Second-order productivity, second-order payoffs, and the Shapley value

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Abstract

We introduce the concepts of the players' second-order productivities in cooperative games with transferable utility (TU games) and of the players' second-order payoffs for one-point solutions for TU games. Second-order productivities are conceptualized as second-order marginal contributions, that is, how one player affects another player's marginal contributions to coalitions containing neither of them by entering these coalitions. Second-order payoffs are conceptualized as the effect of one player leaving the game on the payoff of another player. We show that the Shapley value is the unique efficient one-point solution for TU games that reflects the players' second-order productivities in terms of their second-order payoffs.

Keywords: TU game, Shapley value, second-order marginal contributions, second-order symmetry, second-order marginality
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1. Introduction

A cooperative game with transferable utility for a finite player set (TU game or simply game) is given by a coalition function that assigns a worth to any coalition (subset of the player set), where the empty coalition obtains zero. One-point solutions for TU games assign a payoff to any player in any TU game. The Shapley value (Shapley, 1953) probably is the most eminent one-point solution concept for TU games. Besides its original axiomatic foundation by Shapley himself, alternative foundations of different types have been suggested later on. Important direct axiomatic characterizations are due to Myerson (1980) and Young (1985).

Young (1985) characterizes the Shapley value by three properties of solutions: efficiency, symmetry, and marginality or strong monotonicity. Efficiency: the players' payoffs sum up to the worth generated by the grand coalition. Symmetry: equally productive[1] players

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[1] In this paper, a player's productivity in a game refers to her influence on the generation of worth as expressed by her marginal contributions to coalitions not containing her, that is, the differences between the worth generated after she entered such a coalition and the worth generated before she entered.

obtain the same payoff. Marginality: a player's payoff only depends on her own productivity. Strong monotonicity: whenever a player's productivity in a game weakly increases so does her payoff. Note that strong monotonicity implies marginality. This result indicates that the Shapley value is *the* efficient solution that reflects the players' productivities by their payoffs.

The influence of player i on the productivity of player j can be addressed as player i's *second-order productivity* with respect to player j. It can by expressed by player i's *second-order marginal contributions* with respect to player j, that is, by the difference of player j's marginal contribution to coalitions containing player i and her marginal contribution to coalitions without player i. Further, the influence of player i on the payoff of player j for some solution can be expressed by player i's *second-order payoff* with respect to player j for this solution, that is, the difference of the payoff of player j in the original game and payoff of player j after player i left this game.

In this paper, we discuss efficient solutions that reflect the players' second-order productivities as expressed by their second-order marginal contributionsin terms of their second-order payoffs. To this end, we suggest second-order versions of symmetry, marginality, and strong monotonicity. Second-order symmetry: players who are equally second-order productive with respect to a third player obtain the same second-order payoff with respect to the third player. Second-order marginality: a player's second-order payoff with respect to another player only depends on her own second-order productivity with respect to this other player. Second-order strong monotonicity: whenever a player's second-order productivity with respect to another player weakly increases so does her second-order payoff with respect to this other player. Note that second-order strong monotonicity implies second-order marginality.

As our main result, we show that the Shapley value is the unique solution that satisfies efficiency, second-order symmetry, and second-order marginality or second-order strong monotonicity (Theorem 10). The proof of this result uses the somewhat surprising fact that second-order marginality implies (first-order) marginality (Proposition 6). Since (first-order) symmetry and second-order symmetry do not imply each other (Remarks 3 and 4), our main result is not just a corollary to Young (1985).

The remainder of this paper is organized as follows. In Section 2, we provide basic definitions and notation. In Section 3, we briefly discuss the characterization of the Shapley value by Young (1985). In Section 4, we provide our second-order approach. Some remarks conclude the paper.

2. Basic definitions and notation

Let the universe of players $\mathfrak{U}$ be a countably infinite set, and let $\mathcal{N}$ denote the set of all finite subsets of $\mathfrak{U}$. The cardinalities of $S, T, N \in \mathcal{N}$ are denoted by s, t, and n, respectively. A (finite TU) game for the player set $N \in \mathcal{N}$ is given by a **coalition function** $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**; $v(S)$ is called the worth of coalition S. The set of all games for N is denoted by $\mathbb{V}(N)$; the set of all games is denoted by $\mathbb{V} := \bigcup_{N \in \mathcal{N}} \mathbb{V}(N)$.

For $N \in \mathcal{N}$, $T \subseteq N$, and $v \in \mathbb{V}(N)$, the **subgame** $v|_T \in \mathbb{V}(T)$ is given by $v|_T(S) = v(S)$ for all $S \subseteq T$; for $i \in N$ and $S \subseteq N$, we occasionally write v_{-i} and v_{-S} instead of $v|_{N\setminus\{i\}}$ and $v|_{N\setminus S}$, respectively. The game $\mathbf{0}^N \in \mathbb{V}(N)$ given by $\mathbf{0}^N(S) = 0$ for all $S \subseteq N$ is called the **null game** for N. For $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $\alpha \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}(N)$ and $\alpha \cdot v \in \mathbb{V}(N)$ are given by $(v+w)(S) = v(S) + w(S)$ and $(\alpha \cdot v)(S) = \alpha \cdot v(S)$ for all $S \subseteq N$. For $T \subseteq N$, $T \neq \emptyset$, the game $u_T^N \in \mathbb{V}$ given by $u_T^N(S) = 1$ if $T \subseteq S$ and $u_T^N(S) = 0$ otherwise is called a **unanimity game**. Any $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$ can be uniquely represented by unanimity games. In particular, we have

$$v = \sum_{T \subseteq N : T \neq \emptyset} \lambda_T(v) \cdot u_T^N, \tag{1}$$

where the coefficients $\lambda_T(v)$ are known as the Harsanyi dividends (Harsanyi, 1959) and can be determined recursively by

$$\lambda_T(v) := v(T) - \sum_{S \subsetneq T : S \neq \emptyset} \lambda_S(v). \tag{2}$$

Player $i \in N$ is called a **null player** in $v \in \mathbb{V}(N)$ if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$; players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}(N)$ if $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\}$.

A **solution** for $\mathbb{V}$ is an operator that assigns to any $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$ a payoff $\varphi_i(v)$. The **Shapley value** (solution) (Shapley, 1953) for $\mathbb{V}$, Sh, is given by

$$\mathrm{Sh}_i(v) := \sum_{T \subseteq N : i \in T} \frac{\lambda_T(v)}{t} = \sum_{S \subseteq N \setminus \{i\}} \frac{v(S \cup \{i\}) - v\{S\}}{n \cdot \binom{n-1}{s}} \tag{3}$$

for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i \in N$.

3. Marginal contributions and the Shapley value

The marginal contributions of a player $i \in N$, $N \in \mathcal{N}$ in the game $v \in \mathbb{V}(N)$ given as

$$v(S \cup \{i\}) - v(S), \qquad S \subseteq N \setminus \{i\} \tag{4}$$

indicate her (individual) productivity or contribution to the generation of worth in the game v. These describe how the worth of a coalition not containing her changes when she enters this coalition. The following fact on the relation between marginal contributions and Harsanyi dividends is well-known and easily follows from the recursive definition of the Harsanyi dividends in (2). For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $i \in N$, and $S \subseteq N \setminus \{i\}$, we have

$$v(S \cup \{i\}) - v(S) = \sum_{T \subseteq S \setminus \{i\}} \lambda_{T \cup \{i\}}(v). \tag{5}$$

The right-hand formula of the Shapley value in (3) indicates that the Shapley value reflects the players' productivities in games expressed by their own marginal contributions. Young (1985) shows that the Shapley value is the unique efficient such solution.

Theorem 1 (Young, 1985). *The Shapley value is the unique solution for* $\mathbb{V}$ *that satisfies efficiency (**E**), symmetry (**S**), and marginality (**M**) or strong monotonicity (**Mo**).*[2]

Efficiency, E. For all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$, we have $\sum_{\ell \in N} \varphi_\ell(v) = v(N)$.
Symmetry, S. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$ such that i and j are symmetric in v, we have $\varphi_i(v) = \varphi_j(v)$.
Marginality, M. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) = \varphi_i(w)$.
Strong monotonicity, Mo. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

Symmetry can be paraphrased as follows: players who are equally productive in a game should obtain the same payoff. Marginality: a player who is equally productive in two games should obtain the same payoff in these games. Strong monotonicity: whenever a player becomes (weakly) more productive, her payoff should (weakly) increase. Therefore, a solution that is intended to reflect the players' productivities should satisfy these properties. Note that strong monotonicity implies marginality.

4. Second-order productivities, second-order payoffs, and the Shapley value

In the previous section, we have argued that the Shapley value is the unique efficient solution the payoffs of which reflect the players' productivities. In this section, we show that the Shapley value is the unique efficient solution the second-order payoffs of which reflect the players' second-order productivities. First, we operationalize the notions of second-order productivities and second-order payoffs. Second, we suggest and explore second-order versions of symmetry, marginality, and strong monotonicity. Finally, we show that the Shapley value is the unique efficient solution that satisfies these properties.

4.1. Second-order productivities and second-order payoffs

The contribution of a player $i \in N$, $N \in \mathcal{N}$ to the marginal contributions of another player $j \in N \setminus \{i\}$ in a game $v \in \mathbb{V}(N)$ can be conceptualized as her **second-order marginal contributions** to player j given as

$$[v(S \cup \{i, j\}) - v(S \cup \{i\})] - [v(S \cup \{j\}) - v(S)], \qquad S \subseteq N \setminus \{i, j\}. \tag{6}$$

These describe how the marginal contribution of player j to a coalition not containing both of them changes when player i enters this coalition and can be addressed as player i's **second-order productivity** with respect to player j. In view of (5), second-order

[2] Originally, Young (1985) invokes anonymity (called symmetry by him) instead of symmetry (in our parlance). Although anonymity is stronger than symmetry, it is well-known and easy to check that anonymity can be replaced with symmetry in his characterization. Moreover, his characterization works on fixed player sets.

marginal contributions also can be expressed in terms of Harsanyi dividends: for all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, $i, j \in N$, $i \neq j$ and $S \subseteq N \setminus \{i, j\}$, we have

$$[v(S \cup \{i,j\}) - v(S \cup \{j\})] - [v(S \cup \{i\}) - v(S)] = \sum_{T \subseteq S \setminus \{i,j\}} \lambda_{T \cup \{i,j\}}(v). \tag{7}$$

For a given solution φ, the contribution of a player $i \in N$, $N \in \mathcal{N}$ to the payoff of another player $j \in N \setminus \{i\}$ in a game $v \in \mathbb{V}(N)$ can be conceptualized as the difference between the payoff of j in this game v and her payoff in the game without player i,

$$\varphi_j(v) - \varphi_j(v_{-i}).$$

This difference can be addressed as player i's **second-order payoff** with respect to player j. Depending on its sign, player j would be willing to pay this amount to player i for entering or leaving the game.[3]

Note that the roles of players i and j can be interchanged in (6) and (7). That is, in *any* game $v \in \mathbb{V}(N)$, $N \in \mathcal{N}$ and for *any* two players $i, j \in N$, player i's second-order marginal contributions with respect to player j equal player j's second-order marginal contributions with respect to player i. Therefore, second-order contributions reflect the relation between *a pair of players* with respect to the generation of worth in a game. In contrast, (first order) marginal contributions reflect *an individual players'* impact on the generation of worth in a game. Properties based on marginal contributions as symmetry, marginality, or strong monotonicity draw much of their appeal from this fact. In particular, they establish relations between the players' productivity and their payoffs. In order to adjust to the relational nature of second-order marginal contributions, in the following, we shift attention from the players' (first-order) payoffs to their second-order payoffs, which also are relational in some sense.

4.2. Second-order symmetry

In analogy to symmetric players, we first introduce the notion of second-order symmetric players. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j, k \in N$, $i \neq j \neq k \neq i$, players i and j are called **second-order symmetric with respect to player** k if

$$\begin{aligned} &[v(T \cup \{i,k\}) - v(T \cup \{i\})] - [v(T \cup \{k\}) - v(T)] \\ &\quad = [v(T \cup \{j,k\}) - v(T \cup \{j\})] - [v(T \cup \{k\}) - v(T)] \end{aligned}$$

for all $T \subseteq N \setminus \{i, j, k\}$, that is, if players i and j are equally second-order productive with respect to player k.

If players i and j are second-order symmetric with respect to player k, then player i's and player j's second order-productivity with respect to player k as expressed by their second-order marginal contributions are equal. Therefore, if a solution is intended to reflect the

[3] Second-order (and higher-order) payoffs actually have been introduced by Casajus and Huettner (2018, Definition 9) as higher-order contributions.

players' second-order productivities (see Footnote 1), it seems to be plausible that their second-order payoffs with respect to player k are equal too.

Second-order symmetry, 2S. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$ and $i, j, k \in N$, $i \neq j \neq k \neq i$ such that players i and j are second-order symmetric with respect to player k, we have

$$\varphi_k(v) - \varphi_k(v_{-i}) = \varphi_k(v) - \varphi_k(v_{-j}).$$

Proposition 2. *The Shapley value for $\mathbb{V}$ satisfies second-order symmetry (**2S**).*

Proof. For $N \in \mathcal{N}$, $i, j, k \in N$, $i \neq j \neq k \neq i$, and $v \in \mathbb{V}(N)$, let i and j be second-order symmetric with respect to k in v. Using (6), one easily shows by induction on $|T|$ that

$$\lambda_{T \cup \{i,k\}}(v) = \lambda_{T \cup \{j,k\}}(v) \qquad \text{for all } T \subseteq N \setminus \{i, j, k\}. \tag{8}$$

Moreover, we obtain

$$\mathrm{Sh}_k(v) - \mathrm{Sh}_k(v_{-i}) \overset{(3)}{=} \sum_{T \subseteq N \setminus \{i,j,k\}} \left[\frac{\lambda_{T \cup \{i,k\}}(v)}{t+2} + \frac{\lambda_{T \cup \{i,j,k\}}(v)}{t+3} \right].$$

Analogously for j. In view of (8), this proves the claim. □

We conclude this subsection with a comparison of symmetry and second-order symmetry. As the next two remarks show, the two properties do not imply each other.

Remark 3. *Second-order symmetry does not imply symmetry. For $\beta : \mathfrak{U} \to \mathbb{R}$, let the solution Sh^β be given by*

$$\mathrm{Sh}_i^\beta(v) = \mathrm{Sh}_i(v) + \beta(i) \cdot \lambda_{\{i\}}(v) \qquad \textit{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N),\ \textit{and } i \in N.$$

By construction, these solutions inherit second-order symmetry from the Shapley value. However, they fail symmetry whenever β is not constant.

Remark 4. *Symmetry does not imply second-order symmetry. For $\alpha : \mathcal{N} \to \mathbb{R}$, let the solution Sh^α be given by*

$$\mathrm{Sh}_i^\alpha(v) := \alpha(N) \cdot \mathrm{Sh}_i(v) \qquad \textit{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N),\ \textit{and } i \in N. \tag{9}$$

By construction, these solutions inherit symmetry from the Shapley value.

Let $i, j, k \in \mathfrak{U}$, $i \neq j \neq k \neq i$. By (7), i, j, k, and $u_{\{k\}}^{\{i,j,k\}}$ satisfy the hypothesis of second-order symmetry. Yet, we have

$$\mathrm{Sh}_k^\alpha\left(\left(u_{\{k\}}^{\{i,j,k\}}\right)_{-i}\right) \overset{(9),(3)}{=} \mathrm{Sh}_k^\alpha\left(u_{\{k\}}^{\{j,k\}}\right) = \alpha(\{j,k\})$$

and

$$\mathrm{Sh}_k^\alpha\left(\left(u_{\{k\}}^{\{i,j,k\}}\right)_{-j}\right) \overset{(9),(3)}{=} \mathrm{Sh}_k^\alpha\left(u_{\{k\}}^{\{i,k\}}\right) = \alpha(\{i,k\}).$$

Hence, we can choose α such that $\alpha(\{i,k\}) \neq \alpha(\{j,k\})$ and obtain a solution Sh^α that fails second-order symmetry.

4.3. Second-order marginality and second-order strong monotonicity

In analogy to marginality and strong monotonicity, we first suggest second-order versions of these properties.

Second-order marginality, 2M. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$ such that

$$\begin{aligned}&[v(T \cup \{i,j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)] \\ &\quad = [w(T \cup \{i,j\}) - w(T \cup \{i\})] - [w(T \cup \{j\}) - w(T)]\end{aligned}$$

for all $T \subseteq N \setminus \{i,j\}$, we have

$$\varphi_j(v) - \varphi_j(v_{-i}) = \varphi_j(w) - \varphi_j(w_{-i}).$$

If player i and the games v and w satisfy the hypothesis of second-order marginality, then player i's second-order productivity with respect to player j as expressed by her second-order marginal contributions is the same in both games. Therefore, if a solution is intended to reflect the players' second-order productivities (see Footnote 1), it seems to be plausible that her second-order payoffs with respect to player j are equal too.

Second-order strong monotonicity, 2Mo. For all $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$ and $i, j \in N$, $i \neq j$ such that

$$\begin{aligned}&[v(T \cup \{i,j\}) - v(T \cup \{i\})] - [v(T \cup \{j\}) - v(T)] \\ &\quad \geq [w(T \cup \{i,j\}) - w(T \cup \{i\})] - [w(T \cup \{j\}) - w(T)]\end{aligned}$$

for all $T \subseteq N \setminus \{i,j\}$, we have

$$\varphi_j(v) - \varphi_j(v_{-i}) \geq \varphi_j(w) - \varphi_j(w_{-i}).$$

If player i and the games v and w satisfy the hypothesis of second-order strong monotonicity, then player i's second-order productivity as expressed by her second-order marginal contributions with respect to player j is weakly greater in v than in w. Therefore, if a solution is intended to reflect the players' second-order productivities (see Footnote 1), it seems to be plausible that her second-order payoff with respect to player j is weakly greater in v than in w. Note that second-order strong monotonicity implies second-order marginality.

Proposition 5. *The Shapley value for $\mathbb{V}$ satisfies second-order marginality (**2M**) and second-order strong monotonicity (**2Mo**).*

Proof. For $N \in \mathcal{N}$, $i, j \in N$, $i \neq j$, and $v \in \mathbb{V}(N)$, we obtain

$$\begin{aligned}
\mathrm{Sh}_j(v) - \mathrm{Sh}_j(v_{-i}) &\overset{(3)}{=} \sum_{S \subseteq N \setminus \{j\}} \frac{v(S \cup \{j\}) - v\{S\}}{n \cdot \binom{n-1}{s}} - \sum_{S \subseteq N \setminus \{i,j\}} \frac{v(S \cup \{j\}) - v\{S\}}{(n-1) \cdot \binom{n-2}{s}} \\
&= \sum_{S \subseteq N \setminus \{i,j\}} \left[\frac{v(S \cup \{j\}) - v\{S\}}{n \cdot \binom{n-1}{s}} + \frac{v(S \cup \{i,j\}) - v(S \cup \{i\})}{n \cdot \binom{n-1}{s+1}} \right] \\
&\quad - \sum_{S \subseteq N \setminus \{i,j\}} \frac{v(S \cup \{j\}) - v\{S\}}{(n-1) \cdot \binom{n-2}{s}} \\
&= \sum_{S \subseteq N \setminus \{i,j\}} \frac{[v(S \cup \{i,j\}) - v(S \cup \{i\})] - [v(S \cup \{j\}) - v(S)]}{n \cdot \binom{n-1}{s+1}}
\end{aligned}$$

which shows that Sh satisfies **2M** and **2Mo**. □

Interestingly and perhaps a bit surprisingly, second-order marginality or second-order strong monotonicity imply marginality.

Proposition 6. *If a solution for $\mathbb{V}$ satisfies second-order marginality (**2M**) or second-order strong monotonicity (**2Mo**), then it satisfies marginality (**M**).*

Proof. We know that **2Mo** implies **2M**. Let the solution φ satisfy **2M**. Let further $N \in \mathcal{N}$, $v, w \in \mathbb{V}(N)$, and $i \in N$ satisfy the hypothesis of **M**. If $n = 1$, then $v = w$ and the claim is immediate. Assume now $n > 1$. Our assumptions on v, w, and i imply that v, w, i, and any $j \in N \setminus \{i\}$ satisfy the hypothesis of **2M** not only for v and w themselves but also all their subgames that contain i and at least one other player. Let $N \setminus \{i\} = \{i_1, i_2, \ldots, i_{n-1}\}$. By **2M**, we have

$$\begin{aligned}
\varphi_i(v) - \varphi_i(v_{-i_1}) &= \varphi_i(w) - \varphi_i(w_{-i_1}) \\
\varphi_i(v_{-i_1}) - \varphi_i\left((v_{-i_1})_{-i_2}\right) &= \varphi_i(w_{-i_1}) - \varphi_i\left((w_{-i_1})_{-i_2}\right) \\
&\vdots \\
\varphi_i\left(v|_{\{i,i_{n-1},i_{n-2}\}}\right) - \varphi_i\left(v|_{\{i,i_{n-1}\}}\right) &= \varphi_i\left(w|_{\{i,i_{n-1},i_{n-2}\}}\right) - \varphi_i\left(w|_{\{i,i_{n-1}\}}\right) \\
\varphi_i\left(v|_{\{i,i_{n-1}\}}\right) - \varphi_i\left(v|_{\{i\}}\right) &= \varphi_i\left(w|_{\{i,i_{n-1}\}}\right) - \varphi_i\left(w|_{\{i\}}\right).
\end{aligned}$$

Summing up these equations gives

$$\varphi_i(v) - \varphi_i\left(v|_{\{i\}}\right) = \varphi_i(w) - \varphi_i\left(w|_{\{i\}}\right).$$

Our assumptions on v and w imply $v|_{\{i\}} = w|_{\{i\}}$, which concludes the proof. □

Using Proposition 6, we obtain the following characterization of the Shapley value for $\mathbb{V}$ as a corollary to Theorem 1.

Corollary 7. *The Shapley value is the unique solution for $\mathbb{V}$ that satisfies efficiency (**E**), symmetry (**S**), and second-order marginality (**2M**) or second-order strong monotonicity (**2Mo**).*

This characterization, however, is a bit unbalanced. Whereas second-order marginality and second-order strong monotonicity refer to the players' second-order marginal contributions, symmetry refers to their marginal contributions. In the next subsection, we show that symmetry can be replaced with second-order symmetry in Corollary 7.

We conclude this subsection with a further comparison of marginality and strong monotonicity with their second-order cousins.

Remark 8. *Second-order strong monotonicity does not imply strong monotonicity. Let the solution* $\mathrm{Sh}^{\maltese}$ *be given by*

$$\mathrm{Sh}_i^{\maltese}(v) := \mathrm{Sh}_i(v) - 2 \cdot v(\{i\}) \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N),\ \text{and } i \in N.$$

By construction, this solution inherits second-order strong monotonicity from the Shapley value. Yet, we have

$$\mathrm{Sh}_i^{\maltese}\left(u_{\{i\}}^{\{i\}}\right) = -1 < 0 = \mathrm{Sh}_i^{\maltese}\left(\mathbf{0}^{\{i\}}\right),$$

which contradicts strong monotonicity. Note that the solution $\mathrm{Sh}^{\maltese}$ *fails efficiency. Careful inspection of the proof of Proposition 6 shows that second-order strong monotonicity combined with efficiency (for one-player games), for example, implies strong monotonicity.*

Remark 9. *Marginality or strong monotonicity do not imply second-order marginality. For* $\gamma : \mathbb{N} \to \mathbb{R}_+$*, let the solution* Sh^{γ} *be given by*

$$\mathrm{Sh}_i^{\gamma}(v) := \gamma(|N|) \cdot \mathrm{Sh}_i(v) \qquad \text{for all } N \in \mathcal{N},\ v \in \mathbb{V}(N),\ \text{and } i \in N. \tag{10}$$

By construction, these solutions inherit marginality and strong monotonicity from the Shapley value.

Let $i, j, k \in \mathfrak{U}$*,* $i \neq j \neq k \neq i$ *and* $v, w \in \mathbb{V}(\{i,j,k\})$ *be given by* $v = u_{\{i,j\}}^{\{i,j,k\}} + u_{\{j,k\}}^{\{i,j,k\}}$ *and* $w = u_{\{i,j\}}^{\{i,j,k\}}$*. By (7),* i*,* j*,* v*, and* w *satisfy the hypothesis of second-order marginality. Yet, we have*

$$\mathrm{Sh}_j^{\gamma}(v) - \mathrm{Sh}_j^{\gamma}(v_{-i}) \overset{(10),(3)}{=} \gamma(3) \cdot 1 - \gamma(2) \cdot \frac{1}{2}$$

and

$$\mathrm{Sh}_j^{\gamma}(w) - \mathrm{Sh}_j^{\gamma}(w_{-i}) \overset{(10),(3)}{=} \gamma(3) \cdot \frac{1}{2} - \gamma(2) \cdot 0.$$

Hence, we can choose γ *such that* $\gamma(2) \neq \gamma(3)$ *and obtain a solution* Sh^{γ} *that fails second-order marginality.*

4.4. The Shapley value

In this subsection, we show that the Shapley value is the unique solution that reflects the players' second-order productivities as expressed by their second-order marginal contributions in terms of their second-order payoffs in the following sense. It is the unique solution that satisfies efficiency, second-order symmetry, and second-order marginality or second-order strong monotonicity. The proof of this result uses the somewhat surprising fact that second-order marginality implies (first-order) marginality (Proposition 6). Since (first-order) symmetry and second-order symmetry do not imply each other (Remarks 3 and 4), our main result is not just a corollary to Theorem 1 by Young (1985).

Indeed, symmetry plays a crucial role in the proof of Theorem 1 both in the induction basis and the last part of the induction step. In both places, we have to work much harder using second-order symmetry instead of symmetry. In particular, we have to add a player to the game.

Theorem 10. *The Shapley value is the unique solution for $\mathbb{V}$ that satisfies efficiency (**E**), second-order symmetry (**2S**), and second-order marginality (**2M**) or second-order strong monotonicity (**2Mo**).*

Proof. It is well-known that Sh satisfies **E**. By Propositions 2 and 5, Sh satisfies **2S**, **2M**, and **2Mo**.

For all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$, set

$$\mathcal{C}(v) := \{T \subseteq N \mid T \neq \emptyset,\ \lambda_T(v) \neq \emptyset\} \tag{11}$$

and

$$A(v) := \{i \in N \mid i \in T \text{ for all } T \in \mathcal{C}(v)\}. \tag{12}$$

We know that **2Mo** implies **2M**. Let the solution φ satisfy **E**, **2S**, and **2M**. By Proposition 6, the solution φ satisfies **M**. We show $\varphi = \text{Sh}$ by induction on $|\mathcal{C}(v)|$. If $|N| = 1$, then claim is immediate from (3) and **E**. Let now $|N| > 1$.

Induction basis: Let $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ be such that $|\mathcal{C}(v)| = 0$, i.e., $v = \mathbf{0}^N$. Fix $h \in \mathfrak{U} \setminus N$ and set $M := N \cup \{h\}$. Applying **2S** to $\mathbf{0}^M$ gives

$$\varphi_i\left(\mathbf{0}^{M\setminus\{j\}}\right) = \varphi_i\left(\mathbf{0}^{M\setminus\{k\}}\right) \qquad \text{for all } i, j, k \in M \text{ such that } i \neq j \neq k \neq i. \tag{13}$$

Further, we have

$$\sum_{i \in M\setminus\{k\}} \varphi_i\left(\mathbf{0}^{M\setminus\{k\}}\right) \overset{\mathbf{E}}{=} 0 \qquad \text{for all } k \in M. \tag{14}$$

Equations (13) and (14) give rise to a homogenous system of linear equations involving $n \cdot (n+1)$ unknowns and $n \cdot (n+1)$ linearly independent equations. Hence, this system has exactly one solution, setting all unknowns to 0, and we have $\text{Sh}_i\left(\mathbf{0}^N\right) = 0 = \varphi_i\left(\mathbf{0}^N\right)$ for all $i \in N$.

Induction hypothesis (IH): Let the claim hold for all $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ such that $|\mathcal{C}(v)| \leq t$.

Induction step: Let $N \in \mathcal{N}$ and $v \in \mathbb{V}(N)$ be such that $|\mathcal{C}(v)| = t + 1$. Fix $h \in \mathfrak{U} \setminus N$. For $i \in A(v)$, let $v_{i\to h} \in \mathbb{V}(M \setminus \{i\})$ be given by

$$v_{i\to h} := \sum_{T \in \mathcal{C}(v)} \lambda_T(v) \cdot u^{M\setminus\{i\}}_{(T\setminus\{i\})\cup\{h\}}, \tag{15}$$

that is, one obtains $v_{i\to h}$ from v by renaming i to h. Hence, we have

$$v(N) = v_{i\to h}(M \setminus \{i\}), \tag{16}$$

$|\mathcal{C}(v_{i\to h})| = |\mathcal{C}(v)| = t+1$ and $A(v_{i\to h}) = (A(v) \setminus \{i\}) \cup \{h\}$ for all $i \in A(v)$.

If $i \in N \setminus A(v)$, then there exists some $T_i \in \mathcal{C}(v)$ such that $i \notin T_i$. Hence, we have

$$\varphi_\ell(v) \overset{\mathbf{M}}{=} \varphi_\ell\left(v - \lambda_{T_\ell}(v) \cdot u^N_{T_\ell}\right) \overset{IH}{=} \mathrm{Sh}_\ell\left(v - \lambda_{T_\ell}(v) \cdot u^N_{T_\ell}\right) \overset{\mathbf{M}}{=} \mathrm{Sh}_\ell(v) \qquad \text{for all } \ell \in N \setminus A(v). \tag{17}$$

By (15), we analogously obtain

$$\varphi_\ell(v_{i\to h}) = \mathrm{Sh}_\ell(v_{i\to h}) \qquad \text{for all } i \in A(v) \text{ and } \ell \in N \setminus A(v_{i\to h}). \tag{18}$$

If $|A(v)| = 0$, then we are done. If $|A(v)| = 1$, then (17) and both φ and Sh satisfying **E** imply $\varphi_i(v) = \mathrm{Sh}_i(v)$ for $i \in A(v)$ and we are done.

Let now $|A(v)| > 1$. We obtain

$$\sum_{\ell \in N\setminus A(v_{i\to h})} \varphi_\ell(v_{i\to h}) \overset{(18)}{=} \sum_{\ell \in N\setminus A(v_{i\to h})} \mathrm{Sh}_\ell(v_{i\to h}) \overset{(3),(15)}{=} \sum_{\ell \in N\setminus A(v)} \mathrm{Sh}_\ell(v). \tag{19}$$

Hence, we have

$$\sum_{\ell \in A(v)} \varphi_\ell(v) \overset{\mathbf{E}}{=} v(N) - \sum_{\ell \in N\setminus A(v)} \varphi_\ell(v) \overset{(17)}{=} v(N) - \sum_{\ell \in N\setminus A(v)} \mathrm{Sh}_\ell(v) \tag{20}$$

and

$$\sum_{\ell \in A(v_{i\to h})} \varphi_\ell(v_{i\to h}) \overset{\mathbf{E}}{=} v_{i\to h}(M \setminus \{i\}) - \sum_{\ell \in N\setminus A(v_{i\to h})} \varphi_\ell(v_{i\to h}) \overset{(16),(19)}{=} v(N) - \sum_{\ell \in N\setminus A(v)} \mathrm{Sh}_\ell(v) \tag{21}$$

for all $i \in A(v)$.

Let $w \in \mathbb{V}(M)$ be given by

$$w := \sum_{T \in \mathcal{C}(v)} \lambda_T(v) \cdot u^M_T + \sum_{\ell \in A(v)} \sum_{T \in \mathcal{C}(v)} \lambda_T(v) \cdot u^M_{(T\setminus\{\ell\})\cup\{h\}}. \tag{22}$$

By (15) and (22), we have

$$w_{-h} = v \qquad \text{and} \qquad w_{-i} = v_{i\to h} \qquad \text{for all } i \in A(v) \tag{23}$$

Moreover, equation (22) can be written as

$$w = \sum_{\ell \in A(v)\cup\{h\}} \sum_{T \in \mathcal{C}(v)} \lambda_T(v) \cdot u^M_{(T\cup\{h\})\setminus\{\ell\}},$$

which is "symmetric" with respect to player h and the players in $A(v)$. By (12), we have $A(v) \subseteq T$ for all $T \in \mathcal{C}(v)$. Hence using (11), for all $i, j, k \in A(v) \cup \{h\}$ such that $i \neq j \neq k \neq i$, we obtain

$$\lambda_{T \cup \{i,k\}}(w) = \lambda_{T \cup \{j,k\}}(w) \qquad \text{for all } T \subseteq M \setminus \{i, j, k\}.$$

In view of (7), we obtain

$$\varphi_i(v_{j \to h}) \overset{(23)}{=} \varphi_i(w_{-j}) \overset{\mathbf{2S}}{=} \varphi_i(w_{-k}) \overset{(23)}{=} \varphi_i(v_{k \to h}) \tag{24}$$

for all $i, j, k \in A(v) \cup \{h\}$ such that $i \neq j \neq k \neq i$, where $v_{h \to h}$ stands for v.

Equations (20), (21), and (24) give rise to a system of linear equations involving $|A(v)| \cdot (|A(v)| + 1)$ unknowns and $|A(v)| \cdot (|A(v)| + 1)$ linearly independent equations. Hence, the system has exactly one solution, which is setting all unknowns to

$$\frac{v(N) - \sum_{\ell \in N \setminus A(v)} \mathrm{Sh}_\ell(v)}{|A(v)|}.$$

In particular, we have

$$\varphi_i(v) = \frac{v(N) - \sum_{\ell \in N \setminus A(v)} \mathrm{Sh}_\ell(v)}{|A(v)|} = \mathrm{Sh}_i(v) \qquad \text{for all } i \in A(v),$$

where the second equation drops from the Shapley value satisfying **E** and **S** and any two players in $A(v)$ being symmetric in v. □

5. Concluding remarks

In this paper, we have discussed the notions of second-order productivity and second-order payoffs for TU games. It turned out that the Shapley value is the unique efficient solution that indicates the players' second-order productivities in terms of their second-order payoff. The natural question now arises whether this holds true for higher-order productivities and higher-order payoffs. Whereas the Shapley value satisfies higher-order versions of symmetry, marginality, and strong monotonicity, it is not the unique efficient solution that satisfies the afore-mentioned properties. For details, we refer the reader to Casajus (2020, Appendix A).

In Subsection 4.1, we have noted that player i's second-order marginal contributions with respect to player j equal player j's second-order marginal contributions with respect to player i. Therefore, if a solution is intended to reflect the players' second-order productivities in terms of their second-order payoffs, one find it plausible that their second-order payoffs with respect to each other to coincide in any game. This requirement actually is the balanced contribution property due to Myerson (1980).

Balanced contributions, BC. For all $N \in \mathcal{N}$, $v \in \mathbb{V}(N)$, and $i, j \in N$, $i \neq j$, we have

$$\varphi_i(v) - \varphi_i(v_{-j}) = \varphi_j(v) - \varphi_j(v_{-i}).$$

Myerson (1980) shows that the Shapley value is the unique efficient solution that satisfies the balanced contributions property. This may be viewed as an early indication that the Shapley value is the unique efficient solution that reflects the players' second-order productivities as expressed by their second-order marginal contributions in terms of their second-order payoffs.

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