put positive weight on (conditionally) dominated strategies (Mertens, 1995, Example 1). Hence, isomorphism concepts which genuinely rely on the extensive form seem to be in need.

Successively weakening strong isomorphism (Elmes and Reny, 1994; Peleg, Rosenmüller and Sudhölter, 1999), we introduced weak isomorphism (Casajus 2003, henceforth WI and CA03) and super weak isomorphism (Casajus 2006, henceforth SWI and CA06) under which SEQ is invariant. The latter note observes that ANFI and SWI are generically different (CA06, Example 3.5 and the proof) and then concludes with the question whether there is a concept of isomorphism for extensive games which is generically equivalent to ANFI but under which SEQ remains invariant.

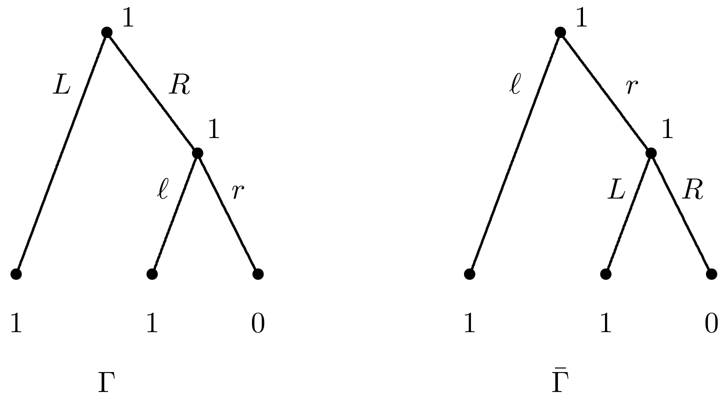

FIGURE 1. QPE is not invariant under isomorphism of the standard form

As an answer to the questions and desiderata above, we propose two new concepts of isomorphism for extensive games both of them being weaker than SWI and both of them relying on the extensive form.

Strong agent normal form isomorphism (SANFI) is concerned with the invariance of SEQ. Besides a sequence of behavior-strategy profiles, the sequence characterization of SEQ (Kreps and Wilson, 1982, Proposition 6) involves a sequence of payoff functions converging to the payoff function of the game under consideration such that any player's SEQ-strategy is a best reply to under these sequences. This indicates that ANFI should/must be strengthened in such a way that it is stable under slight perturbations of payoffs. In essence, this stability property characterizes our concept of a SANFI. It turns out that this strengthening of ANFI does its intended job. While preserving SEQ, generically, it is equivalent to ANFI. Moreover, QPE also is SANFI invariant.

Hyper weak isomorphism (HWI) responds to the invariance of QPE. In view of the motivation of SANFI in the previous paragraph, it seems to be unlikely that SEQ remains invariant under substantial relaxations of SANFI. A closer inspection of the definition of QPE suggests, that in addition to SFI, one might need the preservation of the order of the information sets belonging to the same player. In fact, we show that one can do with less: Just the order of a player's information sets must not be interchanged.

Since this note is an addendum to CA03 and CA06, for expositional parsimony, we rely on the definitions and notation provided there and just explain the most important notation in the next section. The third one introduces the concept of

SANFI and explores its main properties; HWI is dealt with in the fourth section. Some remarks conclude the note. The Appendix contains some lengthier proof.

2. Notation

Assuming a set which contains the labels of players, pure strategies, and nodes, we consider the set $\mathcal{E}$ of finite extensive games Γ with perfect recall and the underlying set $\mathcal{EF}$ of extensive forms γ which have the same constituents as extensive games except for payoff functions and chance probabilities. Associated with $\Gamma \in \mathcal{E}$ are the set I_- of genuine players, the set H_- of their information sets h, and the set $\mathbf{A}$ of pure-strategy profiles $\mathbf{a}$; $i(h)$ denotes the player who controls h. $Z(\mathbf{a})$ denotes the set of terminal nodes reached by $\mathbf{a}$ (CA06, eq. (2.1)). $\mathcal{E}^*$ denotes the set of games where $|A_h| > 1$ for all $h \in H_-$. Since information sets with just one action are not too interesting, we sometimes restrict attention to the set $\mathcal{E}^*$ in order to avoid tedious case distinctions. For $\gamma \in \mathcal{EF}$, $\mathbb{D}(\gamma)$ denotes the set of *assignments* $\delta = (u, p)$ of payoff functions $u \in \mathbb{U}(\gamma)$ and chance probabilities $p \in \mathbb{W}(\gamma)$; $\gamma(\delta)$ denotes the extensive game based on γ and specified by the assignment δ. We define *genericity* as CA06 (Section 2).

An *ANFI* $\mathbf{r}$ from Γ to $\bar{\Gamma}$ is system of bijections $(\nu, (r_h)_{h \in H_-})$ where $\nu : H_- \to \bar{H}_-$, $r_h : A_h \to \bar{A}_{\nu(h)}$ where A_h denotes the set of actions at h. Abusing notation, $\mathbf{r}$ also denotes the induced bijections $\mathbf{A} \to \bar{\mathbf{A}}$ and $B \to \bar{B}$ (CA03 eqs. (2.1) and (3.2)) which preserve the player's preferences. An *SFI* is an ANFI together with a bijection $\pi : I_- \to \bar{I}_-$ such that $\nu(H_i) = \bar{H}_{\pi(i)}$ for all $i \in I_-$. The restrictions to extensive forms also are called ANFI or SFI, respectively.

Since we consider games with perfect recall, a player's information sets are partially ordered. For $i \in I_-$, $h, h' \in H_i$, we write $h \lhd h'$ iff for all (equivalently, some) $x' \in h'$ there is some $x_{x'} \in h$ such that $x_{x'} \lhd x'$. Then, there is unique $a_{(h,h')} \in A_h$ such that all nodes in h' follow nodes in $a_{(h,h')}$. Let $H_i^h := \{h' \in H_i | h \lhd h' \vee h = h'\}$ denote the set comprising h itself and all $h' \in H_i$ that succeed h.

We employ the following characterization of QPE which is quite immediate from the original definition (van Damme, 1984, Definition 1).

Remark 1. *A behavior-strategy profile b is a* QPE *if there is a sequence $(b^k)_{k \in \mathbb{N}}$, $b^k \in B^0$, $\lim_{k \to \infty} b^k = b$ such that*

$$u_i\left(b_{H_i^h} b^k_{H_- \setminus H_i^h}\right) \geq u_i\left(b'_{H_i^h} b^k_{H_- \setminus H_i^h}\right)$$

for all $k \in \mathbb{N}$, $i \in I_-$, $h \in H_i$, and $b' \in B$.

3. Strong agent normal form isomorphism

The proof of CA06 (Theorem 3.7)—SWI invariance of SEQ—employs the sequence characterization of SEQ (Kreps and Wilson, 1982, Proposition 6) and—in fact—rests upon that any SWI is an SFI (CA06, **sISA**, **sPL**, **sPY**) and that for any SWI $\mathbf{r}$ from $\gamma \in \mathcal{EF}$ to $\bar{\gamma} \in \mathcal{EF}$ and any $u \in \mathbb{U}(\gamma)$ there is some $\bar{u} \in \mathbb{U}(\bar{\gamma})$ such that $\bar{u}(\mathbf{r}(\mathbf{a})) = u(\mathbf{a})$ for all $\mathbf{a} \in \mathbf{A}$ (CA06, eqs. (5.6–7)). Note that the Kreps and Wilson (1982, Figures 2 and 13) example does not show this property. Hence, the former feature of SWI seems to drive the above invariance result. The following definition strengthens ANFI in this spirit.

Definition 1. *A* strong ANFI (SANFI) *from* $\Gamma = \gamma(u,p)$ *to* $\bar{\Gamma} = \bar{\gamma}(\bar{u},\bar{p})$, $\gamma, \bar{\gamma} \in \mathcal{EF}$, $(u,p) \in \mathbb{D}(\gamma)$, $(\bar{u},\bar{p}) \in \mathbb{D}(\bar{\gamma})$ *is an ANFI* $\mathbf{r}$ *from* Γ *to* $\bar{\Gamma}$ *such that there are neighborhoods* U *of* u *in* $\mathbb{U}(\gamma)$ *and* $\bar{U}$ *of* $\bar{u}$ *in* $\mathbb{U}(\bar{\gamma})$ *such that for all* $u^\circ \in U$ *and* $\bar{u}^\circ \in \bar{U}$ *there are* $u^\bullet \in \mathbb{U}(\gamma)$ *and* $\bar{u}^\bullet \in \mathbb{U}(\bar{\gamma})$ *such that (i)* $\mathbf{r}$ *is an ANFI from* $\gamma(u^\circ,p)$ *to* $\bar{\gamma}(\bar{u}^\bullet,\bar{p})$ *and (ii)* $\mathbf{r}^{-1}$ *is an ANFI from* $\bar{\gamma}\left(\bar{\delta}^\circ,\bar{p}\right)$ *to* $\gamma(\delta^\bullet,p)$.

SANFI games, SANFI invariant solution concepts, and SANFI invariant behavior-strategy profiles are defined in analogy to their SWI counterparts. From the definition it is clear that the identity on actions as well as composites and inverses of SANFI are SANFI. Non-technically, a SANFI is an ANFI which remains an ANFI under slight perturbations of payoffs. The following theorem shows that one can drop the restriction to neighborhoods in Definition 1. Its proof is referred to the Appendix.

Theorem 1. *If* $\mathbf{r}$ *is an SANFI from* $\Gamma = \gamma(u,p)$ *to* $\bar{\Gamma} = \bar{\gamma}(\bar{u},\bar{p})$, $\gamma, \bar{\gamma} \in \mathcal{EF}$, $(u,p) \in \mathbb{D}(\gamma)$, $(\bar{u},\bar{p}) \in \mathbb{D}(\bar{\gamma})$ *then for all* $u^\circ \in \mathbb{U}(\gamma)$ *and* $\bar{u}^\circ \in \mathbb{U}(\bar{\gamma})$ *there are* $u^\bullet \in \mathbb{U}(\gamma)$ *and* $\bar{u}^\bullet \in \mathbb{U}(\bar{\gamma})$ *such that (i)* $\mathbf{r}$ *is an ANFI from* $\gamma(u^\circ,p)$ *to* $\bar{\gamma}(\bar{u}^\bullet,\bar{p})$ *and (ii)* $\mathbf{r}^{-1}$ *is an ANFI from* $\bar{\gamma}(\bar{u}^\circ,\bar{p})$ *to* $\gamma(u^\bullet,p)$.

Our leading example already reveals that ANFI and its strong cousin do not coincide in general. Generically, however, both concepts coincide. In a sense, the definition of SANFI already incorporates genericity considerations. The lengthy proof of the following theorem is referred to the Appendix.

Theorem 2. *Generically, ANFI and SANFI coincide.*

Since in the neighborhoods U and $\bar{U}$ the players may have pairwise different preferences on the terminal nodes and since ANFI preserve the players' preferences (see also CA03, proof of Theorem 4.8), SANFI respect the assignment of information sets to players and therefore are SFI. This is not case for ANFI in general because ANFI do not directly care about this assignment.

Lemma 3. *(i) Any SANFI* $(\nu, (r_i)_{i \in I_-}) : \Gamma \to \bar{\Gamma}$, $\Gamma, \bar{\Gamma} \in \mathcal{EF}$ *induces a unique bijection* $\pi : I_- \to \bar{I}_-$ *such that* $\nu(H_i) = \bar{H}_{\pi(i)}$ *for all* $i \in I_-$. *(ii) Any SANFI* $(\nu, (r_i)_{i \in I_-})$ *is an SFI* $(\nu, (r_i)_{i \in I_-}, \pi)$ *where* π *is determined by (i).*

The following corollary sheds light on the relation between SWI and SANFI: SWI is non-trivially stronger than SANFI. CA06 show that any SWI induces an ANFI (p. 110) and that (1) can be satified by some $\bar{u}$ for any u (Proof of Theorem 3.7, eqs. (5.6–7)). Since SWI and SANFI can be inverted, this implies claim (ia). Since strong isomorphism and WI imply SWI (CA03, Subsection 3.3; CA06, Theorem 3.3), (ib) and (ic) are immediate. CA06 (Example 3.5 and the proof) establishes a counterexample for claim (ii): The game forms γ and $\bar{\gamma}$ are not SWI, but for any $\delta \in \mathbb{D}(\gamma)$ one can find some $\bar{\delta} \in \mathbb{D}(\bar{\gamma})$ such that $\gamma(\bar{\delta})$ and $\gamma(\delta)$ are ANFI and vice versa.

Corollary 4. *Any (ia) SWI, (ib) WI, or (ic) strong isomorphism is an SANFI. (ii) The converse may fail, even generically.*

Since by definition any SANFI is a ANFI and since perfect equilibrium can be determined via the ANF (Selten, 1975), perfect equilibrium is SANFI invariant. In view of Lemma 3, this also holds true for Nash equilibrium. In contrast, subgame perfect equilibrium is not invariant under SANFI. Consider the games in Figure 3. It is easy to check that the mapping which maps players and actions from Γ to their counterpart with a bar on top in $\bar{\Gamma}$ is an SANFI. Yet, while $(\bar{B}, \bar{b})$ is subgame perfect in $\bar{\Gamma}$, (B, b) is not so in Γ. In view of the remarks at the beginning of this section, SEQ is invariant under SANFI.

Theorem 5. *SEQ is invariant under SANFI.*

The SANFI invariance of QPE is dealt with in the next section. The following property of SANFI—the preservation of the order of a player's information sets—is an important ingredient in the related proof.

Lemma 6. *If* $\mathbf{r} = (\nu, (r_i)_{i \in I_-})$ *is a SANFI from* $\gamma(\delta) \in \mathcal{E}^*$ *to* $\bar{\gamma}(\bar{\delta}) \in \mathcal{E}^*$ *then (i) for all* $i \in I_-$ *and* $h, h' \in H_i$, *we have* $h \lhd h'$ *iff* $\nu(h) \lhd \nu(h')$, *i.e., (ii)* $\nu(H_i^h) = \bar{H}_{\pi(i)}^{\nu(h)}$ *where* $\pi : I_- \to \bar{I}_-$ *is determined by Lemma 3.*

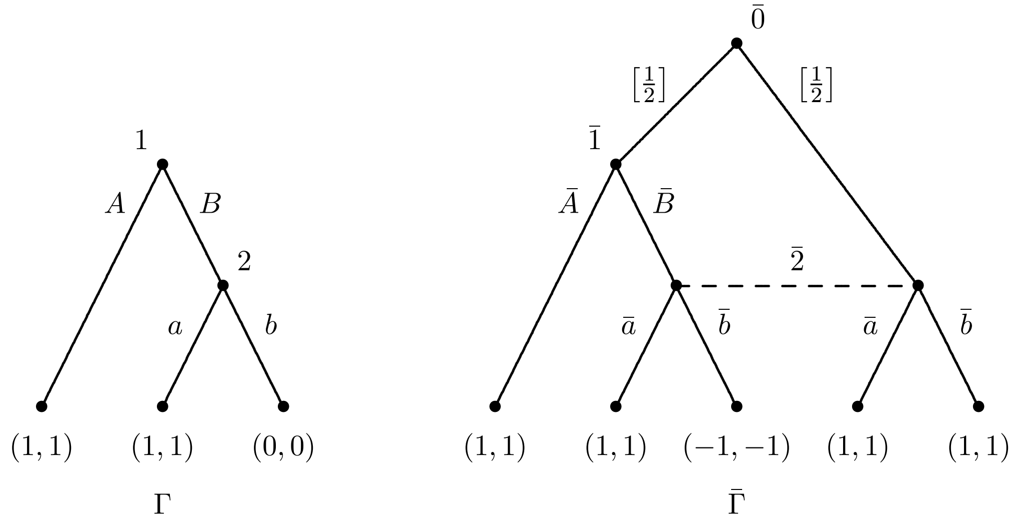

FIGURE 3. Subgame perfect equilibrium is not SANFI invariant

Proof. Let $\mathbf{r}$ be as in the Lemma. Fix $i \in I_-$ and $h, h' \in H_i$ such that $h \lhd h'$. Suppose, $\nu(h) \bar{\ntriangleleft} \nu(h')$. Since $\bar{\gamma}(\bar{\delta}) \in \mathcal{E}^*$, there are $\bar{\mathbf{a}}, \bar{\mathbf{a}}' \in \bar{\mathbf{A}}$ such that $\bar{\mathbf{a}}_{\bar{h}} = \bar{\mathbf{a}}'_{\bar{h}}$ for all $\bar{h} \in H_- \backslash \{\nu(h')\}$, $\bar{\mathbf{a}}_{\nu(h')} = \bar{\mathbf{a}}'_{\nu(h')}$, $\bar{\mathbf{a}}_{\nu(h)} \neq r_h\left(a_{(h,h')}\right)$, and such that $\nu(h')$ is reached under $\bar{\mathbf{a}}$. This implies that in any neighborhood $\bar{U}$ of $\bar{\delta}$ there is some $\bar{\delta}'$ such that $\bar{u}'(\bar{\mathbf{a}}) \neq \bar{u}'(\bar{\mathbf{a}}')$. Yet, since $\mathbf{r}^{-1}(\bar{\mathbf{a}})$ and $\mathbf{r}^{-1}(\bar{\mathbf{a}}')$ differ at h' only and since h' is not reached under $\mathbf{r}^{-1}(\bar{\mathbf{a}})$ and $\mathbf{r}^{-1}(\bar{\mathbf{a}}')$ because $\mathbf{r}_h^{-1}(\bar{\mathbf{a}}) = \mathbf{r}_h^{-1}(\bar{\mathbf{a}}') \neq a_{(h,h')}$, we have $u'(\mathbf{r}^{-1}(\bar{\mathbf{a}})) = u'(\mathbf{r}^{-1}(\bar{\mathbf{a}}'))$ for any $\delta' \in \mathbb{D}(\gamma)$, contradicting $\mathbf{r}$ being an SANFI. Hence, $h \lhd h'$ implies $\nu(h) \bar{\lhd} \nu(h')$. Since $\mathbf{r}^{-1}$ also is an SANFI, this establishes claim (i) which immediately entails claim (ii). □

4. Hyper weak isomorphism

In view of the characterization of QPE in Remark 1, one might be tempted to strengthen an SFI $\left(\pi, \nu, (r_h)_{h \in H_-}\right)$ by the following requirement in order to make QPE invariant. Property **OIS** says that an isomorphism also has to respect the order of information sets of the same player. Note that by Lemma 6, SANFI satisfies **OIS**.

Order of information sets, OIS. For all $i \in I_-$ and $h, h' \in H_i$, we have $h \lhd h'$ iff $\nu(h) \bar{\lhd} \nu(h)$, which is equivalent to $\nu\left(H_i^h\right) = H_{\pi(i)}^{\nu(h)}$.

Yet, it will be shown that one can do with a with a weaker condition. In particular, we just require an isomorphism not to interchange the order of information set of the same player. Note first that this does not preclude that information sets of the same player which were ordered become unordered under isomorphism or vice versa.

Second, the SFI in our leading example interchanges the order of information sets of the single player involved. This gives rise to the following strengthening of SFI, which we call *hyper weak isomorphism (HWI)* because it further weakens SWI.

Definition 2. *A* hyper weak isomorphism (HWI) *from* $\Gamma \in \mathcal{E}^*$ *to* $\bar{\Gamma} \in \mathcal{E}^*$ *is an SFI* $\left(\pi, \nu, (r_h)_{h \in H_-}\right)$ *from* Γ *to* $\bar{\Gamma}$ *which satisfies*
Non-interchange of information sets, NIIS*. For* $i \in I_-$, $h, h \in H_i$, *we do not have* $h \lhd h'$ *and* $\nu(h') \bar{\lhd} \nu(h)$.

HWI games, HWI invariant solution concepts, and HWI invariant behavior-strategy profiles are defined in analogy to their SWI counterparts. From this definition it is clear that the composition and the inverse of the SFI underlying HWI also are HWI. Abusing notation, we also speak of HWI of extensive forms.

In the following, we show that QPE is invariant under HWI. However, this not true for SEQ (see e.g. Kreps and Wilson, 1982, Figures 2 and 13).

Theorem 7. *QPE is HWI invariant.*

The proof is prepared by a technical lemma.

Lemma 8. *If* $\mathbf{r} = (\pi, \nu, (r_i)_{i \in I_-})$ *is an SFI from* $\Gamma \in \mathcal{E}^*$ *to* $\bar{\Gamma} \in \mathcal{E}^*$ *such that for* $i \in I_-$ *and* $h, h' \in H_i$

(i) $h \lhd h'$ *but neither* $\nu(h) \lhd \nu(h')$ *nor* $\nu(h') \bar{\lhd} \nu(h)$ *or*
(ii) *neither* $h \lhd h'$ *nor* $h' \lhd h$ *but* $\nu(h) \bar{\lhd} \nu(h')$

then

(a) *then* $u\left(b_{h'} b_{H_- \backslash h'}\right) = u\left(b'_{h'} b_{H_- \backslash h'}\right)$ *for all* $b, b' \in B$ *and*
(b) $\bar{u}\left(\bar{b}_{\nu(h')} \bar{b}_{\bar{H}_- \backslash \nu(h')}\right) = \bar{u}\left(\bar{b}'_{\nu(h')} \bar{b}_{\bar{H}_- \backslash \nu(h')}\right)$ *for all* $\bar{b}, \bar{b}' \in B$.

Proof. Let $\mathbf{r}$ be as in the Lemma and $i \in I_-$and $h, h' \in H_i$ satisfy (i). Since $\Gamma \in \mathcal{E}^*$, there are $b \in B$ such that $b_h\left(a_{(h,h')}\right) = 0$. Since $h \lhd h'$ and by the definition of $a_{(h,h')}$, $u(b)$ is not affected by changing the local strategies at h'. By CA03 (eqs. (2.1) and (3.2)), $\bar{u}(\mathbf{r}(b))$ also remains unaffected by changing the local strategies at $\nu(h')$. Since $\nu(h)$ and $\nu(h')$ are not ordered by assumption, this holds true even if $\mathbf{r}_{\nu(h)}(b)\left(r_h\left(a_{(h,h')}\right)\right) \neq 0$ or, equivalently, if $b_h\left(a_{(h,h')}\right) \neq 0$. Hence again by CA03 (eqs. (2.1) and (3.2)), we have $u\left(b_{h'} b_{H_- \backslash h'}\right) = u\left(b'_{h'} b_{H_- \backslash h'}\right)$ for all $b \in B$ and $b_{h'} \in B_h$. Since $\mathbf{r}$ involves bijections, claim (b) is immediate. Since SFI are invertible, (ii) also implies (a) and (b). $\square$

Proof. (Theorem 7) Let b be a QPE of Γ. Then there exists a sequence $\left(b^k\right)_{k\in\mathbb{N}}$ as in Remark 1. Since $\mathbf{r}: B \to \bar{B}$ is continuous and both B and $\bar{B}$ are compact, the sequence $\left(\mathbf{r}\left(b^k\right)\right)_{k\in\mathbb{N}}$, $\mathbf{r}\left(b^k\right) \in \bar{B}^0$ converges to $\mathbf{r}(b)$. Further, for all $k \in \mathbb{N}$, $i \in I_-$, $h \in H_i$, and $b' \in B$, we have

$$u_i\left(b_{H_i^h} b^k_{H_-\backslash H_i^h}\right) \geq u_i\left(b'_{H_i^h} b^k_{H_-\backslash H_i^h}\right),$$

hence by Section 2 and CA03 (eqs. (2.1) and (3.2)),

$$\bar{u}_{\pi(i)}\left(\mathbf{r}(b)_{\nu\left(H_i^h\right)} \mathbf{r}\left(b^k\right)_{\bar{H}_-\backslash\nu\left(H_i^h\right)}\right) \geq \bar{u}_{\pi(i)}\left(\mathbf{r}(b')_{\nu\left(H_i^h\right)} \mathbf{r}\left(b^k\right)_{\bar{H}_-\backslash\nu\left(H_i^h\right)}\right).$$

If $\bar{h} \in \nu\left(H_i^h\right) \backslash \bar{H}_{\pi(i)}^{\nu(h)}$ then $h \lhd \nu^{-1}\left(\bar{h}\right)$ but not $\nu(h) \bar{\lhd} \bar{h}$ and, since $\mathbf{r}$ is a HWI, not $\bar{h} \bar{\lhd} \nu(h)$. Lemma 8(i) then implies that the payoffs are not affected by changing the local strategies at $\bar{h}$. Using an analogous argument involving Lemma 8(ii), one shows that the same holds true for $\bar{h} \in \bar{H}_{\pi(i)}^{\nu(h)} \backslash \nu\left(H_i^h\right)$. Hence, the previous equation implies

$$\bar{u}_{\pi(i)}\left(\mathbf{r}(b)_{\bar{H}_{\pi(i)}^{\nu(h)}} \mathbf{r}\left(b^k\right)_{\bar{H}_-\backslash\bar{H}_{\pi(i)}^{\nu(h)}}\right) \geq \bar{u}_{\pi(i)}\left(\mathbf{r}(b')_{\bar{H}_{\pi(i)}^{\nu(h)}} \mathbf{r}\left(b^k\right)_{\bar{H}_-\backslash\bar{H}_{\pi(i)}^{\nu(h)}}\right).$$

Since $\mathbf{r}$ involves bijections, this already shows that $\mathbf{r}(b)$ is a QPE of $\bar{\Gamma}$, i.e., that $\mathbf{r}$ embeds QPE. Since by Definition 2 $\mathbf{r}^{-1}$ is an HWI from $\bar{\Gamma}$ to Γ, $\mathbf{r}$ establishes a bijection between the sets of QPE of Γ and $\bar{\Gamma}$, i.e., QPE is HWI invariant. $\square$

Now, one could think of the following converse of Theorem 7: If $\mathbf{r}$ is an SFI but not an HWI from $\gamma \in \mathcal{EF}^*$ to $\bar{\gamma} \in \mathcal{EF}^*$ then there are $\delta \in \mathbb{D}(\gamma)$ and $\bar{\delta} \in \mathbb{D}(\bar{\gamma})$ such that $\mathbf{r}$ is an SFI from $\gamma(\delta)$ to $\bar{\gamma}\left(\bar{\delta}\right)$ but there is a QPE b of $\gamma(\delta)$ or a QPE $\bar{b}$ of $\bar{\gamma}\left(\bar{\delta}\right)$ such that $\mathbf{r}(b)$ or $\mathbf{r}^{-1}\left(\bar{b}\right)$ is not a QPE, respectively. If this were true, HWI could be characterized as those SFI that preserve QPE in the above sense. However, the following example shows that this may not be the case.

Example 1. *Consider the forms γ and $\bar{\gamma}$ in Figure 5 and the SFI $\mathbf{r}$ which maps players and actions in γ to their relatives with a bar on top in $\bar{\gamma}$. Since player 1's information sets are interchanged this is not an HWI. Let u and $\bar{u}$ be payoff functions for γ and $\bar{\gamma}$ such that $\mathbf{r}$ is an SFI from $\gamma(u)$ to $\gamma(\bar{u})$. Then, any pure-strategy profile involving $\bar{\lambda}$ leads to $\bar{z}_1$. Hence by CA03 (eqs. (2.1) and (3.2)), all pure-strategy profiles in γ involving λ must result in the same payoffs, i.e., $u(z_1) = u(z_3) = u(z_4)$. Similarly, considering strategy profiles in $\bar{\gamma}$ involving $\bar{\rho}$ and $\bar{\ell}$, one concludes that $u(z_2) = u(z_3)$. Hence, both payoff functions are constant which implies that all behavior-strategy profiles in $\gamma(u)$ and $\gamma(\bar{u})$ are a QPE.*

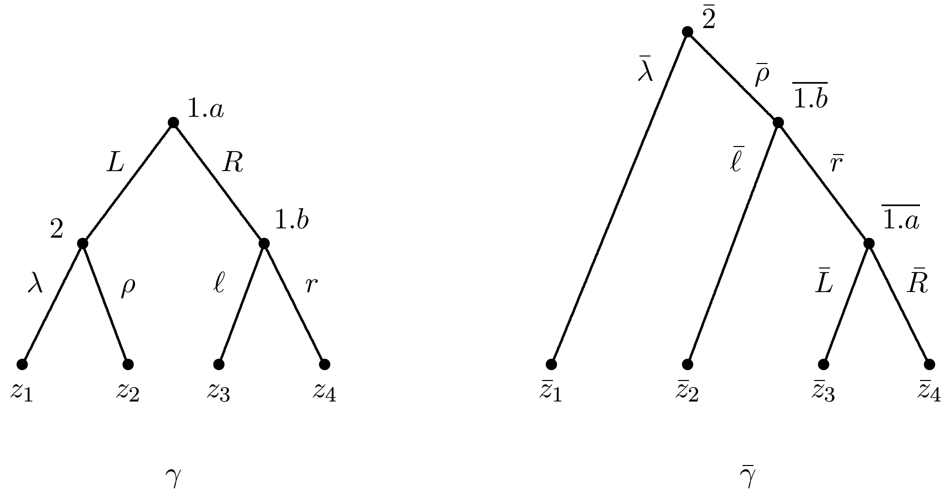

FIGURE 5. Non-SWI does not imply non-invariance of QPE

Corollary 9. *QPE is invariant under SANFI, SWI, WI, and strong isomorphism.*

Proof. By Lemma 3(ii), any SANFI is an SFI, and by Lemma 6, any SANFI also satisfies **OIS**, hence **NIIS**; i.e., any SANFI is an HWI. Since strong isomorphism implies WI (CA03, Section 3.3), WI implies SWI (CA06, Theorem 3.3) and SWI implies SANFI (Corollary 4(ia)), Theorem 7 implies the claim. □

5. Conclusion

In this note, we go some way towards identifying those features of extensive games which determine SEQ and QPE. In particular, we relax SWI and strengthen SFI/ANFI. A slight strengthening of SFI—HWI—already suffices to leave QPE invariant. This invariance, however, cannot be used in order to characterize those SFI that also are HWI (see Example 1).

In contrast, it is not yet clear whether SANFI are those ANFI that leave SEQ invariant in the following sense: If $\mathbf{r} = \left(\nu, (r_i)_{i\in I_-}\right) : \gamma(\delta) \to \bar{\gamma}\left(\bar{\delta}\right)$ is an ANFI but not an SANFI then there are $\delta' \in \mathbb{D}(\gamma)$, $\bar{\delta}' \in \mathbb{D}(\bar{\gamma})$, $b \in B$, and $\bar{b} \in \bar{B}$ such that (a) b is an SEQ in $\gamma(\delta')$ and $\bar{b}$ is not so in $\bar{\gamma}\left(\bar{\delta}'\right)$ or (b) vice versa. Further, a characterization of SANFI which explicitly refers to the structure of extensive games seems to be desirable. Of course, the latter question is related to the former.

Appendix

Proof of Theorem 1. Let $\mathbf{r} = \left(\nu, (r_h)_{h\in H_-}\right)$ be an SANFI from $\Gamma = \gamma(u', p')$ to $\bar{\Gamma} = \bar{\gamma}(\bar{u}', \bar{p}')$ and let U and $\bar{U}$ be neighborhoods of u' or $\bar{u}'$, respectively, as in Definition 1. Let $\pi : I_- \to \bar{I}_-$ be the mapping associated with $\mathbf{r}$ according to Lemma 8. By Section 2 and CA03 (eqs. (2.1) and (3.2)), there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$, $i \in I_-$ such that

$$\bar{u}^{\bar{p}}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u^p_i(\mathbf{a}) + \beta_i \tag{1}$$

for all $\mathbf{a} \in \mathbf{A}$ and $h \in H_-$ for all $u \in U$ and some $\bar{u} \in \mathbb{U}(\bar{\gamma})$. Note that we attach the superscript p to u in order to indicate that $u(\mathbf{a})$ depends on both u and p. Since the affine transformations of payoffs preserves neighborhoods, we are allowed to assume that $\alpha_i = 1$ and $\beta_i = 0$ for all $i \in I_-$.

Using CA06 (eq. (5.1)), (1) can be written as

$$\sum_{\bar{z}\in\bar{Z}(\mathbf{r}(\mathbf{a}))} \mathrm{prob}_{\bar{p}}(\bar{z})\, \bar{u}_{\pi(i)}(\bar{z}) = \sum_{z\in Z(\mathbf{a})} \mathrm{prob}_p(z)\, u_i(z) \tag{2}$$

where $\mathrm{prob}_p(z)$ denotes $\mathrm{prob}(z)$ (CA06, p. 113) derived from p. Since U is a neighborhood of u', any $u'' \in \mathbb{U}(\gamma)$ can be written as $u'' = u' + \xi \cdot (u''' - u')$ for some $u''' \in U$ and $\xi \in \mathbb{R}$. By assumption, there are $\bar{u}', \bar{u}''' \in \mathbb{U}(\bar{\gamma})$ such that (2) is satisfied for the pair $(u', \bar{u}')$ and the pair $(u''', \bar{u}''')$, respectivly. Since (2) is linear in u and $\bar{u}$, (2) can be satisfied for any $u'' \in \mathbb{U}(\gamma)$. This proves claim (i). Claim (ii) then follows from the fact that $\mathbf{r}^{-1}$ is an SANFI from $\bar{\Gamma}$ to Γ.

Proof of Theorem 2. The proof makes use of the following observation.

Observation 10. *Let* $\mathrm{Mat}_{\alpha,\beta}(N)$, $\alpha, \beta \in \mathbb{N}$ *denote the set of all matrices with* α *rows,* β *columns, and with entries from the set* N. *For* $\alpha \in \mathbb{N}$, *set* $\mathbb{Z}_\alpha := \{k \in \mathbb{Z} | \, |k| \le \alpha\}$. *For* $n, m, \bar{m} \in \mathbb{N}$, $M \in \mathrm{Mat}_{n,m}(\{0,1\})$, $\bar{M} \in \mathrm{Mat}_{n,\bar{m}}(\{0,1\})$, *there is some* $\eta(n) \in \mathbb{N}$ *such that there is a regular matrix* $R \in \mathrm{Mat}_{n,n}(\mathbb{Z})$ *such that* RM *is in the row echelon form, not necessarily reduced and except for that leading numbers of non-zero rows may not equal* 1, *and* $R\bar{M} \in \mathrm{Mat}_{n,m}\left(\mathbb{Z}_{\eta(n)}\right)$.

First, we construct subsets $\mathbb{D}^{\mathrm{gen}}(\gamma) \subseteq \mathbb{D}(\gamma)$, $\gamma \in \mathcal{EF}$ as in the definition of genericity (CA06, Section 2). Second, we show that any ANFI is an SANFI "in" $\mathbb{D}^{\mathrm{gen}}$. Let $\eta(\gamma) := \eta(|\mathbf{A}|)$ be some number as in Observation 10. The set $\mathbb{D}^{\neq}(\gamma) := \bigcup_{p\in\mathbb{W}(\gamma)} \left(\mathbb{U}^{\neq}_p(\gamma) \times \{p\}\right) \subseteq \mathbb{D}(\gamma)$,

$$\mathbb{U}^{\neq}_p(\gamma) := \left\{u \in \mathbb{U}(\gamma) \,|\forall i \in I_- : \max_{\mathbf{a}\in\mathbf{A}} u^p_i(\mathbf{a}) \neq \min_{\mathbf{a}\in\mathbf{A}} u^p_i(\mathbf{a})\right\}$$

is open and dense in $\mathbb{D}(\gamma)$. Let $\tau : \mathbb{D}^{\neq}(\gamma) \to \mathbb{D}^{\neq}(\gamma)$, $(u,p) \mapsto (\tau u, p)$ given by

$$\tau u_i(z) = \frac{u_i(z) - \min_{\mathbf{a}\in\mathbf{A}} u_i^p(\mathbf{a})}{\max_{\mathbf{a}\in\mathbf{A}} u_i^p(\mathbf{a}) - \min_{\mathbf{a}\in\mathbf{A}} u_i^p(\mathbf{a})}, \qquad i \in I_-, z \in Z \tag{3}$$

denote the 0-1-normalization, i.e., we have $\max_{\mathbf{a}\in\mathbf{A}} \tau u_i^p(\mathbf{a}) = 1$ and $\min_{\mathbf{a}\in\mathbf{A}} \tau u_i^p(\mathbf{a}) = 0$ for all $i \in I_-$ and $u \in \mathbb{U}_p^{\neq}(\gamma)$. We consider the set

$$\mathbb{D}^{\text{gen}}(\gamma) := \bigcup_{p\in\mathbb{W}(\gamma)} \left(\mathbb{U}_p^{\text{gen}}(\gamma) \times \{p\}\right) \subseteq \mathbb{D}(\gamma)$$

where $\mathbb{U}_p^{\text{gen}}(\gamma)$, $p \in \mathbb{W}(\gamma)$ is given by

(4)

$$\mathbb{U}_p^{\text{gen}}(\gamma) := \left\{ u \in \mathbb{U}(\gamma) \,|\, \forall \zeta \in \mathbb{Z}^Z_{\eta(\gamma)}, \zeta \neq \mathbf{0}, i \in I_- : \sum_{z\in Z} \zeta(z)\, \text{prob}_p(z)\, \tau u_i(z) \neq 0 \right\};$$

$\mathbf{0} \in \mathbb{Z}^Z_{\eta(\gamma)}$, $\mathbf{0}(z) = 0$ for all $z \in Z$. Since $\mathbb{Z}^Z_{\eta(\gamma)}$ is finite, $\text{prob}_p(z) \neq 0$ for all $z \in Z$, and by (3), it is clear that all $\mathbb{U}_p^{\text{gen}}(\gamma)$ are open and dense in $\mathbb{U}(\gamma)$; hence, $\mathbb{D}^{\text{gen}}(\gamma)$ is so in $\mathbb{D}(\gamma)$.

Let $\mathbf{r} = \left(\nu, (r_h)_{h\in H_-}\right)$ be an ANFI from $\Gamma = \gamma(\delta)$ to $\bar{\Gamma} = \bar{\gamma}(\bar{\delta})$ where $\delta = (u,p) \in \mathbb{D}^{\text{gen}}(\gamma)$ and $\bar{\delta} = (\bar{u},\bar{p}) \in \mathbb{D}^{\text{gen}}(\bar{\gamma})$. Let $\pi : I_- \to \bar{I}_-$ be the mapping associated with $\mathbf{r}$ according to Lemma 8. By Section 2 and CA03 (eqs. (2.1) and (3.2)), there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ for all $i \in I_-$ such that (1) holds for all $\mathbf{a} \in \mathbf{A}$ and $h \in H_-$. By (3) and (4), the payoff function u' given by $u_i'(z) = \alpha_i u_i(z) + \beta_i$ for all $i \in I$ and $z \in Z$ also is in $\mathbb{U}_p^{\text{gen}}(\gamma)$. Hence, we are allowed to assume that $\alpha_i = 1$ and $\beta_i = 0$ for all $i \in I$. Using CA06 (eq. (5.1)), (1) can be written as (2).

Enumerate Z and $\bar{Z}$ (indicated by superscript numbers in braces) and enumerate $\mathbf{A}$ and $\bar{\mathbf{A}}$ such that $\mathbf{r}\left(\mathbf{a}^{(k)}\right) = \bar{\mathbf{a}}^{(k)}$. Consider the $|\mathbf{A}| \times |Z|$ and $\left|\bar{\mathbf{A}}\right| \times \left|\bar{Z}\right|$ matrices $\Phi = (\Phi_{k\ell})$ and $\bar{\Phi} = \left(\bar{\Phi}_{k\ell}\right)$, respectively, given by

$$\Phi_{k\ell} = \begin{cases} 1, & z^{(\ell)} \in Z\left(\mathbf{a}^{(k)}\right), \\ 0, & z^{(\ell)} \notin Z\left(\mathbf{a}^{(k)}\right), \end{cases} \qquad \text{and} \qquad \bar{\Phi}_{k\ell} = \begin{cases} 1, & \bar{z}^{(\ell)} \in \bar{Z}\left(\bar{\mathbf{a}}^{(k)}\right), \\ 0, & \bar{z}^{(\ell)} \notin Z\left(\bar{\mathbf{a}}^{(k)}\right). \end{cases} \tag{5}$$

In Φ, for example, the entry in the kth row and the ℓth column indicates whether terminal node $z^{(\ell)}$ can be reached by $\mathbf{a}^{(k)}$, value 1, or not, value 0. Given this, by CA06 (eq. (5.1)), (2) can be written as

$$\bar{\Phi}\bar{v}_{\pi(i)} = \Phi v_i, \qquad i \in I_-, \tag{6}$$

$v_i \in \mathbb{R}^Z$, $\bar{v}_{\pi(i)} \in \mathbb{R}^{\bar{Z}}$, $v_i(z) = \text{prob}_p(z)\, u_i(z)$, $z \in Z$, and $\bar{v}_{\pi(i)}(\bar{z}) = \text{prob}_{\bar{p}}(\bar{z})\, \bar{u}_{\pi(i)}(\bar{z})$, $\bar{z} \in \bar{Z}$. Hence, the systems of linear equations in $x \in \mathbb{R}^{\bar{Z}}$

$$\bar{\Phi} x = \Phi v_i, \qquad i \in I_- \tag{7}$$

have a solution, $x = \bar{v}_{\pi(i)}$. From basic linear algebra, we then know that

$$\text{rank}\left(\bar{\Phi}\right) = \text{rank}\left(\begin{array}{cc} \bar{\Phi} & \Phi v_i \end{array}\right), \tag{8}$$

where rank assigns the rank to any matrix. By Observation 10, there is a regular $\left|\bar{\mathbf{A}}\right| \times \left|\bar{\mathbf{A}}\right|$ matrix $\bar{C}$ with entries in $\mathbb{Z}$ such that $\bar{C}\bar{\Phi}$ is in row echelon form, not necessarily reduced and except for that leading numbers of non-zero rows may not equal 1. In $\bar{C}\bar{\Phi}$, the last $\left|\bar{\mathbf{A}}\right| - \text{rank}\left(\bar{\Phi}\right)$ rows are zero; by (8), this also holds for the last $\left|\bar{\mathbf{A}}\right| - \text{rank}\left(\bar{\Phi}\right)$ entries of $\bar{C}\Phi v_i$. By Observation 10, $\bar{C}\Phi$ has entries in $\mathbb{Z}_{\eta(\gamma)}$ only. Hence, since $(u,p) \in \mathbb{D}^{\text{gen}}(\gamma)$, by (4), the last $\left|\bar{\mathbf{A}}\right| - \text{rank}\left(\bar{\Phi}\right)$ rows of $\bar{C}\Phi$ are zero too. This already implies that (1) can be satisfied for arbitrary $(p,u) \in \mathbb{D}(\gamma)$. This shows part (i) of Definition 1. Part (ii) follows from the fact that $\mathbf{r}^{-1}$ is an ANFI from $\bar{\Gamma}$ to Γ. Hence, generically, any ANFI is an SANFI.

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