On the partitional core

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André Casajus*† and Andreas Tutic*

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Abstract

We advocate the PCore as a solution concept for TU games that remedies certain deficiencies of the core. The PCore retains the stability property of the core, while waiving efficiency. We provide an axiomatization as well as a non-emptiness criterion.

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*Universität Leipzig, Wirtschaftswissenschaftliche Fakultät, Professur für Mikroökonomik, PF 100920, D-04009 Leipzig, Germany

†corresponding author, e-mail: casajus@wifa.uni-leipzig.de

1. Introduction

Suppose, as it sometimes might happen, that two men, Max (M) and Onno (O), love the same woman, Ada (A). The story can be summarized by the following coalition function:

$$v(S)=\begin{cases}0, & |S|\leq 1,\\ 6, & S=\{M,A\},\\ 4, & S=\{O,A\},\\ 1, & S=\{M,O\},\\ 2, & S=\{M,O,A\}.\end{cases}$$

What would you expect as the outcome of this particular mating game? We hypothesize that the couple with the most intense feelings sticks together, that is, Max and Ada will form the productive coalition $\{M,A\}$, while miserable Onno stays alone. Furthermore, we suspect that the commonly desired female achieves a higher payoff than her endogenously determined partner because of her better outside options (Casajus, 2007; Wiese, 2007).

In this paper, we explore the characteristics of a set-valued solution concept that is a very close relative of the core. Like the core, it demands that a payoff vector cannot be blocked by any coalition. Unlike the core, it does not confine itself to payoff vectors which distribute the worth of the grand coalition. Consider the mating game above. In view of the restriction to efficient payoff vectors, that is, since the core distributes the total worth of $v(N)=2$, it is empty. The proposed concept instead distributes the worth of the efficient partition. In this example, there is a unique efficient partition, $\{\{M,A\},\{O\}\}$, which generates a worth of 6 that is entirely distributed between M and A; O gets nothing, because he is a member of a non-productive coalition. Note that Ada's better options outside the partnership with Max guarantee her at least 4 units of worth, while he can only be sure to get at least a worth of 1. Thus, a slight modification of the core gives a quite reasonable prediction on the outcome of the game.

The main problem with the Core is that it distributes the worth of the grand coalition irrespective of the game being superadditive or not. In games that are not superadditive there might be a coalition structure that yields a higher worth than the grand coalition. But then it is not too intuitive that the players confine themselves to a Pareto inferior outcome (Aumann and Drèze, 1974, p. 129). Instead, the advocated concept, called partitional core (PCore), distributes the worth of an efficient partition, which might not be the grand one. Aumann and Drèze (1974) introduced the similar notion of a coalition structure core (CS-core), which refers to a particular coalition structure as a primitive notion. In contrast, the PCore endogenously determines which coalition structures may form. Aumann and Drèze (1974) already intimate this concept, which recently also appeared as the set of economically stable payoff vectors (Herings, van der Laan and Talman, 2007, p. 91). Pauly (1970) seems to have aimed at the same concept. Yet, its basic properties seem to be unexplored.

We argue that the PCore is the most basic notion, which can be utilized for the prediction of group formation in a TU setting. The core assumes that the grand coalition will form and therefore is of little use in this regard. It is rather a tool to test whether the grand coalition is stable, and if so, which payoff vectors that distribute its worth can be sustained. But all other well introduced concepts combine the idea of endogenizing coalition structures with other ideas, such as second-order stability or specific fairness considerations.

For example, the bargaining set (Maschler, 1992) considers coalitions structures and disregards some deviations by coalitions because of counter-objections. Component efficient point-solution concepts like the AD-value (Aumann and Drèze, 1974), the Wiese value (Wiese, 2007), or the χ-value (Casajus, 2007) can be combined with stability notions like those of Hart and Kurz (1983) to predict the formation of coalition structures. While this kind of analysis undoubtedly has its merits these concepts buy their character as point-solutions with the acceptance of further axioms that are less intuitive and less basic than the requirement that no coalition should be able to deviate profitably. It can be argued (Moulin, 1995) that these additional axioms cannot be grounded solely on the underlying economic structure, but reflect exogenous considerations, i.e., ideas about fair division. Furthermore, these point-solutions rule out many ways of distributing the worth a coalition generates, thereby oversimplifying the problem of building stable coalition structures. In a nutshell, we advocate the PCore as the most basic and intuitive way of analyzing group formation in TU games.

This paper is organized as follows: First, we introduce basic notation and then define the PCore in terms of feasibility and stability. In the third section, we give an axiomatic characterization of the PCore. The fourth section provides a necessary and sufficient condition for the non-emptiness of the PCore. Some remarks conclude the paper.

2. Basic definitions and notation

A TU game is a pair (N, v) consisting of a non-empty and finite set of players N and the coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$; the set of all games on N is denoted by $\mathbb{G}(N)$. Let $\mathcal{N}$ denote the set of all player names; $\mathbb{G} := \bigcup_{N \subseteq \mathcal{N}, \text{finite}} \mathbb{G}(N)$ stands for the set of all games. Subsets of N are called coalitions; $v(K)$ is called the worth of coalition K. The restriction of v to $N' \subseteq N$ is denoted $v|_{N'}$. For $x \in \mathbb{R}^N$, v^x denotes the coalition function given by $v^x(S) = x(S)$ for all $S \subseteq N$. Let $\mathbb{G}^{\text{mod}}(N) = \left\{(N, v^x) \,|x \in \mathbb{R}^N\right\}$ denote the set of modular games.

For any player set N, let $\mathbb{P}(N)$ denote the set of partitions on N; $\mathcal{P}(i)$ denotes the component of $\mathcal{P} \in \mathbb{P}(N)$ containing player $i \in N$. The superadditive hull $\bar{v}$ of some coalition function v for N is given by $\bar{v}(S) = \max_{\mathcal{P} \in \mathbb{P}(N)} \sum_{C \in \mathcal{P}} v(P)$ for all $S \subseteq N$. Let $\mathbb{P}(N, v) = \left\{\mathcal{P} \in \mathbb{P}(N) \,|\, \sum_{C \in \mathcal{P}} v(P) = \bar{v}(N)\right\}$ denote the set of **efficient partitions**; a coalition is called **productive** if it is contained in some efficient coalition structure; $\mathbb{N}(N, v) := \{C \subseteq |\exists \mathcal{P} \in \mathbb{P}(N, v) : C \in \mathcal{P}\}$ denotes the set of productive coalitions. A solution concept is a correspondence φ that assigns to any $(N, v) \in \mathbb{G}$ a set $\varphi(N, v) \subseteq \mathbb{R}^N$; for $x \in \mathbb{R}^N$ and $S \subseteq N$, we write $x(C)$ for $\sum_{i \in C} x_i$ where x_i denotes the entry corresponding to player $i \in N$; $x_S = (x_i)_{i \in S}$ denotes the restriction of x to $S \subseteq N$.

The **Core** of a game (N, v) is the set $\text{Core}(N, v)$ containing those payoff vectors $x \in \mathbb{R}^N$ which satisfy the following properties:

1. $x(N) \leq v(N)$ (**Feasibility, F**).
2. $x(S) \geq v(S)$ for all $S \subseteq N$ (**Stability, S**).

Of course, **F** and **S** imply $x(N) = v(N)$ (**Efficiency, E**).

3. The partitional core (PCore)

Definition. The **PCore** of a game (N, v) is the set $\text{PCore}(N, v)$ containing those payoff vectors $x \in \mathbb{R}^N$ which satisfy the following properties:

1. $x(N) \leq \bar{v}(N)$ (**Hull feasibility, HF**).
2. $x(S) \geq v(S)$ for all $S \subseteq N$ (**Stability, S**).

As the reader can see, we retain the stability notion of the core while substituting hull feasibility for feasibility. From the definition, it is quite immediate that, together with stability, hull feasibility implies hull efficiency in the following sense.

Lemma 1. *If $x \in \text{PCore}(N, v)$ then $x(N) = \bar{v}(N)$ **(Hull efficiency).***

While hull feasibility alone does not imply component efficiency, both axioms together rule out cross-subsidies between productive coalitions. Again, the proof is immediate.

Lemma 2. *If* $x \in \mathrm{PCore}(N, v)$ *and* $C \in \mathbb{N}(N, v)$ *then* $x(C) = v(C)$ *(**Component efficiency**).*

These two lemmas make clear that the PCore predicts that some efficient coalition structure prevails and that the productive coalitions obtain their worth.

Remark 1. The PCore is the core of the superadditive hull: $\mathrm{PCore}(N, v) = \mathrm{Core}(N, \bar{v})$.

Remark 2. For superadditive games, we have $v = \bar{v}$, hence, $\mathrm{PCore}(N, v) = \mathrm{Core}(N, v)$.

In view of Remark 2, the necessity of a concept like the PCore depends on the reasonableness of non-superadditive coalition functions. In line with Aumann and Drèze (1974), we argue for the acceptance of non-superadditivity. Superadditivity implies a certain interpretation of what a coalition is or actually can do. Yet, there is no reason to restrict, ex ante, the set of potential applications by disregarding alternative interpretations. Reconsider our leading example; the low worth of the grand coalition could be due to several circumstances. For example, a candle light dinner might be more pleasant à deux than à trois (internal difficulties). Or, the state could enforce a law against promiscuity (external difficulties). In a nutshell, non-superadditivity results from difficulties in "acting together" for one or another reason.

4. A characterization of the PCore

Since the concepts are very close relatives, one would like to have some similar characterizations of the core and the PCore. There are many axiomatizations (for example Peleg (1986), Voorneveld and van den Nouweland (1998), and Llerena (2007)). First, we adapt the Llerena characterization in order the axiomatize the PCore and then give some remarks on the other ones. We employ the following axioms:

Hull efficiency, HE. For all $(N, v) \in \mathbb{G}$ and $x \in \varphi(N, v)$, we have $x(N) = \bar{v}(N)$.

Anti-monotonicity, AM. For all $(N, v), (N, v') \in \mathbb{G}$, if $v'(S) \geq v(S)$ for all $S \subseteq N$ and $\bar{v}(N) = \bar{v}'(N)$ then $\varphi(N, v') \subseteq \varphi(N, v)$.

Reasonableness from above, R. For all $(N, v) \in \mathbb{G}$, $x \in \varphi(N, v)$, we have $x_i \leq \max_{i \in S \subseteq N} \{v(S) - v(S \setminus \{i\})\}$ for all $i \in N$

Modularity, M. For all $(N, v^x) \in \mathbb{G}^{\text{mod}}(N)$, we have $x \in \varphi(N, v^x)$.

Consistency, C. For all $(N, v) \in \mathbb{G}$, $S \subseteq N$, $|S| = 2$, and $x \in \varphi(N, v)$, we have $x_S \in \varphi(S, v_x)$. For $(N, v) \in \mathbb{G}(N)$, $x \in R^N$, and $S \subseteq N$, $S \neq \emptyset$, the **reduced game** (S, v_x) relative to x and S is defined by

$$v_x(T) = \begin{cases} 0, & T = \emptyset, \\ \bar{v}(N) - x(N \backslash S) & T = S, \\ \max_{Q \subseteq N \backslash S} \{v(T \cup Q) - x(Q)\}, & \emptyset \neq T \subsetneq S. \end{cases}$$

R and **M** are as in the original characterization. **HE**, **AM** and **C** were obtained by replacing the worth of the grand coalition by its worth in the superadditive hull.

The fundamental change from **E** to **HE** entails the specific adaptions of the other axioms. **AM** states that a distribution of a given pie which satisfies a group of people also should satisfy less greedy ones. Note that **HE** determines the size of the pie. **R** sets the upper bound that no player can obtain more than his maximal marginal contribution. Since **R** does not refer to the size of the pie, it is unrelated to feasibility and therefore remained unchanged. The idea behind **M** is that if cooperation neither pays nor harms the players should accept their stand-alone payoffs. **C** states that a solution should be consistent regarding hypothetical re-bargaining within two-player coalitions. These coalitions may reconsider the distribution of their share of the pie, which is, given **HE**, just $\bar{v}(N) - x(N \backslash S)$. Bargaining on this pie, both parties invoke their maximal net outside options,$\max_{Q \subseteq N \backslash S} \{v(T \cup Q) - x(Q)\}$. Now, **C** demands that the players accept their actual payoffs also in the fictional re-bargaining game. In nuce, re-bargaining does not pay.

Theorem 1. *The PCore is the unique solution that satisfies **HE**, **C**, **AM**, **R**, and **M**.*

This theorem is implied by the following two lemmas.

Lemma 3. *The PCore satisfies **HE**, **C**, **AM**, **R**, and **M**.*

Since the proof of the PCore satisfying **C** does not involve the coalition size, **C** could be sharpened to re-bargaining groups of any size.

Proof. By Lemma 1, we know that the PCore satisfies **HE**. Further, it is easy to see that the PCore satisfies **M**. Let v and v' be as in **AM**. If $x \in \text{PCore}(N, v')$ then x is feasible and stable in (N, v), i.e., $x \in \text{PCore}(N, v)$.

Consider **C**. Let $x \in \text{PCore}(N,v)$ and $S \subseteq N$. First, we check stability. We consider two cases: $T \subsetneq S$ and $T = S$. For $T = S$, note that $v_x(S) = \bar{v}(N) - x(N\backslash S) = x(S) = x_S(S)$. Turning to the case $T \subsetneq S$, suppose there is some $Q \subseteq N\backslash S$ such that $x_S(T) < v(T \cup Q) - x(Q)$. But then the coalition $T \cup Q$ could block x, so $x \notin \text{PCore}(N,v)$, contradicting the assumption. Now, we turn to feasibility. By Lemma 1, we have to show that x_S is efficient in (S, v_x). To see this, just observe $x_S(S) = \sum_{C \in \mathcal{P}} x(C) \geq \sum_{C \in \mathcal{P}} v_x(C)$ for all $\mathcal{P} \in \mathbb{P}(S)$, where the inequality follows from the fact that $T \subseteq S$ cannot block x.

Finally, we show that the PCore satisfies **R**. Let $x \in \text{PCore}(N,v)$, and suppose there is some $i \in N$ such that $x_i > \max_{i \in S \subseteq N} \{v(S) - v(S\backslash\{i\})\}$. By Lemma 2, this implies $x_i > v(\mathcal{P}(i)) - v(\mathcal{P}(i)\backslash\{i\}) = x(\mathcal{P}(i)) - v(\mathcal{P}(i)\backslash\{i\})$ for all $\mathcal{P} \in \mathbb{P}(N,v)$. But then $\mathcal{P}(i)\backslash\{i\}$ could block x. ∎

Lemma 4. *There is at most one solution that satisfies **HE**, **C**, **AM**, **R**, and **M**.*

Proof. Let φ be a solution that satisfies these axioms. First, we show $\text{PCore}(N,v) \subseteq \varphi(N,v)$, afterwards, we turn to the converse inclusion. Let $(N,v) \in \mathbb{G}$. There are two possibilities: Either there exists some $x \in \text{PCore}(N,v)$ or not. Consider the first case. Construct the modular game (N, v^x). We obtain $v^x(S) \geq v(S)$, for all $S \subseteq N$ and $\bar{v}^x(N) = \bar{v}(N)$. Combining **M** and **AM**, we get $x \in \varphi(N, v^x) \subseteq \varphi(N,v)$. Thus, $\text{PCore}(N,v) \subseteq \varphi(N,v)$, which also is true in the case of $\text{PCore}(N,v) = \emptyset$.

Now, we show $\varphi(N,v) \subseteq \text{PCore}(N,v)$. For games with $|N| = 1$, **HE** implies $\text{PCore}(N,v) = \varphi(N,v)$.

For $|N| = 2$, there are two cases: (a) $v(\{1,2\}) \geq v(\{1\}) + v(\{2\})$ or (b) $v(\{1,2\}) < v(\{1\}) + v(\{2\})$. In case (a), **HE** implies $x(N) = v(\{1,2\})$ for all $x \in \varphi(N,v)$. Note that $v(\{1,2\}) \geq v(\{1\}) + v(\{2\})$ implies that $v(\{i,j\}) - v(\{j\}) \geq v(\{i\})$, $i,j = 1,2$, $i \neq j$. Then, **R** gives $x_i \leq x(N) - v(\{j\})$, $i,j = 1,2$, $i \neq j$, which is equivalent to $v(\{j\}) \leq x_j$, $j = 1,2$. So, $x \in \text{PCore}(N,v)$. In case (b), **HE** implies $x(N) = v(\{1\}) + v(\{2\})$ for all $x \in \varphi(N,v)$. By **R**, we immediately obtain $x_i \leq v(\{i\})$, $i = 1,2$. So, $x \in \text{PCore}(N,v)$.

Finally, we consider all games with $|N| \geq 3$. Let $x \in \varphi(N,v)$ and let $T \subsetneq N$. There is some $\{i,j\} \subsetneq N$ such that $i \in T$, $j \in N\backslash T$. **C** implies $x_{\{i,j\}} \in \varphi(\{i,j\}, v_x)$. But this is a game with just two players for which we already have proved that $x \in$

$\varphi(N, v)$ iff $x \in \text{PCore}(N, v)$. Thus,

$$x_i \geq \max_{Q \subseteq N \setminus S} \{v(\{i\} \cup Q) - x(Q)\} \geq v(T) - x(T \setminus \{i\}).$$

So, $x(T) \geq v(T)$ for all $T \subseteq N$. If $T = N$, $x(N) = \bar{v}(N) \geq v(N)$ implies that stability is trivially satisfied. Invoking **HE** establishes that $\varphi(N, v) \subseteq \text{PCore}(N, v)$. Thus, $\varphi(N, v) = \text{PCore}(N, v)$. ∎

Remark 3. The Peleg (1986) axiomatization employs the superadditivity axiom, which is not satisfied by the PCore. Thus, it cannot be adapted in an obvious way.

Remark 4. Adding individual rationality, $x_i \geq v(\{i\})$, one could relax **HE** into **HF**. The proof is analogous to Voorneveld and van den Nouweland (1998, Proposition 3.4).

5. Non-emptiness of the PCore

Adapting the well-known criterion for the non-emptiness of the core (Bondareva, 1963; Shapley, 1967), one obtains a similar criterion for the PCore. Our proof, of course, mimics those of Bondareva and Shapley.

Theorem 2. *The PCore of (N, v) is non-empty iff for all $\lambda : 2^N \to [0, 1]$ such that $\sum_{i \in S \subseteq N} \lambda(S) = 1$ for all $i \in N$, we have $\sum_{S \subseteq N} \lambda(S) v(S) \leq \bar{v}(N)$.*

Proof. We employ the following version of Farkas' Lemma which is well-known in linear programming: The system $Ax \leq b$ has a solution iff $y^T b \geq 0$ for all y such that $y \geq 0$ and $y^T A = 0$, where $b, y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, the superscript T denotes the transpose, and A is a real $m \times n$-matrix for some $m, n \in \mathbb{N}$.

Consider the real matrix $A = (a_{k\ell})$ with $|2^N|$ rows indexed by the coalitions $S \subseteq N$ and with $|N|$ columns indexed by the players $i \in N$ such that $a_{\emptyset,i} = 1$, $a_{S,i} = -1$, and $a_{N \setminus S,i} = 0$ for all $i \in N$, $i \in S \subseteq N$. Further, consider $b \in \mathbb{R}^{(2^N)}$, $b_S = v(S)$ if $S \neq \emptyset$ and $b_\emptyset = \bar{v}(N)$. By definition, we have $\text{PCore}(N, v) = \{x \in \mathbb{R}^N | Ax \leq b\}$. Farkas' Lemma then implies that $\text{PCore}(N, v) \neq \emptyset$ iff for all $y \in \mathbb{R}^{2^N}$, $y \geq 0$ such that $y^T A$ $= 0$, we have $y^T b \geq 0$. By construction of A, $y \geq 0$ and $yA = 0$ implies $y_\emptyset > 0$. Hence, we are allowed to restrict attention to those y where $y_\emptyset = 1$. Then, $y^T A = 0$ implies $y_S \in [0, 1]$ for all $\emptyset \neq S \subseteq N$. Further, $y^T b \geq 0$ becomes $\sum_{S \subseteq N} y_S v(S) \leq \bar{v}(N)$ which proves the claim. ∎

6. Conclusion

In this paper, we explore the basic properties of the PCore which is obtained from the core by substituting hull feasibility for feasibility. By stability, this entails hull efficiency instead of efficiency. One may be tempted to modify other efficient solution concepts in an analogous way. In particular, one could look for a hull efficient version of the Shapley value; the Shapley formula could be applied to the superadditive hull or some modification of the coalition function were just the worth of the grand coalition is altered. Though it is clear that the resulting solutions fail additivity, it may be worthwhile to explore their basic properties.

The PCore complements other research in cooperative game theory on group formation as in the framework of hedonic games (see for example Banerjee, Konishi and Sönmez, 2001) or in a TU setting. Within the TU setting, the diverse concepts differ on the extent they take into account outside options. The AD-value completely disregards outside options, which are captured by the PCore and the χ-value. For example, one may wonder under which conditions χ-stable payoffs (Casajus, 2007, Section 6) lie in the PCore. More generally, it remains to explore the relation between the seemingly different ways of incorporating outside options.

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