An efficient value for TU games with a cooperation structure

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Abstract

In this note, we introduce and characterize an efficient value for TU games with a cooperation structure which generalizes the Owen (1977) value for games with a coalition structure but which does not deviate too much from the Myerson (1977) value.

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1. Introduction

Consider the TU game with the player set $N = \{P1, P2, P3, A\}$ and the coalition function given by

$$v(K) = \begin{cases} 1 & , |K \cap \{P1, P2, P3\}| > 1, \\ 0 & , |K \cap \{P1, P2, P3\}| \leq 1, \end{cases} \qquad , K \subseteq N.$$

A is a Null player and the presence of any two of the productive players $P1, P2$, and $P3$ already suffices to produce the worth of 1. Suppose all these players cooperate in order to create the grand coalition's worth of $v(N) = 1$. If the players do not form any coalitions when bargaining on the distribution of $v(N)$, then, for symmetry reasons, one would expect an equal split between the three productive players. Would/should this split change if $P1$ and $P2$ formed a bargaining bloc? What if these players could form this bloc only via the Null player A?

As an answer to questions like the first one, Owen (1977) introduces and axiomatizes an efficient value for games with coalition structure (partition of the player set). Hart and Kurz (1983; 1984) provide alternative axiomatizations and explore stability issues with respect to the Owen value. In our leading example, the Owen value assigns the payoff $\frac{1}{2}$ to $P1$ and to $P2$ while $P3$ and A get nothing. Since the players $P1$ and $P2$ already produce the grand coalition's worth and since they bargain as one person as well as for symmetry reasons, this fits nicely with our intuitions.

Yet, the Owen value may not give an adequate answer to the second type of questions. If $P1$ and $P2$ need A in order to form a bargaining bloc then one could argue that—despite being a Null player—A should obtain a positive payoff. However, adding A to the bloc formed by $P1$ and $P2$ does not affect the Owen payoffs. One reason for this is that coalition structures are too coarse structures. From the coalition $\{P1, P2, A\}$ alone one cannot infer whether A is necessary to connect the productive players $P1$ and $P2$ or not. The necessity of A can be modelled by the undirected graph

$$\underset{\bullet}{P1} \quad — \quad \underset{\bullet}{A} \quad — \quad \underset{\bullet}{P2} \qquad\qquad \underset{\bullet}{P3} \tag{1.1}$$

where $P1$ and $P2$ are connected but only via a chain of links involving A. Of course, this transcends the world of coalition structures and leads into the realm of cooperation structures (undirected graphs).

Generalizing the Shapley (1953) value for TU games and the Aumann and Drèze (1974) value for TU games with a coalition structure, Myerson (1977) introduced a value for TU games with a cooperation structure (henceforth CO-games and CO-value). As an alternative, Meessen (1988) suggests the position value for CO-games which was popularized by Borm, Owen and Tijs (1992). Yet another CO-value has been introduced by Hamiache (1999) which was discussed by Bilbao, Jiménez and López (2006). All these CO-values have in common that they are component efficient. In contrast to efficiency, this corresponds to the interpretation of connected components as productive units. In the following, we focus on the Myerson value as the most eminent one of these CO-values.

Since in our leading example the connected component $\{P1, P2, A\}$ already produces $v(N)$, the Myerson payoffs for the graph in (1.1) actually are efficient, but this is rather accidental. Just increase $v(N)$ by a small amount. Moreover, for the empty graph, the Myerson payoffs vanish due to component efficiency. Hence one would like to have an efficient CO-value which recognizes, for example, the coordinating role of player A in the situation above.

This is what this paper aims at. We introduce and axiomatize a CO-value that generalizes the Owen value to the class of CO-games and which, in a sense, does not deviate too much from the Myerson value. More specific, our CO-value coincides with the Owen value for completely connected components and coincides with the Myerson value for connected graphs. For the graph (1.1) in our leading example, that CO-value assigns the payoffs $\varphi_{P1} = \varphi_{P2} = \frac{5}{12}$, $\varphi_A = \frac{1}{6}$, and $\varphi_{P3} = 0$ which meet our intuitions concerning player A.

The axiomatization involves four axioms. Besides efficiency, we require our CO-value to assign the same payoffs for the complete graph as for the empty graph. Further, merging connected components into single players should not affect the component's payoffs. Finally, we modify the Myerson fairness axiom such that the number of components involved is not affected by removing a link. Yet, the player set involved may shrink.

The plan of this paper is as follows: Basic definitions and notation are given in second section. In the third section, we discuss some axioms related to CO-values. Our CO-value is introduced and axiomatized in the fourth section. The fifth section explores the relation of our CO-value to the Myerson value and to the Owen value as well as consistency properties, and touches stability issues. A few remarks conclude the paper.

2. Basic definitions and notation

In order to avoid set theoretic complications, we assume that there is a large enough set $\mathcal{U}$ that contains the names of the players. A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players $N \subset \mathcal{U}$ and a coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. In general, we consider the set of all TU games, possibly equipped with some additional structure. $v(K)$ is called the worth of $K \subseteq N$; subsets of N are called coalitions. For $\emptyset \neq T \subseteq N$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a unanimity game. The sum $v + v'$ of two coalition functions on N is given by $(v + v')(K) = v(K) + v'(K)$ for all $K \subseteq N$; $v|_{N'}$ denotes the restriction of v to $N' \subseteq N$. A game is called superadditive iff $v(K \cup K') \geq v(K) + v(K')$ for all $K, K' \subseteq N$, $K \cap K' = \emptyset$.

A value is an operator φ that assigns payoff vectors $\varphi(N, v) \in \mathbb{R}^N$ to all games (N, v), $N \subset \mathcal{U}$. An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted by $K_i(\sigma) = \{j \in N : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is defined by

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma) \quad , i \in N. \tag{2.1}$$

For $K \subseteq N$, we denote by $\varphi_K(N, v, \cdot)$ the sum $\sum_{i \in K} \varphi_i(N, v, \cdot)$.

A coalition structure for (N, v) is a partition $\mathcal{P} \subseteq 2^N$ where $\mathcal{P}(i)$ denotes the cell containing player i. We denote by $\langle S \rangle$, $S \subseteq N$ the atomistic partition on S, $\langle S \rangle := \{\{i\} | i \in S\}$. By $\mathcal{P}|_{N'}$ we denote the restriction of the partition $\mathcal{P}$ on N to $N' \subseteq N$, $\mathcal{P}|_{N'} := \{\mathcal{P}(i) \cap N' | i \in N'\}$. A CS-game is a game together with a coalition structure, $(N, v, \mathcal{P})$. A CS-value is an operator φ that assigns payoff vectors $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to all CS-games $(N, v, \mathcal{P})$, $N \subset \mathcal{U}$. For any coalition structure $\mathcal{P}$ on N, we define a subset

$$\Sigma(N, \mathcal{P}) := \{\sigma \in \Sigma(N) | \forall i, j \in \mathcal{P}(i) : |\sigma(i) - \sigma(j)| < |\mathcal{P}(i)|\} \tag{2.2}$$

of $\Sigma(N)$. The Owen (1977) value is given by

$$\text{Ow}_i(N,v,\mathcal{P}) := |\Sigma(N,\mathcal{P})|^{-1} \sum_{\sigma\in\Sigma(N,\mathcal{P})} MC_i^v(\sigma) \quad , i\in N. \tag{2.3}$$

Any $\sigma \in \Sigma(N,\mathcal{P})$ uniquely determines some $\sigma|_{\mathcal{P}} \in \Sigma(\mathcal{P})$ and $\sigma|_P \in \Sigma(P)$, $P \in \mathcal{P}$ such that $\sigma|_{\mathcal{P}}(\mathcal{P}(i)) < \sigma|_{\mathcal{P}}(\mathcal{P}(j))$ iff $\sigma(i) < \sigma(j)$ for all $i,j \in N$ or $\sigma|_P(i) < \sigma|_P(j)$ iff $\sigma(i) < \sigma(j)$ for all $i,j \in P$, respectively. For $\sigma_i \in \Sigma(\mathcal{P}(i))$ and $\rho \in \Sigma(\mathcal{P})$, we set

$$\Sigma(N,\mathcal{P},\sigma_i,\rho) := \left\{\sigma \in \Sigma(N,\mathcal{P}) \,|\sigma|_{\mathcal{P}} = \rho \wedge \sigma|_{\mathcal{P}(i)} = \sigma_i\right\}. \tag{2.4}$$

A cooperation structure for (N,v) is an undirected graph (N,L), $L \subseteq L^N := \{\{i,j\}\,|i,j\in N, i\neq j\}$. A typical element of L is written as ij. Given any graph (N,L), N splits into (maximal connected) components the set of which is denoted by $C(N,L)$; $C_i(N,L) \in C(N,L)$ denotes the component containing $i \in N$. $L|_{N'} = \{\{i,j\} \in L|i,j \in N'\}$ denotes the restriction of L to $N' \subseteq N$. For any partition $\mathcal{P} \subseteq 2^N$, $L^{\mathcal{P}}$ denotes the graph $\bigcup_{P\in\mathcal{P}} L^P$ which splits in the completely connected components in $C\left(N, L^{\mathcal{P}}\right) = \mathcal{P}$.

A CO-game is a game together with a cooperation structure. A CO-value is an operator φ that assigns payoff vectors $\varphi(N,v,L) \in \mathbb{R}^N$ to all CO-games (N,v,L), $N \subset \mathcal{U}$. The Myerson (1977) value μ is defined by

$$\mu(N,v,L) := \text{Sh}\left(N,v^L\right) \qquad , v^L(K) := \sum_{S\in C(K,L|_K)} v(S) \qquad , K\subseteq N. \tag{2.5}$$

3. Axioms for CO-values

In this section, we consider several axioms for CO-values with respect to bargaining within the grand coalition.

Axiom 3.1 (Additivity, A)**.** $\varphi(N,v+v',L) = \varphi(N,v,L) + \varphi(N,v',L)$.

From a mathematical viewpoint, additivity is nice axiom which is satisfied by quite a lot of values for TU games with or without additional structures and which is part of many axiomatizations. Nevertheless, additivity does not reflect any fairness considerations and therefore one may wish to avoid explicit reference to this property.

Axiom 3.2 (Efficiency, E)**.** $\varphi_N(N,v,L) = v(N)$.

We feel that the efficiency axiom presupposes the grand coalition to be the productive unit which creates its worth $v(N)$. This corresponds to the interpretation of the connected components of L as bargaining blocs which are formed via bilateral agreements or communication channels.

Axiom 3.3 (Component efficiency, CE)**.** *For all* $C \in C(N, L)$, *we have*

$$\varphi_C(N, v, L) = v(C).$$

Component efficiency evokes another interpretation of the graph L. In order to cooperate in the production of worth, players have to be connected via a chain of links. Hence, the connected components C of L are the productive units which produce their worth $v(C)$.

Axiom 3.4 (Fairness, F)**.** *For all* $ij \in L$, *we have*

$$\varphi_i(N, v, L) - \varphi_i(N, v, L - ij) = \varphi_j(N, v, L) - \varphi_j(N, v, L - ij).$$

CE and **F** are the original axioms that characterize the Myerson value. The very nice fairness axiom **F** has strong consequences far beyond pure fairness considerations. In particular, van den Nouweland (1993, pp. 28) shows that μ satisfies the following axiom which says that (distribution of) payoffs within a component is not affected by the players outside. In general, of course, **CD** and **E** are incompatible.

Axiom 3.5 (Component decomposability, CD)**.** *For all* $i \in N$,

$$\varphi_i(N, v, L) = \varphi_i\left(C_i(N, L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right).$$

Axiom 3.6 (Equivalence, Q)**.** $\varphi\left(N, v, L^N\right) = \varphi(N, v, \emptyset)$.

This axiom says that—from the bargaining viewpoint—it does not make a difference whether the players do not form any (bargaining) components ($L = \emptyset$) or they form just one such component where all players are completely connected ($L = L^N$). Note that the Owen value has a similar property: The Owen payoffs for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N \rangle$ coincide. We feel that **Q** as a natural generalization of that property should be satisfied by an efficient CO-value.

Axiom 3.7 (Modified fairness, MF)**.** *For all* $ij \in L$,

$$\begin{aligned}\varphi_i(N, v, L) - \varphi_i\left(N_i(L, ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) \\ = \varphi_j(N, v, L) - \varphi_j\left(N_j(L, ij), v|_{N_j(L,ij)} L|_{N_j(L,ij)}\right)\end{aligned}$$

where

$$N_i(L, ij) := N \backslash \left(C_i(N, L) \backslash C_i(N, L - ij)\right). \tag{3.1}$$

MF is intended to replace the fairness axiom **F**. It is trivially satisfied if $C_i(N,L)$ does not split by removing the link ij since then $N_i(L,ij) = N$. Otherwise, $C_i(N,L)$ splits into two disjoint components. In this case, $N_i(L,ij) = N \backslash C_j(N, L - ij)$, i.e. the players in j's component are removed from N. Hence, and this seems to be one important thing about **MF**, all graphs involved have the same number of connected components while the number of players may differ. We feel that this modification of **F** fits nicely with the interpretation of the graph L as a device to model structured bargaining blocs. Compare this with **F**. There, the player set involved is fixed at N but removing a link may increase the number of components. Note also the role of **MF** in the proof of the consistency property of the CO-value to be introduced in Theorem 5.5. Further, compare the player set in (3.1) with those in (5.2) and (5.4).

Axiom 3.8 (Component merging, CM)**.** *For all $C \in C(N,L)$, we have*

$$\varphi_C(N,v,L) = \varphi_C(C(N,L), v \circ \cup, \emptyset)$$

where $v \circ \cup (K) = v\left(\bigcup_{S \in K} S\right)$ for all $K \subseteq C(N,L)$.

CM says the distribution of worth among the components depends only on the game between coalitions, $(C(N,L), v \circ \cup)$, which are completely disconnected. This could be paraphrased as that merging all connected components into single players does not affect the component's payoffs. I.e. the inner structure of the components does not matter in this respect. What matters is just the fact that they are connected. Note that **CM** is very similar to Owen's (1977) axiom A3.

Of course, instead of **CM** one could think of a more graph-related axiom which requires the component's payoffs not to be affected by inflating links, i.e. by merging directly connected players i and j, i.e. $ij \in L$, removing the resulting loop at ij, and identifying parallel links. Yet, this would imply **CM** by successively merging links. The other way round, inflating links is equivalent to **CM** if one assumes invariance under the renaming of players.

4. A generalization of the Owen value

In this section, we show that some of the axioms advocated in the previous section, in particular **E**, **Q**, **MF**, and **CM**, already characterize a CO-value which satisfies the remaining such axioms. Further, the non-redundancy of our axiomatization is established.

4.1. **Uniqueness.** We first consider connected graphs, i.e. all players are contained in one bargaining bloc. In this case, one could argue that the distribution of the grand coalition's worth should be governed by the inner structure of that single bloc and the fairness considerations embodied in the Myerson value. Yet, this is already implied by **E** and **MF**.

Lemma 4.1. *If a CO-value φ satisfies **E** and **MF** then it coincides with μ on all connected graphs.*

Proof. We first note that for connected graphs **MF** involves connected graphs only. μ satisfies **CE** which for connected graphs becomes **E**. We also have

$$\begin{aligned} & \mu_i(N,v,L) - \mu_j(N,v,L) \\ & \quad \overset{\mathbf{CD}}{=} \quad \mu_i\left(C_i(N,L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L), v|_{C_j(N,L)}, L|_{C_j(N,L)}\right) \\ & \quad = \quad \mu_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \mu_j\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \end{aligned}$$

for $ij \in L$ where the second equation follows from $C_i(N,L) = C_i(N_i(L,ij), L|_{N_i(L,ij)})$ and the analogon for j. Hence, μ satisfies **MF**.

We mimic the Myerson (1977) proof of uniqueness. Suppose φ and $\bar{\varphi}$ both satisfy **E** and **MF**. Suppose N is a minimal player set such that φ and $\bar{\varphi}$ differ on a connected graph. Further, suppose L is a minimal connected graph on N such that they do so. By **CE**, L contains at least one edge. If $j \in C_i(N, L-ij)$ then **MF** and the minimality of L imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. And if $j \notin C_i(N, L-ij)$ then again **MF** and the minimality of N imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. Since L is connected, we have $\varphi_i(N,v,L) - \bar{\varphi}_i(N,v,L) = \Delta$ for some Δ and all $i \in N$. **E** then implies $\Delta = 0$. Contradiction. ∎

Applying this Lemma and again the Myerson technique, we are now able to approach the general case.

Theorem 4.2. *There is at most one CO-value that satisfies **E**, **Q**, **MF**, and **CM**.*

In view of their role in the proof below, one could of course merge **Q** and **CM** into a single axiom. However, we feel that the two axioms refer to essentially different considerations. While **Q** basically is as a very weak expression of invariance with respect to the renaming of players, **CM** requires the payoff of the components to be independent of their inner structure.

Proof. Let φ be a CO-value that satisfies **E**, **Q**, **MF**, and **CM**. By **CM** and **Q**, we have

$$\varphi_C(N,v,L) = \varphi_C\left(C(N,L), v \circ \cup, L^{C(N,L)}\right)$$

for all $C \in C(N,L)$. Since $\left(C(N,L), L^{C(N,L)}\right)$ is connected, Lemma 4.1 implies

$$\varphi_C(N,v,L) = \mu_C\left(C(N,L), v \circ \cup, L^{C(N,L)}\right). \tag{4.1}$$

Again, we mimic the Myerson (1977) proof of uniqueness. Suppose there were two CO-values, φ and $\bar{\varphi}$, that satisfy **E**, **Q**, **MF**, and **CM**. Let N be a minimal player set such that $\varphi \neq \bar{\varphi}$ and let L be a minimal graph on N such that they do so. By **E**, N then contains more than one player, and by **Q** and Lemma 4.1, L contains at least one link. If $C_i(N,L) = \{i\}$, then $\varphi_i(N,v,L) = \bar{\varphi}_i(N,v,L)$ by (4.1). For $ij \in L|_{C_i(N,L)}$, we have

$$\begin{aligned}
&\varphi_i(N,v,L) - \varphi_j(N,v,L) \\
&\quad = \varphi_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \varphi_j\left(N_j(L,ij), v|_{N_j(L,ij)} L|_{N_j(L,ij)}\right) \\
&\quad = \bar{\varphi}_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \bar{\varphi}_j\left(N_j(L,ij), v|_{N_j(L,ij)} L|_{N_j(L,ij)}\right) \\
&\quad = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)
\end{aligned}$$

by **MF**, the minimality of N and L, and again **MF**. Thus, we have $\varphi_j(N,v,L) - \bar{\varphi}_j(N,v,L) = \varphi_k(N,v,L) - \bar{\varphi}_k(N,v,L)$ for all $j,k \in C_i(N,L)$. In view of (4.1), this implies $\varphi_j(N,v,L) = \bar{\varphi}_j(N,v,L)$ for all $j \in C_i(N,L)$. A contradiction. ∎

4.2. **Existence.** We show that there exists a CO-value that combines the Owen value (distribution between components) and the Myerson value (distribution within components) which satisfies our set of axioms.

Theorem 4.3. *There is a CO-value that satisfies **E**, **Q**, **MF**, and **CM**.*

Proof. Consider the CO-value φ given by

$$\varphi_i(N,v,L) = |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \mu_i\left(C_i(N,L), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L)}\right) \tag{4.2}$$

where v^{σ}_C is given by

$$v^{\sigma}_C(S) = v\left(S \cup \bigcup_{\substack{C' \in C(N,L): \\ \sigma(C') < \sigma(C)}} C'\right) - v\left(\bigcup_{\substack{C' \in C(N,L): \\ \sigma(C') < \sigma(C)}} C'\right) \qquad, S \subseteq C \tag{4.3}$$

for all $C \in C(N, L)$. We then have

$$\begin{aligned}
\sum_{i \in N} \varphi_i (N, v, L) &= |\Sigma (C (N, L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \sum_{C \in C(N,L)} \mu_C (C, v_C^\sigma, L|_C) \\
&= |\Sigma (C (N, L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \sum_{C \in C(N,L)} v_C^\sigma (C) \\
&= |\Sigma (C (N, L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} v (N) \\
&= v (N)
\end{aligned}$$

by (4.2) and changing the order of summation, by the fact that $(C, L|_C)$ is connected for $C \in C(L, N)$ and that μ is component efficient, and by (4.3). Hence, φ satisfies **E**.

For $L = \emptyset$, we have $C_i(N, L) = \{i\}$, hence $C(N, L) \cong N$ and $\Sigma(C(N, L)) \cong \Sigma(N)$, and therefore

$$\begin{aligned}
\varphi_i (N, v, \emptyset) &= |\Sigma (C (N, L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \mu_i (\{i\}, v_i^\sigma, \emptyset) \\
&= |\Sigma (N)|^{-1} \sum_{\sigma \in \Sigma(N)} v_i^\sigma (i) \\
&= |\Sigma (N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v (\sigma) \\
&= \mathrm{Sh}_i (N, v)
\end{aligned}$$

by (4.2), again by the fact that $(\{i\}, \emptyset)$ is connected and that μ is component efficient, by (4.3) and the definition of $MC_i^v(\sigma)$, and by (2.1). Further, by (4.2) and (4.3), we have $\varphi\left(N, v, L^N\right) = \mu\left(N, v, L^N\right)$ since $\left(N, L^N\right)$ is connected. Since $\mu\left(N, v, L^N\right) = \mathrm{Sh}(N, v)$ (Myerson, 1977), φ also satisfies **Q**.

Next, we show that φ satisfies **CM**. In view of (4.2), it suffices to show that

$$\mu_C (C, v_C^\sigma, L|_C) = \mu_{\{C\}} \left(\{C\}, (v \circ \cup)_{\{C\}}^\sigma, \emptyset\right)$$

for all $C \in C(N, L)$ and $\sigma \in \Sigma(C(N, L))$. Since μ is component efficient and $(C, L|_C)$ as well as $(\{C\}, \emptyset)$ are connected for $C \in C(N, L)$ in general, this is equivalent to $v_C^\sigma(C) = (v \circ \cup)_{\{C\}}^\sigma(\{C\})$ which holds by (4.3).

Finally, we show that φ satisfies **MF**. In view of (4.2) and (4.3), it suffices to show that

$$\mu_i\left(C_i(N,L), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_i\left(C_i(N,L-ij), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L-ij)}\right)$$
$$= \mu_j\left(C_j(N,L), v^{\sigma}_{C_j(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L-ij), v^{\sigma}_{C_j(N,L)}, L|_{C_i(N,L-ij)}\right)$$

holds for all $ij \in L$. Yet, this follows from μ satisfying **F** and **CD**. ∎

Below, we show that the value defined by (4.2) and (4.3) is a generalization of the Owen value. This may justify the notation $\mathrm{Ow}^{\sharp}$ where the musical "sharp" symbol $\sharp$ is intended to indicate a graph.

4.3. **Non-redundancy.** Next, we show that our axiomatization is non-redundant. Since by Theorem 4.2 and 4.3 $\mathrm{Ow}^{\sharp}$ is characterized by **E**, **Q**, **MF**, and **CM**, it suffices to show that there are CO-values that are different from $\mathrm{Ow}^{\sharp}$ but satisfy any three of theses axioms. The CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by $\varphi_i(N,v,L) = 0$ for all $i \in N$ satisfies **MF**, **CM**, and **Q**. From our leading example it is clear that $\mathrm{Ow}^{\sharp}$ and Ow applied to the coalition structure $C(N,L)$ do not coincide. Yet, the latter satisfies **E**, **CM**, and **Q**. Also, the CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by $\varphi_i(N,v,L)$ $= |N|^{-1} v(N)$ for all $i \in N$ satisfies **E**, **MF**, and **Q**. Finally, consider the CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by

$$\varphi_i(N,v,L) = \mu_i(N,v,L) + \frac{v(N) - \sum_{C\in C(N,L)} v(C)}{|C(N,L)|\,|C_i(N,L)|}. \tag{4.4}$$

Since μ satisfies **CE**, we have

$$\varphi_{C_i(N,L)}(N,v,L) = v(C_i(N,L)) + \frac{v(N) - \sum_{C\in C(N,L)} v(C)}{|C(N,L)|}, \tag{4.5}$$

i.e. $\varphi_{C_i(N,L)}(N,v,L)$ depends only on the worth of the components in $C(N,L)$ and $|C(N,L)|$ which are not affected by considering components as players. Hence, φ satisfies **CM**. Summing up (4.5) over $C(N,L)$ then shows $\varphi_N(N,v,L) = v(N)$, i.e. φ satisfies **E**. Further, we have

$$\varphi_i(N,v,L) - \varphi_j(N,v,L)$$

$$\begin{aligned}
&\overset{(4.4)}{=} \mu_i(N,v,L) - \mu_j(N,v,L) \\
&\overset{\mu,\mathbf{CD}}{=} \mu_i\left(C_i(N,L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L), v|_{C_j(N,L)}, L|_{C_j(N,L)}\right) \\
&\overset{\mu,\mathbf{F}}{=} \mu_i\left(C_i(N,L-ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad\qquad -\mu_j\left(C_j(N,L-ij), v|_{C_j(N,L-ij)}, L|_{C_j(N,L-ij)}\right) \\
&\overset{\mu,\mathbf{CD}}{=} \mu_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \mu_j\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \\
&\overset{(4.4)}{=} \varphi_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \varphi_j\left(N_j(L,ij), v|_{N_j(L,ij)} L|_{N_j(L,ij)}\right)
\end{aligned}$$

which finally shows that φ also satisfies **MF**.

4.4. **An example.** Concluding this section, we reconsider our leading example with the graph in (1.1). There are two orders, σ and ρ, on $C(N,L) = \{C, \{P3\}\}$, $C = \{P1, P2, A\}$ where $\sigma(C) = 1$ and $\rho(C) = 2$. By (4.3), this gives the payoff functions $v_C^\sigma(S) = v(S)$, $S \subseteq C$ and

$$v_C^\rho(S) = v(S \cup \{P3\}) - v(\{P3\}) = \begin{cases} 1 & , |S \cap \{P1, P2\}| > 0 \\ 0 & , |S \cap \{P1, P2\}| = 0 \end{cases} \qquad S \subseteq C.$$

Straightforward calculations in accordance with (2.5) yield the Myerson payoffs

$$\mu_{(P1,P2,A)}(C, v_C^\sigma, L|_C) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \quad \text{and} \quad \mu_{(P1,P2,A)}(C, v_C^\rho, L|_C) = \left(\frac{1}{2}, \frac{1}{2}, 0\right).$$

By (4.2), this gives the payoffs

$$\mathrm{Ow}^\sharp_{(P1,P2,P3,A)}(N,v,L) = \left(\frac{5}{12}, \frac{5}{12}, 0, \frac{1}{6}\right)$$

as in the Introduction where the payoff for $P3$ is immediate from **E**.

5. Properties

5.1. **Relation to the Myerson value and to the Owen value.** From (4.2) and (4.3) it is easy to see that $\mathrm{Ow}^\sharp$ and μ coincide on connected graphs and that $\mathrm{Ow}^\sharp$ inherits additivity from the Myerson value in general. The axiomatizations for the Owen value of Owen (1977) itself as well as those of Hart and Kurz (1983) involve the additivity axiom. Khmelnitskaya and Yanovskaya (2006) characterize the Owen value without additivity by employing the Young (1985) marginality axiom. Vázquez-Brage, García-Jurado and Carreras (1996) suggest a generalization of both the Owen and the Myerson value the axiomatization of which also does not involve the additivity axiom. However, their value refers to TU games with both a coalition structure and a cooperation structure. For the complete graph, this value coincides with the Owen value, but if the coalition structure equals the set of the graph's components then the Myerson value results. Hence, that value and our value are essentially different. Yet, in view of the following Theorem, our value indeed extends the Owen value to CO-games and therefore provides another justification of the Owen value without the additivity axiom.

Theorem 5.1. $\mathrm{Ow}^{\sharp}\left(N, v, L^{\mathcal{P}}\right) = \mathrm{Ow}\left(N, v, \mathcal{P}\right).$

Proof. Since $C_i\left(N, L^{\mathcal{P}}\right) = \mathcal{P}\left(i\right)$, we have

$$
\begin{aligned}
&\mathrm{Ow}_i^{\sharp}\left(N, v, L^{\mathcal{P}}\right) \\
&\overset{(4.2)}{=} \quad \left|\Sigma\left(\mathcal{P}\right)\right|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \mu_i\left(\mathcal{P}\left(i\right), v^{\rho}_{\mathcal{P}(i)}, L^{\mathcal{P}(i)}\right) \\
&\overset{(5.2)}{=} \quad \left|\Sigma\left(\mathcal{P}\right)\right|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \mathrm{Sh}_i\left(\mathcal{P}\left(i\right), v^{\rho}_{\mathcal{P}(i)}, L^{\mathcal{P}(i)}\right) \\
&\overset{(2.1)}{=} \quad \left|\Sigma\left(\mathcal{P}\right)\right|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \left|\Sigma\left(\mathcal{P}\left(i\right)\right)\right|^{-1} \sum_{\sigma_i \in \Sigma(\mathcal{P}(i))} MC_i^{v^{\rho}_{\mathcal{P}(i)}}\left(\sigma_i\right) \\
&\overset{(4.3),(2.4)}{=} \quad \left|\Sigma\left(\mathcal{P}\right)\right|^{-1} \sum_{\rho \in \Sigma(\mathcal{P})} \left|\Sigma\left(\mathcal{P}\left(i\right)\right)\right|^{-1} \sum_{\sigma_i \in \Sigma(\mathcal{P}(i))} \left|\Sigma\left(N, \mathcal{P}, \sigma_i, \rho\right)\right|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P}, \sigma_i, \rho)} MC_i^{v}\left(\sigma\right) \\
&\overset{(2.4)}{=} \quad \left|\Sigma\left(N, \mathcal{P}\right)\right|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^{v}\left(\sigma\right) \\
&\overset{(2.3)}{=} \quad \mathrm{Ow}_i\left(N, v, \mathcal{P}\right)
\end{aligned}
$$

for all $i \in N$. ∎

Since the Owen value and the Shapley value coincide for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N \rangle$ the following property is immediate.

Corollary 5.2. $\mathrm{Ow}^{\sharp}\left(N, v, L^{N}\right) = \mathrm{Ow}^{\sharp}\left(N, v, \emptyset\right) = \mathrm{Sh}\left(N, v\right).$

Finally, **CM** and **Q** then imply that the distribution of the grand coalition's worth between components is governed by the same principles for $\mathrm{Ow}^{\sharp}$ and Ow.

Corollary 5.3. *For all* $C \in C\left(N, L\right)$, $\mathrm{Ow}^{\sharp}_C\left(N, v, L\right) = \mathrm{Ow}_C\left(N, v, C\left(N, L\right)\right).$

5.2. **Consistency.** Owen (1977) shows that for Ow the distribution of worth between coalitions and within coalitions is governed by the same principles. In particular, he shows that his value satisfies the following consistency property:

Theorem 5.4 (Owen 1977)**.** *For all $i \in N$, we have*

$$\mathrm{Ow}_i(N, v, \mathcal{P}) = \mathrm{Ow}_i\left(\mathcal{P}(i), v^{N,\mathcal{P}}_{\mathcal{P}(i)}, \{\mathcal{P}(i)\}\right) = \mathrm{Ow}_i\left(\mathcal{P}(i), v^{N,\mathcal{P}}_{\mathcal{P}(i)}, \emptyset\right) \tag{5.1}$$

where the coalition function $v^{N,\mathcal{P}}_P$ on $P \in \mathcal{P}$ is defined by

$$v^{N,\mathcal{P}}_P(S) := \mathrm{Ow}^\sharp_S\left(N\backslash(P\backslash S), v|_{N\backslash(P\backslash S)}, \mathcal{P}|_{N\backslash(P\backslash S)}\right) \qquad , S \subseteq P. \tag{5.2}$$

$\mathrm{Ow}^\sharp$ satisfies a similar consistency property. In view of $\{\mathcal{P}(i)\} = \mathcal{P}|_{\mathcal{P}(i)}$, the following Theorem is the obvious analogon to Theorem 5.4. Since the components of $\mathcal{P}$ have no inner structure, however, there is no such analogon to the second equation in (5.1).

Theorem 5.5. *We have* $\mathrm{Ow}^\sharp = \mathrm{Ow}^\#$ *where the CO-value* $\mathrm{Ow}^\#$ *is defined by*

$$\mathrm{Ow}^\#_i(N, v, L) = \mathrm{Ow}^\sharp_i\left(C_i(N, L), v^{N,L}_{C_i(N,L)}, L|_{C_i(N,L)}\right) \qquad , i \in N \tag{5.3}$$

where the coalition functions $v^{N,L}_C$ on $C \in C(N, L)$ are defined by

$$v^{N,L}_C(S) := \mathrm{Ow}^\sharp_S\left(N\backslash(C\backslash S), v|_{N\backslash(C\backslash S)}, L|_{N\backslash(C\backslash S)}\right) \qquad , S \subseteq C. \tag{5.4}$$

Proof. By Theorems 4.2 and 4.3, it suffices to show that $\mathrm{Ow}^\#$ satisfies **E**, **Q**, **MF**, and **CM**. Since $\mathrm{Ow}^\sharp$ satisfies **E** and by (5.3) and (5.4), we have $\mathrm{Ow}^\#_C(N, v, L) = \mathrm{Ow}^\sharp_C(N, v, L)$ for all $C \in C(N, L)$. Therefore, $\mathrm{Ow}^\#$ inherits **E** and **CM** from $\mathrm{Ow}^\sharp$.

In order to show **Q**, we prove $\mathrm{Ow}^\#(N, v, \emptyset) = \mathrm{Sh}(N, v) = \mathrm{Ow}^\#\left(N, v, L^N\right)$. The first equation follows from

$$\begin{aligned}\mathrm{Ow}^\#_i(N, v, \emptyset) &\overset{(5.3)}{=} \mathrm{Ow}^\sharp_i\left(\{i\}, v^{N,L}_{\{i\}}, \emptyset\right) \overset{(5.4)}{=} v^{N,L}_{\{i\}}(\{i\}) \\ &\overset{(5.4)}{=} \mathrm{Ow}^\sharp_i(N, v, \emptyset) \overset{\text{Thm. 5.1}}{=} \mathrm{Ow}_i(N, v, \langle N\rangle) \overset{\text{Cor. 5.2}}{=} \mathrm{Sh}_i(N, v).\end{aligned}$$

By (5.3), Theorem 5.1, and (5.4), we have $v^{N,\{N\}}_N(S) = v^{N,L^N}_N(S)$ for all $S \subseteq N$. Since

$$\mathrm{Ow}^\#_i\left(N, v, L^N\right) \overset{(5.3)}{=} \mathrm{Ow}^\sharp_i\left(N, v^{N,L^N}_N, L^N\right) \overset{\text{Thm. 5.1}}{=} \mathrm{Ow}_i\left(N, v^{N,L^N}_N, \{N\}\right),$$

Theorem 5.4 implies $\mathrm{Ow}^\#_i\left(N, v, L^N\right) = \mathrm{Ow}_i(N, v, \{N\})$. Hence, $\mathrm{Ow}^\#\left(N, v, L^N\right) = \mathrm{Sh}(N, v)$ by Theorem 5.1 and Corollary 5.2.

Let now $ij \in L$ and $C := C_i(N, L)$. We then have

$$
\begin{aligned}
&\mathrm{Ow}_i^{\#}(N, v, L) - \mathrm{Ow}_j^{\#}(N, v, L) \\
&\quad \overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(C, v_C^{N,L}, L|_C\right) - \mathrm{Ow}_j^{\sharp}\left(C, v_C^{N,L}, L|_C\right) \\
&\quad \overset{\mathbf{MF}}{=} \mathrm{Ow}_i^{\sharp}\left(C_i(N, L-ij), v_C^{N,L}|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad\qquad - \mathrm{Ow}_j^{\sharp}\left(C_j(N, L-ij), v_C^{N,L}|_{C_j(N,L-ij)}, L|_{C_j(N,L-ij)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathrm{Ow}_i^{\#}\left(N_i(L, ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \mathrm{Ow}_j^{\#}\left(N_j(L, ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \\
&\quad \overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(C_i(N, L-ij), v_{C_i(N,L-ij)}^{N_i(L,ij), L|_{N_i(L,ij)}}, L|_{C_i(N,L-ij)}\right) \\
&\qquad\qquad - \mathrm{Ow}_j^{\sharp}\left(C_j(N, L-ij), v_{C_j(N,L-ij)}^{N_j(L,ij), L|_{N_i(L,ij)}}, L|_{C_j(N,L-ij)}\right).
\end{aligned}
$$

Since $C_i(N, L-ij) \subseteq C$ and by (5.4), we also have

$$
v_C^{N,L}|_{C_i(N,L-ij)}(S) = \mathrm{Ow}_S^{\sharp}\left(N \backslash (C \backslash S), v|_{N \backslash (C \backslash S)}, L|_{N \backslash (C \backslash S)}\right) = v_{C_i(N,L-ij)}^{N_i(L,ij), L|_{N_i(L,ij)}}(S)
$$

for all $S \subseteq C_i(N, L-ij)$, analogously for j. Hence, $\mathrm{Ow}^{\#}$ satisfies **MF**. ∎

In addition, Hart and Kurz (1983) show that for Ow the distribution of worth between coalitions is consistent with the distribution within coalitions in the following sense.

Theorem 5.6 (Hart and Kurz 1983)**.** *Theorem 5.4 remains true if we replace the coalition function $v_P^{N,\mathcal{P}}$, $P \in \mathcal{P}$ by either of the following ones: For all $S \subseteq P$*

$$
{}^{(1)}v_P^{N,\mathcal{P}} := \mathrm{Ow}_S(N, v, (\mathcal{P} \backslash \{P\}) \cup \{S, P \backslash S\}) \tag{5.5}
$$

$$
{}^{(2)}v_P^{N,\mathcal{P}} := \mathrm{Ow}_S(N, v, (\mathcal{P} \backslash \{P\}) \cup \{S\} \cup \langle P \backslash S \rangle) \tag{5.6}
$$

The following conjecture tries to transfer the results of Theorem 5.6 to $\mathrm{Ow}^{\sharp}$. In (5.5), the component $P \in \mathcal{P}$ is split into the components $S, P \backslash S \subseteq P$. In a sense, all "links" between the players in S and those in $P \backslash S$ have been removed. This is the idea of (5.7): Now, the links between players in $S \subseteq C \in C(N, L)$ and $C \backslash S$ have been removed indeed. The idea of (5.8) is the same except that the players in $C \backslash S$ are completely connected which, of course, did not make a difference for coalition

S	$L\vert_{N\backslash(C\backslash S)} \cup L\vert_{C\backslash S}$	$L\vert_{N\backslash(C\backslash S)} \cup L^{C\backslash S}$	$L\vert_{N\backslash(C\backslash S)}$	${}^{(k)}v_N^{N,L}(S)$
$\{1\}$	$\{23\}$	$\{23\}$	$\emptyset$	1
$\{2\}$	$\emptyset$	$\{13\}$	$\emptyset$	$\frac{1}{2}$
$\{3\}$	$\{12\}$	$\{12\}$	$\emptyset$	$\frac{1}{2}$
$\{1,2\}$	$\{12\}$	$\{12\}$	$\{12\}$	$\frac{3}{2}$
$\{1,3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\frac{3}{2}$
$\{2,3\}$	$\{23\}$	$\{23\}$	$\{23\}$	1
N	L	L	L	2

TABLE 5.1. Graphs and worths for the counterexample

functions. In (5.6), the players in S are also separated from those in $P\backslash S$ but the players in $P\backslash S$ are isolated, i.e. they form singleton coalitions. (5.9) mimics this by removing all links outside $N\backslash(C\backslash S)$.

Conjecture 5.7. *Theorem 5.5 remains true is we replace the coalition function $v_C^{N,L}$, $C \in C(N,L)$ by either of the following ones: For all $S \subseteq C$*

$$ {}^{(1)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp \left(N, v, L|_{N\backslash(C\backslash S)} \cup L|_{C\backslash S}\right) \tag{5.7} $$

$$ {}^{(2)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp \left(N, v, L|_{N\backslash(C\backslash S)} \cup L^{C\backslash S}\right) \tag{5.8} $$

$$ {}^{(3)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp \left(N, v, L|_{N\backslash(C\backslash S)}\right) \tag{5.9} $$

As the following example reveals, however, this conjecture is wrong.

Example 5.8. Set $N = \{1,2,3\}$, $L = \{12,23\}$ and $v = u_{\{1,2\}} + u_{\{1,3\}}$. Since L is connected, one easily obtains $\mathrm{Ow}^\sharp(N,v,L) = \mu(N,v,L) = \left(\frac{5}{6}, \frac{5}{6}, \frac{1}{3}\right)$. Table 5.1 lists the graphs and worths involved in Conjecture 5.7 where the payoff functions coincide. Again, one easily obtains $\mathrm{Ow}^\#(N,v,L) = \mathrm{Ow}_N^\sharp\left(N, {}^{(k)}v_N^{N,L}, L\right) = \mu\left(N, {}^{(k)}v_N^{N,L}, L\right) = \left(1, \frac{1}{2}, \frac{1}{2}\right)$. I.e., $\mathrm{Ow}^\# \neq \mathrm{Ow}^\sharp$.

5.3. **Stability issues.** Employing the Owen value, Hart and Kurz (1983) study coalition formation in CS-games by strong equilibria of simultaneous coalition formation games. Yet, Hart and Kurz (1984) provide examples of TU games that do not allow for stable coalition structures. Dutta, van den Nouweland and Tijs

(1998) study link formation in CO-games by simultaneous link formation games which involve the Myerson value. For superadditive games, they show that the complete network can be supported by undominated Nash equilibria and coalition proof Nash equilibria and that any such equilibrium yields the same payoffs. Partly, this result rests on the following axiom which μ satisfies for superadditive games (Myerson, 1977).

Axiom 5.9 (Link monotonicity, LM)**.** *For all* $i, j \in N$,

$$\varphi_i (N, v, L + ij) \geq \varphi_i (N, v, L).$$

As the following example reveals, $\mathrm{Ow}^\sharp$ fails this axiom.

Example 5.10. Consider the game (N, u_N), $N = \{1, 2, 3\}$ which is superadditive and the graph $L = \emptyset$. It is easy to check that we then have $\mathrm{Ow}_1^\sharp \left(N, u_{\{1,2\}}, L\right) = \frac{1}{3}$ but $\mathrm{Ow}_1^\sharp (N, u_N, L + 12) = \frac{1}{4}$. Note that $2 \notin C_1 (N, L)$.

Hence, since $\mathrm{Ow}^\sharp$ combines the Owen value and the Myerson value, it seems to us that one cannot reasonably expect general stability results for $\mathrm{Ow}^\sharp$. Nevertheless, in view of (4.2) and (4.3), it is immediate that $\mathrm{Ow}^\sharp$ satisfies the following component restricted version of **LM** for superadditive games.

Theorem 5.11 (Component restricted link monotonicity, CLM)**.** *If* (N, v) *is superadditive then* $\mathrm{Ow}^\sharp$ *satisfies the following axiom: For all* $i \in N$ *and* $j \in C_i (N, L)$, $\varphi_i (N, v, L + ij) \geq \varphi_i (N, v, L)$.

6. Conclusion

In this paper, we introduced and advocated an efficient CO-value, $\mathrm{Ow}^\sharp$, which combines the ideas underlying the Owen and the component efficient Myerson value. In contrast to the Owen value, this value is capable to exploit the inner structure of the bargaining blocs modelled by the connected components of a graph. This way, $\mathrm{Ow}^\sharp$ may recognize e.g. the role of a coordinating player who keeps a bloc together. As mentioned above, this may be an additional source for instability in network formation. Nevertheless, it seems to be worthwhile to study implications of $\mathrm{Ow}^\sharp$ in this regard, both in general and in specific applications.

van den Nouweland, Borm and Tijs (1992) extend the Myerson value to the class of TU games with a conference structure (hypergraph on the player set) (henceforth CF-games and CF-value) which we will call the Myerson CF-value. Remember, a hypergraph is a pair (N, H) consisting of a set N and a subset H of the power set 2^N the elements h of which are called hyperlinks or conference structures. Let $C(N, H)$ denote the set of connected components of (N, H) and $C_i(N, H)$ the component that hosts player i. Since the characterization of the Myerson CF-value is analogous to that of the Myerson value, one may wonder whether the results of this paper could be extended to CF-games.

Indeed, slightly adapting the arguments from this paper and van den Nouweland et al. (1992), it is hardly more than a five-finger exercise to extend our CO-value into a CF-value with analogous properties: In the definition, i.e. in (4.2) and (4.3), the graph has to be replaced by a hypergraph, and in (4.2), the Myerson value has to replaced by the Myerson CF-value. The characterization then involves extensions of **CE**, **Q**, **CF**, and **CM**. Those of **CE** and **CM** are natural. The obvious extension of **Q** would require $\varphi\left(N, v, 2^N\right) = \varphi(N, v, \emptyset)$, but in view of the definition of the Myerson CF-value, the complete hypergraph 2^N could be replaced by the complete graph L^N as a subset of 2^N. Besides **CE**, the Myerson CF-value is characterized by the following modification of **F**: For all $i, j \in h \in H$, we have

$$\varphi_i(N, v, H) - \varphi_i(N, v, H\backslash\{h\}) = \varphi_j(N, v, H) - \varphi_j(N, v, H\backslash\{h\}).$$

This translates into the following extension of **MF**: For all $i, j \in h \in H$,

$$\begin{aligned}\varphi_i(N, v, H) - \varphi_i\left(N_i(H, h), v|_{N_i(H,h)}, H|_{N_i(H,h)}\right) \\ = \varphi_j(N, v, H) - \varphi_j\left(N_j(H, h), v|_{N_j(H,h)}, H|_{N_j(H,h)}\right)\end{aligned}$$

where

$$N_i(H, h) := N\backslash\left(C_i(N, H)\backslash C_i(N, H\backslash\{h\})\right).$$

It is easy to see that for hypergraphs containing just two-player hyperlinks the modified axioms become the original ones.

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