Null players, solidarity, and the egalitarian Shapley values

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Abstract

The Shapley value certainly is the most eminent single-point solution concept for TU-games. In its standard characterization, the null player property indicates the absence of solidarity among the players. First, we replace the null player property by a new axiom that guarantees null players non-negative payoffs whenever the grand coalition’s worth is non-negative. Second, the equal treatment property is strengthened into desirability. This way, we obtain a new characterization of the class of egalitarian Shapley values, i.e., of convex combinations of the Shapley value and the equal division solution.

Key Words: Solidarity; egalitarian Shapley value; equal division value; desirability

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1. Introduction

A cooperative game with transferable utility (TU-game) consists of a non-empty and finite set of players, $N$, and a coalition function, $v \in \{ f \mid f : 2^N \to \mathbb{R}, \ f(\emptyset) = 0 \}$, which describes the worths $v(S)$ that can be generated by individuals that are willing to cooperate within coalitions $S \subseteq N$. Assuming that the grand coalition eventually forms, the question arises how to distribute the grand coalition’s worth, $v(N)$. The most prominent single-point solution concept that answers this question certainly is the Shapley value (Shapley, 1953).

The Shapley value obeys marginality (Young, 1985), i.e., a player’s payoff depends only on his own productivity measured by marginal contributions. Hence, the Shapley value does not allow for solidarity between the players. This can also be seen from the fact that unproductive players (null players) are assigned zero payoffs. Therefore, we depart from the standard characterization of the Shapley value by dropping the null player property\(^1\) in order to create space for solidarity (as e.g. Nowak and Radzik (1994), van den Brink (2007), and van den Brink, Funaki and Ju (2011)).

Given that we no longer require the payoffs of null players to be zero, the nature of solidarity manifests itself in the way null players are treated. Since null players are completely unproductive, their “selfish” payoffs, i.e., their Shapley payoffs are zero. Consequently, any non-zero payoff must be due to solidarity among the players. Note that Nowak and Radzik (1994), van den Brink (2007), and Chameni-Nembua (2012) pursue another approach. Instead of dealing with the treatment of null players, they consider other notions of “null players”. In fact, they identify certain types of players that are supposed to obtain a zero payoff.

We suggest the null player in a productive environment property that requires a null player to obtain a non-negative payoff whenever the worth generated by the grand coalition is non-negative. This property rules out that one could find oneself in a wealthy society that drains out the unproductive players in order to make other players better off. We feel that this would stretch our sense of solidarity too far. Obviously, this property is implied by the null player property and therefore it is satisfied by the Shapley value.

Our main result states that the null player in a productive environment property together with additivity, efficiency, and desirability already characterizes the class of egalitarian Shapley values introduced by Joosten (1996). Note that desirability

\(^{1}\)Ruiz, Valenciano and Zarzuelo (1998), Driessen and Radzik (2003), Hernandez-Lamoneda, Juarez and Sanchez-Sanchez (2008), Chameni-Nembua and Andjiga (2008), and Chameni-Nembua (2012) provide formulae for the class of efficient, linear, and symmetric values.
demands more productive players to obtain no lower payoffs than less productive ones (Maschler and Peleg, 1966). Other characterizations of this class have been given by Joosten (1996), Malawski (2004), and van den Brink et al. (2011) where the latter also provide a non-cooperative implementation of the egalitarian Shapley values as the subgame perfect Nash equilibrium of an extensive form game.

This paper is organized as follows. Further definitions and notations are given in Section 2. Section 3 provides our characterization of the egalitarian Shapley values. Finally, the appendix contains all the proofs.

2. Basic definitions and notation

The set of all coalition functions on $N$ is denoted by $V(N)$. Since we work within a fixed player set, it is dropped as an argument. In particular, we address $v \in V$ as a game. Subsets of $N$ are called coalitions, and $v(S)$ is called the worth of coalition $S$. For $v, w \in V, \alpha \in \mathbb{R}$, the coalition functions $v + w \in V$ and $\alpha \cdot v \in V$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $w_T \in V$, $w_T(S) = 1$ if $T \subseteq S$ and $w_T(S) = 0$ for $T \not\subseteq S$, is called a unanimity game. For $T \subseteq N$, $T \neq \emptyset$, the game $e_T \in V$, $e_T(S) = 1$ if $T = S$ and $e_T(S) = 0$ for $T \neq S$, is called a standard game. Any $v \in V$ can be uniquely represented by unanimity games,

$$v = \sum_{T \subseteq N: T \neq \emptyset} \lambda_T(v) \cdot u_T, \quad \lambda_T(v) := \sum_{S \subseteq T: S \neq \emptyset} (-1)^{|T|-|S|} \cdot v(S).$$

Player $i \in N$ is called a null player in $v \in V$ if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$; players $i, j \in N$ are called symmetric in $v \in V$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A value on $N$ is an operator $\varphi$ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in V$. The equal division value, ED, is given by

$$\text{ED}_i(v) := \frac{v(N)}{|N|}, \quad \text{for all } i \in N.$$ 

The Shapley value (Shapley, 1953), Sh, given by

$$\text{Sh}_i(v) := \sum_{T \subseteq N: i \in T} \frac{\lambda_T(v)}{|T|}, \quad \text{for all } i \in N,$$

is characterized by the axioms $\mathbf{E}$, $\mathbf{N}$, $[\mathbf{A} \text{ or } \mathbf{L}]$, and $[\mathbf{ET} \text{ or } \mathbf{S}]$ below.

Efficiency, $\mathbf{E}$. For all $v \in V$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Null player, $\mathbf{N}$. For all $v \in V$ and every $i \in N$, who is a null player in $v$, $\varphi_i(v) = 0$.

Additivity, $\mathbf{A}$. For all $v, w \in V$, $\varphi(v + w) = \varphi(v) + \varphi(w)$. 

3. Treatment of null players in a productive environment

The “mixture” of the Shapley value and the equal division value, $\text{Sh}^\alpha$, $\alpha \in \mathbb{R}$, is given by

$$
\text{Sh}^\alpha_i (v) = \alpha \cdot \varphi_i (v) + (1 - \alpha) \cdot \text{ED}_i (v), \quad \text{for all } i \in N \text{ and all } v \in \mathcal{V}.
$$

For $0 \leq \alpha \leq 1$, the value $\text{Sh}^\alpha$ is said to be an egalitarian Shapley value (Joosten, 1996). The egalitarian Shapley values obey efficiency, symmetry, and additivity. For $\alpha \neq 1$, they do not satisfy the null player property.\(^2\) But, they satisfy the following weaker one.

**Null player in a productive environment, NPE.** For all $v \in \mathcal{V}$ and $i \in N$ such that $i$ is a null player in $v$ and $v(N) \geq 0$, we have $\varphi_i (v) \geq 0$.

This property guarantees null players to obtain a non-negative payoff whenever the worth generated by the grand coalition is non-negative. This is quite plausible. For we consider deviations from the Shapley payoffs as an expression of a certain degree of solidarity among players. When the whole society is productive, $v(N) \geq 0$, then it is not necessary that any player ends up with a negative payoff. In particular, null players need not receive a negative payoff. Since they do not do any harm to society, they actually should not obtain negative payoffs. It turns out that this axiom has strong implication on the nature of solidarity among the players within the class of efficient, linear, and symmetric values; one is almost down to the class of egalitarian Shapley values.

**Proposition 1.** A $\text{TU-value } \varphi$ satisfies efficiency, linearity, the equal treatment property, and the null player in a productive environment property if and only if there is an $\alpha \leq 1$ such that $\varphi = \text{Sh}^\alpha$.

\(^2\) Instead, each $\text{Sh}^\alpha, \alpha \in \mathbb{R}$ satisfies the $\alpha$-egalitarian null player property, stating that $\varphi_i (v) = (1 - \alpha) \cdot v(N) / |N|$ for all null players $i$ in $v$ (Joosten, 1996). In a sense, the parameter $\alpha$ determines the extent of solidarity among the players.
The null player in a productive environment property in combination with standard axioms does not rule out that a higher productivity translates into lower payoffs. For example, the TU-value $\text{Sh}^{-1} = 2 \cdot \text{ED} - \text{Sh}$ satisfies this weak requirement on null players as well as additivity, efficiency and the equal treatment property. Yet, for $1, 2 \in N$, we have $\text{Sh}_1^{-1} (u_{(1)}) = \frac{2}{|N|} - 1 < \frac{2}{|N|} = \text{Sh}_2^{-1} (u_{(1)})$ even though player 1 is more productive than player 2 in $u_{(1)}$. In a sense, the transfers from the productive player, 1, to the unproductive one, 2, are too high. We feel that this is also not in line with sound solidarity considerations. In order to rule out such awkward transfers, we invoke the following axiom.

Desirability\(^3\), D. For all $v \in \mathcal{V}$ and all $i, j \in N$, if $v(\{S \cup \{i\}) \geq v(\{S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$, then $\varphi_i(v) \geq \varphi_j(v)$.

Desirability compares two players in a game and ensures that their payoffs are not opposite to their productivities measured by marginal contributions. Replacing the equal treatment property by desirability does not only rule out adverse incentives, but also allows to weaken linearity into additivity.\(^4\) Next, we state the main result of this paper, a characterization of the class of egalitarian Shapley values.

Theorem 2. A TU-value $\varphi$ satisfies additivity, efficiency, desirability, and the null player in a productive environment property if and only if there exists an $\alpha \in [0, 1]$ such that $\varphi = \text{Sh}^\alpha$.

Our characterization is non-redundant. The solution $\varphi^A$, given by $\varphi^A(v) = \text{Sh}(v)$ if $v(N) < 1$ and $\varphi^A(v) = \text{ED}(v)$ if $v(N) \geq 1$, satisfies all axioms but additivity. The solution given by $\varphi^E_i(v) = 0$ for all $i \in N$ and $v \in \mathcal{V}$ satisfies all axioms but efficiency. The solution given by $\varphi^D = \text{Sh}^{-1}$ satisfies all axioms but desirability. The solution given by $\varphi^{\text{NPE}} = \text{Sh}^2$ satisfies all axioms but the null player in a productive environment property.

Recently, van den Brink et al. (2011) characterize the class of egalitarian Shapley values employing efficiency, linearity, symmetry, and weak monotonicity below.

Weak monotonicity, $M^-$. For all $v, w \in \mathcal{V}$ and all $i \in N$ such that $v(N) \geq w(N)$ and $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

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\(^3\)Desirability is also known as local monotonicity (e.g. van den Brink et al., 2011).

\(^4\)Employing that fact that there are functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$ but $f(x \cdot y) \neq x \cdot f(y)$ for some $x, y \in \mathbb{R}$ (Macho-Stadler, Pérez-Castrillo and Wettstein, 2007), it is easy to show that there are solutions that obey additivity, efficiency, and symmetry but not linearity.
This axiom is a relaxation of the well-known monotonicity axiom (Young, 1985) in the sense that its implication only is supposed to hold if \( v(N) \geq w(N) \). Their characterization and ours are related as follows: van den Brink et al. (2011, Lemma 4.2) show that symmetry and weak monotonicity entail the null player in a productive environment property. Hence, we also provide an alternative proof of their characterization (van den Brink et al., 2011, Theorem 4.3).

4. APPENDIX

Proof of Proposition 1. It is clear that \( Sh^\alpha, \alpha \leq 1 \), inherits \( L, E, ET \), and \( NPE \) from \( Sh \).

Let the TU-value \( \varphi \) satisfy \( L, E, ET \), and \( NPE \). By Malawski (2008, Theorem 2), \( \varphi \) meets \( S \).

For the rest of the proof, we set \( n := |N| \). Chameni-Nembua (2012) shows that every TU-value \( \varphi \) satisfying \( E, L \), and \( S \) is characterized by a sequence \( \alpha = \alpha(s)_{s=2}^{n} \in \mathbb{R}^{n-1} \) such that

\[
\varphi_i^\alpha (v) = v(\{i\}) + \sum_{s=2}^{n} \sum_{S \supseteq \{i\}, |S|=s} \frac{(n-s)! (s-1)!}{n!} A_i^{\alpha(s)} (S),
\]

where \( A_i^{\alpha(s)} (S) = \alpha(s) \cdot [v(S) - v(S \setminus \{i\})] + \frac{1-\alpha(s)}{|S|-1} \sum_{j \in S \setminus \{i\}} [v(S) - v(S \setminus \{j\})] \).

Note that (i) \( \varphi^\alpha(v) + \varphi^\beta(v) = \varphi^{\alpha+\beta}(v) \) and \( \varphi^{r\cdot \alpha}(v) = r \cdot \varphi^\alpha(v) \) for all \( \alpha, \beta \in \mathbb{R}^{n-1} \) and \( r \in \mathbb{R} \), (ii) \( \varphi^\alpha(v) = Sh(v) \) if \( \alpha(s) = 1 \), for \( s = 2, \ldots, n \), and (iii) \( \varphi^\alpha(v) = ED(v) \) if \( \alpha(s) = 0 \), for \( s = 2, \ldots, n \). Therefore, it suffices to show that our conditions impose \( \alpha(s) = c \leq 1 \) for \( s = 2, \ldots, n \).

For \( T \neq N, e_T \), and some \( i \notin T \), the term \( A_i^{\alpha(S)}(S) \) vanishes except for \( S = T \cup \{i\} \). Hence, \( \varphi_i^\alpha(e_T) = - (n - |T| - 1)! \cdot |T|! \cdot \alpha(|T| + 1) \). From \( E \) and \( S \), we conclude \( \varphi_i^\alpha(e_N) = 1/|N| \) for all \( i \in N \) and

\[
\varphi_i^\alpha(e_T) = \begin{cases} 
\frac{(n-|T|)! \cdot (|T|-1)!}{n!} \cdot \alpha(|T| + 1), & i \in T, \\
- \frac{(n-|T|)! \cdot |T|!}{n!} \cdot \alpha(|T| + 1), & i \notin T,
\end{cases}
\] (2)

for all \( T \subset N, T \neq \emptyset \). Thus, \( \varphi_i^\alpha(u_{N \setminus \{i\}}) = \varphi_i^\alpha(e_{N \setminus \{i\}} + e_N) = (1 - \alpha(n))/n \) for all \( i \in N \). Since \( i \) is a null player in \( u_{N \setminus \{i\}} \) and by \( NPE \), we conclude \( \alpha(n) \leq 1 \).

Further, we need to show \( \alpha(s) = \alpha(n) \) for \( s = 2, \ldots, n - 1 \). Define

\[
k := \min \{ s \mid \alpha(s + \ell) = \alpha(n) \text{ for all } 0 \leq \ell \leq n - s \}.\]
Suppose $k > 2$. Let $K \subseteq N$ be such that $|K| = k$ and $i, j \in K$. Then, we have
\begin{equation}
  u_{K \setminus \{i\}} - u_{K \setminus \{i, j\}} = -\sum_{S \subseteq N \setminus K} e_{K \setminus \{i, j\} \cup S} - \sum_{S \subseteq N \setminus K} e_{K \setminus \{j\} \cup S}
\end{equation}
and
\begin{equation}
  \varphi_i^\alpha (u_{K \setminus \{i\}} - u_{K \setminus \{i, j\}}) \equiv \sum_{S \subseteq N \setminus K} \varphi_i^\alpha (e_{K \setminus \{i, j\} \cup S}) - \sum_{S \subseteq N \setminus K} \varphi_i^\alpha (e_{K \setminus \{j\} \cup S})
\end{equation}
\begin{equation}
  = \sum_{S \subseteq N \setminus K} \frac{(n - (k - 2 + s) - 1)! \cdot (k - 2 + s)!}{n!} \cdot \alpha((k - 2 + s) + 1)
\end{equation}
\begin{equation}
  - \sum_{S \subseteq N \setminus K} \frac{(n - (k - 1 + s))! \cdot ((k - 1 + s) - 1)!}{n!} \cdot \alpha((k - 1 + s) + 1)
\end{equation}
\begin{equation}
  = \sum_{S \subseteq N \setminus K} \frac{(n - k + 1 - s)! \cdot (k - 2 + s)!}{n!} \cdot \frac{\alpha(k + s - 1) - \alpha(k + s)}{s}
\end{equation}
\begin{equation}
  = \frac{(n - k + 1)! \cdot (k - 2)!}{n!} \cdot \left[ \alpha(k - 1) - \alpha(k) \right],
\end{equation}
where in the last equation we use $\alpha(k + s - 1) = \alpha(k + s)$ for all $1 \leq s \leq n - k$.
Define $w := \alpha(k) - \alpha(k - 1) \cdot \left( u_{K \setminus \{i\}} - u_{K \setminus \{i, j\}} \right)$. We obtain $\varphi_i^\alpha (w) < 0$ although $i$ is a null player in $w$, a contradiction to NPE.

We prepare the proof of Theorem 2 by a lemma showing that E, A, and D imply L.

**Lemma 3.** Every TU-value $\varphi$ that satisfies efficiency, additivity, and desirability is linear.

**Proof.** Let $\varphi$ meet E, A, and D. It suffices to show $\varphi(\lambda \cdot v) = \lambda \cdot \varphi(v)$ for all $\lambda \in \mathbb{R}$, $v \in V$. First, we establish that for all $v, w \in V$ and $i, j \in N$ such that
\begin{equation}
  v(K \cup \{i\}) - v(K \cup \{j\}) \geq w(K \cup \{i\}) - w(K \cup \{j\}) \quad \text{for all } K \subseteq N \setminus \{i, j\},
\end{equation}
we have $\varphi_i(v) - \varphi_j(v) \geq \varphi_i(w) - \varphi_j(w)$.

By (4), we have $(v - w)(K \cup \{i\}) \geq (v - w)(K \cup \{j\})$ for all $K \subseteq N \setminus \{i, j\}$, i.e., $v - w, i$, and $j$ meet the hypothesis of D. This entails
\begin{equation}
  \varphi_i(v) - \varphi_j(v) = \varphi_i(w + (v - w)) - \varphi_j(w + (v - w))
\end{equation}
\begin{equation}
  \overset{D}{=} \varphi_i(w) - \varphi_j(w) + \varphi_i(v - w) - \varphi_j(v - w) \geq \varphi_i(w) - \varphi_j(w)
\end{equation}
as claimed above.
It is well known that A implies homogeneity for rational scalars, i.e., \( \varphi(\rho \cdot v) = \rho \cdot \varphi(v) \) for all \( \rho \in \mathbb{Q} \). By A, it suffices to consider coalition functions of the form \( \lambda \cdot u_T, T \subseteq N, T \neq \emptyset, \lambda \in \mathbb{R} \setminus \mathbb{Q} \).

Let \( \lambda > 0 \) and \( T \subseteq N, T \neq \emptyset \). Since the rational numbers are dense in the reals, there are rational sequences \( (\lambda_k^-)_{k \in \mathbb{N}} \) and \( (\lambda_k^+)_{k \in \mathbb{N}} \) such that \( 0 < \lambda_k^- \leq \lambda \leq \lambda_k^+ \) for all \( k \in \mathbb{N} \) and \( \lim_{k \to \infty} \lambda_k^- = \lim_{k \to \infty} \lambda_k^+ = \lambda \). For all \( i \in T \), it is clear that
\[
\lambda_k^- \cdot (u_T(K \cup \{i\}) - u_T(K \cup \{j\})) \leq \lambda \cdot (u_T(K \cup \{i\}) - u_T(K \cup \{j\})) \leq \lambda_k^+ \cdot (u_T(K \cup \{i\}) - u_T(K \cup \{j\}))
\]
for all \( K \subseteq N \setminus \{i,j\} \) and \( j \in N \). Hence, \( i, j, \lambda_k^- \cdot u_T, \) and \( \lambda \cdot u_T \) (or \( \lambda \cdot u_T \) and \( \lambda_k^+ \cdot u_T \)) satisfy the condition stated in (4). Thus, we have
\[
(\varphi_i(\lambda_k^- \cdot u_T) - \varphi_j(\lambda_k^- \cdot u_T)) \leq \varphi_i(\lambda \cdot u_T) - \varphi_j(\lambda \cdot u_T) \leq (\varphi_i(\lambda_k^+ \cdot u_T) - \varphi_j(\lambda_k^+ \cdot u_T)) .
\]
Since A implies homogeneity for rational scalars, we obtain
\[
\lambda_k^- \cdot (\varphi_i(u_T) - \varphi_j(u_T)) \leq \varphi_i(\lambda \cdot u_T) - \varphi_j(\lambda \cdot u_T) \leq \lambda_k^+ \cdot (\varphi_i(u_T) - \varphi_j(u_T)).
\]
Taking the limit and by assumption, we thus have
\[
\varphi_i(\lambda \cdot u_T) - \varphi_j(\lambda \cdot u_T) = \lambda \cdot (\varphi_i(u_T) - \varphi_j(u_T)).
\]
Summing up over \( j \in N \) gives
\[
|N| \cdot \varphi_i(\lambda \cdot u_T) - \sum_{\ell \in N} \varphi_\ell(\lambda \cdot u_T) = |N| \cdot \lambda \cdot \varphi_i(u_T) - \lambda \cdot \sum_{\ell \in N} \varphi_\ell(u_T) .
\]
Using E, one obtains \( \varphi_i(\lambda \cdot u_T) = \lambda \cdot \varphi_i(u_T) \). Analogously, this can be seen for \( i \in N \setminus T \) or \( \lambda < 0 \).

**Proof of Theorem 2.** It is clear that Sh\(^\alpha\), \( \alpha \in [0,1] \) inherits A, E, D, and NPE from Sh. Let the TU-value \( \varphi \) satisfy A, E, D, and NPE. By Lemma 3, \( \varphi \) meets L. Since D implies ET, Proposition 1 already entails \( \varphi = \text{Sh}^\alpha \) for some \( \alpha \leq 1 \). Suppose \( 0 > \alpha \). This entails
\[
\varphi_i^\alpha(u_{N\setminus\{i\}}) = \frac{1 - \alpha}{n} > \frac{1 - \alpha}{n} + \frac{\alpha}{n - 1} = \varphi_j^\alpha(u_{N\setminus\{i\}}) \quad \text{for } i \in N, j \in N \setminus \{i\}.
\]
Yet, the marginal contributions of \( i \) are not greater than those of \( j \) in \( u_{N\setminus\{i\}} \), contradicting D. Hence \( 0 \leq \alpha \leq 1 \).
References


