Efficient extensions of communication values[☆]

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Abstract

Most of the values for TU-games with a communication graph are characterized by a link deletion property, which describes specific payoff variations induced by the removal of some links from the graph, but are only efficient if the underlying graph is connected. In this article, we study efficient extensions of values for TU-games with a connected graphs, *i.e.*, values which are efficient, still satisfy the link deletion property characterizing the targeted value, and which coincide with the targeted value on connected graphs. We show the uniqueness of an efficient extension for the Myerson value (Myerson, 1977) and the Average tree solution (Herings et al., 2008), and provide axiomatic characterizations as well.

Keywords: communication graph, fairness, component fairness, efficiency, efficient extension, Shapley value, Myerson value, Average tree solution
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1. Introduction

The players involved in a cooperative game with transferable utilities, or simply TU-game, only differ with respect to the worth that the coalitions they belong to can obtain from cooperation. Nonetheless, an essential characteristic of many natural situations is that the players organize themselves into some hierarchical, technical, or communicational structure. It is therefore crucial to understand how the distribution of payoffs among the players can be affected by their social organization. Myerson (1977) proposes to model the affinities between the players by an undirected graph. The combination of a TU-game and a graph is called a communication game, or simply CO-game, and a communication value,

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henceforth CO-value, evaluates the payoff that each player can claim for his participation in a CO-game.

Myerson (1977) assumes that a coalition is feasible if and only if its members are connected directly or indirectly through their own links in the graph. Thus, the communication between the players is necessary to enable their cooperation. This interpretation leads to CO-values that are component efficient, i.e., the worth of each component of the graph is distributed among its members. The most prominent component efficient CO-values are the Myerson value (Myerson, 1977) and the Average tree solution Herings et al. (2008). The Myerson value can be characterized by component efficiency and fairness, i.e., adding a link to the graph changes the payoffs of the players forming this link by the same amount. The Average tree solution is introduced for CO-games with a cycle-free graph and can be characterized by component efficiency and component fairness. The latter property requires that deleting a link yields for both resulting new components of the graph the same average change in payoff, where the average is taken over the players in each new component.

An alternative natural interpretation of the communication graph is that the players use their social links in order to improve their bargaining position in the negotiation process underlying a CO-game, as highlighted by Hart and Kurz (1983) in the framework of TU-games with a coalition structure. Communication among players is therefore not regarded to be necessary for establishing cooperation. As a consequence, this interpretation supports CO-values that are efficient, i.e., the worth of the grand coalition is distributed among its members. Efficient CO-values have been introduced by Casajus (2007) and very recently by Hamiache (2012), Béal et al. (2012b), and van den Brink et al. (2012).

In this article, further developments on the design of efficient CO-values are investigated. Our study is motivated by two facts. Firstly, even though the two interpretations of a communication graph are incompatible for CO-games with unconnected graphs, they generally agree on the payoff allocation for CO-games with connected CO-graphs. Secondly, we feel that the fairness properties are also reasonable if a communcation graph is mainly understood as a means of bargaining. This is particularly desirable because these properties often convey natural principles of distributive justice. A first step towards the analysis of efficient CO-values for CO-games is then to consider CO-values on CO-games with connected graphs and to look for their *efficient extensions* to the class of CO-games with arbitrary graphs. By an efficient extension, we mean an *(a)* efficient CO-value that *(b)* coincides with the targeted CO-value on connected graphs, and *(c)* satisfies the fairness property characterizing the targeted CO-value.

We support the argument that the Myerson value is an efficient and fair CO-value for CO-games with connected graphs by providing a new characterization of the Myerson value on this class of CO-games. Then, we turn to the only efficient and fair extension of the Myerson value that has been provided in the literature (van den Brink et al., 2012). We prove its uniqueness as a corollary of the following more general result: There exists a unique extension of any fair and efficient CO-value for CO-games with connected graphs to the class of all CO-games. Based on these results, we then present a new characterization of the CO-value introduced by van den Brink et al. (2012).

The second part of the article provides analogous results for the Average tree solution.

We show that there exists a unique efficient and component-fair extension of the Average tree solution from the class of CO-games with connected cycle-free graphs to the class of all CO-games with cycle-free graphs. This CO-value gives every player his payoff according to the Average tree solution plus an equal share of the difference of the worth of the grand coalition and the overall payoff assigned by the Average tree solution. We further give a new characterization of the Average tree solution on the class of CO-games with connected graphs.

It is worth to mention that our study exhibits some similarities with the literature on TU-games with a coalition structure, in which two analogous conflicting interpretations of the coalition structure coexist. The Owen value (Owen, 1977) is efficient while the value of Aumann and Drèze (1974) is component efficient, i.e., the worth of each component of the coalition structure is distributed among its members. For an efficient CO-value that is in the spirit of the Owen value, the reader is referred to Casajus (2007).

The article is organized as follows. Section 2 gives basic definitions and notations. The the (component-)fair and efficient extensions of the Myerson value and of the Average tree solution are studied in Sections 3 and 4, respectively. Section 5 concludes with some remarks on similar results for other CO-values, in particular, for the compensation solution for CO-games with cycle-free graphs (Béal et al., 2012a) and the position value (Borm et al., 1992).

2. Cooperative games and graphs

Fix an infinite set $\mathfrak{U}$, the universe of players, and let $\mathcal{N}$ denote the set of non-empty and finite subsets of $\mathfrak{U}$.

2.1. Cooperative games with transferable utilities

A **TU-game** is a pair (N,v) consisting of a set of players $N \in \mathcal{N}$ and a **coalition function** $v \in \{f : 2^N \longrightarrow \mathbb{R} \mid f(\emptyset) = 0\}$, where 2^N denotes the power set of N. Subsets of N are called **coalitions**, and $v(S)$ is called the worth of coalition S. For any TU-game (N,v) and any $S \subseteq N$, the sub-game of (N,v) induced by S is denoted by $(S, v|_S)$, where $v|_S$ is the restriction of v to 2^S.

A **value** on $\mathcal{N}$ is an operator φ that assigns a payoff vector $\varphi(N,v) \in \mathbb{R}^N$ to any TU-game (N,v). The **Shapley value** (Shapley, 1953) is the value given by

$$\mathrm{SH}_i\,(N,v) = \sum_{S \subseteq N\setminus\{i\}} \frac{1}{|N|} \cdot \binom{|N|-1}{|S|}^{-1} \cdot \left(v\left(S \cup \{i\}\right) - v\left(S\right)\right)$$

for all TU-games (N,v), and $i \in N$.

2.2. Graphs

A **communication graph** for $N \in \mathcal{N}$ is an undirected graph (N,L), $L \subseteq \mathcal{L}^N := \{\{i,j\}\,|i,j \in N, i \neq j\}$; a typical element (**link**) of L is written as $ij := \{i,j\}$. Given a graph (N,L), N splits into (maximal connected) **components** the set of which is denoted by $\mathcal{C}\,(N,L)$; $C_i\,(N,L) \in \mathcal{C}\,(N,L)$ denotes the component containing $i \in N$. The graph (N,L)

is called **connected** if $\mathcal{C}(N,L)=\{N\}$. A link $ij \in L$ is called a **bridge** in (N,L) if $\mathcal{C}(N,L) \neq \mathcal{C}(N,L \setminus \{ij\})$. For $S \subseteq N$ and $L \subseteq \mathcal{L}^N$, set $L|_S := \{ij \in L | i,j \in S\}$. A graph (N,L) is **cycle-free** if each $ij \in L$ is a bridge. Each component of a cycle-free graph is called a **tree**. For any cycle-free graph (N,L) and any $C \in \mathcal{C}(N,L)$, each player $r \in C$ induces a **rooted spanning tree** on C, i.e., a directed graph that arises from the tree $(C, L|_C)$ by directing all links away from the root r. If a spanning tree rooted at r contains a directed link (i,j), then j is a called a **successor** of i. Denote by $s_r(i)$ the possibly empty set of successors of player $i \in C$ in the spanning tree rooted at r. A player j is a **subordinate** of i if there is a directed path from i to j, i.e., if there is a sequence of distinct players $(i_1,\ldots,i_k)$ such that $i_1=i$, $i_k=j$, and, for each $q=1,\ldots,k-1$, $i_{q+1} \in s_r(i_q)$. The set $S_r(i)$ denotes the union of all subordinates of i in the spanning tree rooted at r and $\{i\}$.

2.3. Communication games

A **CO-game** is a triple (N,v,L), where (N,v) is a TU-game and $L \subseteq \mathcal{L}^N$. We denote by $\mathcal{G}$ the set of all such CO-games. A CO-game is called connected if the associated graph is connected, and cycle-free is the associated graph is cycle-free. We denote by $\mathcal{G}_C \subseteq \mathcal{G}$ and $\mathcal{G}_{CF} \subseteq \mathcal{G}$ the classes of all **connected CO-games** and of all **cycle-free CO-games**, respectively. A **CO-value** on some class of CO-games $\mathcal{G}^* \subseteq \mathcal{G}$ is an operator φ that assigns a payoff vector $\varphi(N,v,L) \in \mathbb{R}^N$ to every CO-game $(N,v,L) \in \mathcal{G}^*$.

The **Myerson value** (Myerson, 1977) is the CO-value on $\mathcal{G}$ given by

$$\mathrm{MY}(N,v,L) := \mathrm{SH}\left(N,v^L\right), \qquad v^L(S) := \sum_{T \in \mathcal{C}(S,L|_S)} v(T), \quad S \subseteq N$$

It is characterized by component efficiency and fairness. Throughout this article we sometimes invoke axioms on different subclasses of CO-games indicated by "$|_{\mathcal{G}^*}$" in their definition. In case $\mathcal{G}^*=\mathcal{G}$, we omit this indicator. For any such subclass, all the CO-games used in the axiom belong to the subclass.

Component efficiency, CE$|_{\mathcal{G}^*}$. For all $(N,v,L) \in \mathcal{G}^*$, and $C \in \mathcal{C}(N,L)$,

$$\sum_{i \in C} \varphi_i(N,v,L) = v(C).$$

Fairness, F$|_{\mathcal{G}^*}$. For all $(N,v,L) \in \mathcal{G}^*$, and $ij \in L$,

$$\varphi_i(N,v,L) - \varphi_i(N,v,L \setminus \{ij\}) = \varphi_j(N,v,L) - \varphi_j(N,v,L \setminus \{ij\}).$$

Component efficiency states that the worth of each component of the graph is distributed among its members. Fairness requires that removing a link from the graph changes the payoffs of the players forming this link by the same amount.

Theorem 1. (Myerson, 1977) *The Myerson value is the unique CO-value on $\mathcal{G}$ that satisfies component efficiency (**CE**) and fairness (**F**).*

For all $(N, v, L) \in \mathcal{G}_{CF}$, all $C \in \mathcal{C}(N, L)$, and $r \in C$, Demange (2004) defines the **hierarchical outcome** for the spanning tree on C rooted at r as:

$$m_i^r(N, v, L) = v(S_r(i)) - \sum_{j \in s_r(i)} v(S_r(j))$$

for each $i \in C$. The **Average tree solution** AT introduced by Herings et al. (2008) is the CO-value on $\mathcal{G}_{CF}$ that assigns to each cycle-free CO-game and to each player the average of his hierarchical outcomes:

$$\mathrm{AT}_i(N, v, L) = \frac{1}{|C_i(N, L)|} \sum_{r \in C_i(N,L)} m_i^r(N, v, L)$$

for all $(N, v, L) \in \mathcal{G}_{CF}$, and $i \in N$. They use component efficiency and component fairness below in order to characterize the Average tree solution for cycle-free CO-games.

Component fairness, CF. For all $(N, v, L) \in \mathcal{G}_{CF}$, and $ij \in L$,

$$\begin{aligned} &\sum_{k \in C_i(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L \setminus \{ij\})}{|C_i(N, L \setminus \{ij\})|} \\ = &\sum_{k \in C_j(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L \setminus \{ij\})}{|C_j(N, L \setminus \{ij\})|}. \end{aligned}$$

Component fairness states that deleting a link between two players yields for both resulting components the same average change in payoffs.

Theorem 2. (Herings et al., 2008) *The Average tree solution is the unique CO-value on $\mathcal{G}_{CF}$ which satisfies component efficiency (**CE**$|_{\mathcal{G}_{CF}}$) and component fairness (**CF**).*

3. Efficient and fair extension of the Myerson value

In this section, we explore the possibility to extend the Myerson value from the class of connected CO-games to the class of all CO-games under the preservation of efficiency. This extension shall be in the same spirit as the Myerson value. In this article, we take the view to measure this similarity by two properties. Firstly, since the Myerson value is efficient on the class of connected CO-games, one possibly would like this CO-value to coincide with the Myerson value for connected CO-games. Secondly, Myerson (1977) characterizes his CO-value via component efficiency and fairness, i.e., fairness can be considered to be the soul of his value. Thus, fairness seems to be a desirable property for the efficient version of the Myerson value.

In order to clarify that the Myerson value is an efficient and fair CO-value for connected CO-games, we invoke the following relative of fairness that has been suggested by Casajus (2009).

Connected fairness, CNF. For all $(N, v, L) \in \mathcal{G}_C$ and $ij \in L$, we have

$$\begin{aligned}\varphi_i(N, v, L) - \varphi_i\left(C_i(N, L \setminus \{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L|_{C_i(N,L\setminus\{ij\})}\right) \\ = \varphi_j(N, v, L) - \varphi_j\left(C_j(N, L \setminus \{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L|_{C_j(N,L\setminus\{ij\})}\right).\end{aligned}$$

Similarly to fairness, connected fairness considers the change of the payoffs of two players i and j if the link ij is removed. Either the players remain in the same component and connected fairness imposes the same condition as fairness. Or the players end up in different components. In this case, connected fairness compares the original payoffs with the payoffs obtained if the CO-game is restricted to each player's component, respectively, and imposes an equal change of the payoffs. Note that all graphs involved in this axiom are connected. We provide the following characterization of the Myerson value on the class of connected CO-games.[1]

Proposition 1. *A CO-value φ on $\mathcal{G}_C$ satisfies efficiency (**E**$|_{\mathcal{G}_C}$) and connected fairness (**CNF**) if and only if $\varphi = \mathrm{MY}$ on $\mathcal{G}_C$.*

Proof. Since the Myerson value on $\mathcal{G}_C$ satisfies $\mathbf{E}|_{\mathcal{G}_C}$ and **CNF**, we only have to prove that at most one value satisfies these two axioms. By contradiction, assume that two different CO-values φ and ψ on $\mathcal{G}_C$ satisfy $\mathbf{E}|_{\mathcal{G}_C}$ and **CNF**. Consider a minimal N and a minimal $L \subseteq \mathcal{L}^N$ such that $\varphi(N, v, L) \neq \psi(N, v, L)$. By $\mathbf{E}|_{\mathcal{G}_C}$, it holds that $|N| \geq 2$. Furthermore, $L \neq \emptyset$ since $(N, v, L) \in \mathcal{G}_C$. For each $ij \in L$, **CNF** and the minimality of both N and L yield that

$$\begin{aligned}\varphi_i(N, v, L) - \varphi_j(N, v, L) &= \varphi_i\left(C_i(N, L \setminus \{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L|_{C_i(N,L\setminus\{ij\})}\right) \\ &\quad -\varphi_j\left(C_j(N, L \setminus \{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L|_{C_j(N,L\setminus\{ij\})}\right) \\ &= \psi_i\left(C_i(N, L \setminus \{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L|_{C_i(N,L\setminus\{ij\})}\right) \\ &\quad -\psi_j\left(C_j(N, L \setminus \{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L|_{C_j(N,L\setminus\{ij\})}\right) \\ &= \psi_i(N, v, L) - \psi_j(N, v, L).\end{aligned}$$

Since (N, L) is connected, this implies that $\varphi_i(N, v, L) - \psi_j(N, v, L) = \Delta$ for some $\Delta \in \mathbb{R}$ and all $i \in N$. But $\mathbf{E}|_{\mathcal{G}_C}$ implies $\Delta = 0$, a contradiction. □

Among the efficient CO-values that coincide with the Myerson value on connected CO-games and that have been dicussed in the literature, only one CO-value satisfies fairness. In the following, we call this value the **efficient Myerson value**, given by

$$\mathrm{EMY}_i(N, v, L) := \mathrm{MY}_i(N, v, L) + \frac{v(N) - v^L(N)}{|N|}.$$

van den Brink et al. (2012) introduce and characterize the efficient Myerson value using the following axiom.

[1] According to Proposition 1, there exists a redundancy in the similar but weaker result due to Casajus (2009, Lemma 4.2).

Fair distribution of surplus, FDS. For all $(N, v, L) \in \mathcal{G}$, and $C, C' \in \mathcal{C}(N, L)$, we have

$$\sum_{i \in C} \frac{\varphi_i(N, v, L) - \varphi_i(C, v|_C, L|_C)}{|C|} = \sum_{i \in C'} \frac{\varphi_i(N, v, L) - \varphi_i(C', v|_{C'}, L|_{C'})}{|C'|}.$$

Fair distribution of surplus requires that the average change of the payoffs of the players in a component $C \in \mathcal{C}(N, L)$ equals the average change of the players in any other component $C' \in \mathcal{C}(N, L)$ if one compares the payoffs in the restriction of the CO-game to the respective component with the payoffs of the original CO-game.

Theorem 3. (van den Brink et al., 2012) *A CO-value φ on $\mathcal{G}$ satisfies efficiency (**E**), fairness (**F**), and fair distribution of surplus (**FDS**) if and only if $\varphi = \mathrm{EMY}$.*

Note that the efficient Myerson value distributes the surplus generated between the components in the most egalitarien way, i.e., every player obtains the same share from $v(N) - v^L(N)$. From Theorem 3, this egalitarism is particularly explicit in the fair distribution of surplus property. Therefore, we try to avoid this property. Instead, we look for other (possibly less egalitarian) efficient and fair extensions of the Myerson value from connected CO-games to the class of all CO-games. Unfortunately, such extensions do not exist. This is a direct consequence of the next result, which shows that for every fair and efficient CO-value on the class of connected CO-games, there exists at most one fair and efficient extension to the class of all CO-games.[2]

Proposition 2. *Let φ and ψ be two CO-values on $\mathcal{G}$ that satisfy efficiency (**E**) and fairness (**F**). If $\varphi(N, v, L) = \psi(N, v, L)$ for all $(N, v, L) \in \mathcal{G}_C$, then $\varphi = \psi$ for all $(N, v, L) \in \mathcal{G}$.*

Proof. Let the CO-values φ and ψ be as in the proposition. Suppose $\varphi \neq \psi$ and consider any $N \in \mathcal{N}$ such that $\varphi(N, v, L) \neq \psi(N, v, L)$ for some $(N, v, L) \in \mathcal{G}$. There is some maximal $L \subseteq \mathcal{L}^N$ and some $i \in N$ such that $\varphi_i(N, v, L) \neq \psi_i(N, v, L)$. By assumption, $(N, v, L) \in \mathcal{G} \setminus \mathcal{G}_C$ so that $|\mathcal{C}(N, L)| > 1$. Let $C := C_i(N, L)$ and choose any $j \in N \setminus C$. By **F** and the maximality of L, we have

$$\begin{aligned} \varphi_i(N, v, L) - \varphi_k(N, v, L) &= \varphi_i(N, v, L \cup \{ik\}) - \varphi_k(N, v, L \cup \{ik\}) \\ &= \psi_i(N, v, L \cup \{ik\}) - \psi_k(N, v, L \cup \{ik\}) \\ &= \psi_i(N, v, L) - \psi_k(N, v, L) \end{aligned}$$

for all $k \in N \setminus C$. Analogously, one shows

$$\varphi_j(N, v, L) - \varphi_\ell(N, v, L) = \psi_j(N, v, L) - \psi_\ell(N, v, L)$$

for all $\ell \in C \setminus \{i\}$. Hence, we have

$$\varphi_i(N, v, L) - \psi_i(N, v, L) = \varphi_j(N, v, L) - \psi_j(N, v, L)$$

[2] The proof of Proposition 2 indicates that it can sharpened by just requiring $\varphi(N, v, L) = \psi(N, v, L)$ for all $L \subseteq \mathcal{L}^N$ such that $\left|\mathcal{C}\left(N, \mathcal{L}^N \backslash L\right)\right| > 1$. Note that $\left|\mathcal{C}\left(N, \mathcal{L}^N \backslash L\right)\right| > 1$ entails $|\mathcal{C}(N, L)| = 1$.

for all $j \in N$. Summing up over $j \in N$ gives

$$|N| \cdot (\varphi_i(N,v,L) - \psi_i(N,v,L)) = \sum_{j \in N} \varphi_j(N,v,L) - \sum_{j \in N} \psi_j(N,v,L) \overset{\mathbf{E}}{=} 0,$$

i.e., $\varphi_i(N,v,L) = \psi_i(N,v,L)$. Contradiction. □

Applying the previous proposition to the efficient Myerson value yields the following corollary.

Corollary 1. *The efficient Myerson value* EMY *is the unique efficient and fair extension of the Myerson value, i.e., φ satisfies $\varphi = \mathrm{MY}$ on $\mathcal{G}_C$ and meets efficiency (**E**) and fairness (**F**), if and only if $\varphi = \mathrm{EMY}$ on $\mathcal{G}$.*

Finally, we obtain a new characterization of the efficient Myerson value.

Theorem 4. *The efficient Myerson value* EMY *is the unique CO-value on $\mathcal{G}$ that satisfies efficiency (**E**), fairness (**F**), and connected fairness (**CNF**).*

Proof. *Existence*: By Theorem 3, EMY satisfies **E** and **F**. From Proposition 1, it is immidiate that EMY satisfies **CF**. *Uniqueness*: Let φ satisfy **E**, **F**, and **CNF**. By Proposition 1, $\varphi = \mathrm{MY} = \mathrm{EMY}$ on $\mathcal{G}_C$. Using Theorem 2, one obtains $\varphi = \mathrm{EMY}$ on $\mathcal{G}$. □

Remark 1. *Among all the efficient CO-values, the efficient Myerson value is the unique value that minimizes the euclidean distance to the Myerson value.*

4. Efficient and component-fair extension of the Average tree solution

In this section, we prove analogous results of the previous section for the Average tree solution. In order to understand the Average tree solution as an efficient CO-value for connected and cycle-free CO-games, we invoke the following property.

Connected component fairness, CCF. For all $(N,v,L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ and $ij \in L$, we have

$$\sum_{k \in C_i(N,L\setminus\{ij\})} \frac{\varphi_k(N,v,L) - \varphi_k(C_i(N,L\setminus\{ij\}), v|_{C_i(N,L\setminus\{ij\})}, L|_{C_i(N,L\setminus\{ij\})})}{|C_i(N,L\setminus\{ij\})|}$$
$$= \sum_{k \in C_j(N,L\setminus\{ij\})} \frac{\varphi_k(N,v,L) - \varphi_k(C_j(N,L\setminus\{ij\}), v|_{C_j(N,L\setminus\{ij\})}, L|_{C_j(N,L\setminus\{ij\})})}{|C_j(N,L\setminus\{ij\})|}.$$

Similarly to component fairness, connected component fairness considers the average change of the payoffs of the components of two players i and j if the link ij is removed. Connected component fairness compares the original payoffs with the payoffs obtained if the CO-game is restricted to each player's component, respectively, and imposes an equal average change of the payoffs. Note that all graphs involved in this axiom are connected and cycle-free. We suggest the following characterization of the Average tree solution on the class of connected and cycle-free CO-games.

Proposition 3. *A CO-value φ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}\cap\mathcal{G}_C}$) and connected component fairness (**CCF**) if and only if $\varphi = \mathrm{AT}$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$.*

Proof. Since AT satisfies the two axioms on $\mathcal{G}_{CF} \cap \mathcal{G}_C$, we shall only prove that if a CO-value φ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$ satisfies the two axioms, then it is uniquely determined. This part of the proof is similar to those of Theorem 3.4 of Herings et al. (2008). Thus, we only sketch the proof by induction on the cardinality of the player set. For a CO-game $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| = 1$, $\mathbf{E}|_{\mathcal{G}_{CF}\cap\mathcal{G}_C}$ uniquely determines φ. Now, suppose that φ is uniquely determined for all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| < n$ and consider a CO-game $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$ such that $|N| = n$. Applying **CCF**, $\mathbf{E}|_{\mathcal{G}_{CF}\cap\mathcal{G}_C}$, and the induction hypothesis as in proof of Theorem 3.4 of Herings et al. (2008), we get a system of linearly independent equations as desired. $\square$

In the next Theorem, we prove that there exists a unique efficient extension of the Average tree solution for cycle-free CO-games. In other words, there exists a unique CO-value that satisfies efficiency and component fairness and that coincides with the Average tree solution for connected cycle-free CO-games. We call this value the **efficient Average tree solution** EAT, which is defined by

$$\mathrm{EAT}_i(N, v, L) = \mathrm{AT}_i(N, v, L) + \frac{v(N) - v^L(N)}{|N|}$$

for all $(N, v, L) \in \mathcal{G}_{CF}$, and $i \in N$. Moreover, we give a concise characterization of the efficient Average tree solution.

Theorem 5. *(i) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), component fairness (**CF**), and $\varphi = \mathrm{AT}$ on $\mathcal{G}_{CF\cap\mathcal{G}_C}$ if and only if $\varphi = \mathrm{EAT}$ on $\mathcal{G}_{CF}$.*

*(ii) A CO-value φ on $\mathcal{G}_{CF}$ satisfies efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$), component fairness (**CF**), and connected component fairness (**CCF**) if and only if $\varphi = \mathrm{EAT}$.*

The proof of Theorem 5 is based on the following lemma, which states that if two CO-values for cycle-free CO-games satisfy efficiency, component fairness, and agree on connected cycle-free CO-games, then they must assign the same total payoff to all components of a graph in all cycle-free CO-games.

Lemma 1. *Let φ and ψ be two CO-values on $\mathcal{G}_{CF}$ that satisfy efficiency ($\boldsymbol{E}|_{\mathcal{G}_{CF}}$) and component fairness (**CF**). If $\varphi = \psi$ on $\mathcal{G}_{CF} \cap \mathcal{G}_C$, then $\sum_{i\in C} \varphi_i(N, v, L) = \sum_{i\in C} \psi_i(N, v, L)$ for all $(N, v, L) \in \mathcal{G}_{CF}$ and all $C \in \mathcal{C}(N, L)$.*

Proof. Let φ and ψ be two CO-values for cycle-free CO-games that satisfy $\mathbf{E}|_{\mathcal{G}_{CF}}$ and **CF** and suppose that $\varphi(N, v, L) = \psi(N, v, L)$ for all $(N, v, L) \in \mathcal{G}_{CF} \cap \mathcal{G}_C$. The result follows by $\mathbf{E}|_{\mathcal{G}_{CF}}$ on connected cycle-free CO-games. So, consider any cycle-free CO-game $(N, v, L) \in \mathcal{G}_{CF} \setminus \mathcal{G}_C$, which implies $|\mathcal{C}(N, L)| > 1$. By way of contradiction, assume that there is a cycle-free CO-game in which the φ and ψ do not assign the same total payoff to some component. More specifically, consider $(N, v, L) \in \mathcal{G}_{CF} \setminus \mathcal{G}_C$ with a maximal $L \subseteq \mathcal{L}^N$

such that $\sum_{i\in C}\varphi_i(N,v,L)\neq\sum_{i\in C}\psi_i(N,v,L)$ for some $C\in\mathcal{C}(N,L)$. Since $|\mathcal{C}(N,L)|>1$, we can consider distinct players $i,j\in N$ such that $i\in C$ and $j\in N\setminus C$. By **CF**, the maximality of L, and the initial assumption, we get

$$\begin{aligned}
& \frac{\sum_{k\in C}\varphi_k(N,v,L)}{|C|}-\frac{\sum_{k\in C_j(N,L)}\varphi_k(N,v,L)}{|C_j(N,L)|} \\
= & \frac{\sum_{k\in C}\varphi_k(N,v,L\cup\{ij\})}{|C|}-\frac{\sum_{k\in C_j(N,L)}\varphi_k(N,v,L\cup\{ij\})}{|C_j(N,L)|} \\
= & \frac{\sum_{k\in C}\psi_k(N,v,L\cup\{ij\})}{|C|}-\frac{\sum_{k\in C_j(N,L)}\psi_k(N,v,L\cup\{ij\})}{|C_j(N,L)|} \\
= & \frac{\sum_{k\in C}\varphi_k(N,v,L)}{|C|}-\frac{\sum_{k\in C_j(N,L)}\varphi_k(N,v,L)}{|C_j(N,L)|}.
\end{aligned}$$

Equivalently, the latter equality can be written as

$$\frac{|C_j(N,L)|}{|C|}\sum_{k\in C}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right)=\sum_{k\in C_j(N,L)}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right).$$

Summing the last expression on all $C_j(N,L)$ in $\mathcal{C}(N,L)$ and using $\mathbf{E}|_{\mathcal{G}_{CF}}$ yield

$$\begin{aligned}
& \sum_{C_j(N,L)\in\mathcal{C}(N,L)}\frac{|C_j(N,L)|}{|C|}\sum_{k\in C}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right) \\
= & \sum_{C_j(N,L)\in\mathcal{C}(N,L)}\sum_{k\in C_j(N,L)}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right) \\
\iff & \frac{|N|}{|C|}\sum_{k\in C}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right)=\sum_{k\in N}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right) \\
\iff & \sum_{k\in C}\left(\varphi_k(N,v,L)-\psi_k(N,v,L)\right)\overset{\mathbf{E}|_{\mathcal{G}_{CF}}}{=}0,
\end{aligned}$$

a contradiction that proves the result. □

We are now ready to prove Theorem 5.

Proof. (Theorem 5) (i) It is easy to check that EAT satisfies the two axioms. For the uniqueness part, consider any CO-value φ on $\mathcal{G}_{CF}$ that satisfies the two axioms and that coincides with AT for connected cycle-free CO-games. By definition of EAT, for any $(N,v,L)\in\mathcal{G}_{CF}\cap\mathcal{G}_C$, it holds that $\varphi(N,v,L)=\text{AT}(N,v,L)=\text{EAT}(N,v,L)$. Therefore, Lemma 1 implies that for all $(N,v,L)\in\mathcal{G}_{CF}$ and all $C\in\mathcal{C}(N,L)$, we have $\sum_{i\in C}\varphi_i(N,v,L)=\sum_{i\in C}\text{EAT}_i(N,v,L)$. In particular, if $L=\emptyset$, then $\mathcal{C}(N,L)=\{\{i\},i\in N\}$ and thus $\varphi(N,v,L)=\text{EAT}(N,v,L)$. This proves that φ is uniquely determined for all cycle-free CO-games with an empty graph. It remains to consider cycle-free CO-games with a nonempty graph. So pick any $(N, v, L) \in \mathcal{G}_{CF}$, $L \neq \emptyset$, and any $ij \in L$. By **CF**, it holds that

$$\begin{aligned} &\sum_{k \in C_i(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L \setminus \{ij\})}{|C_i(N, L \setminus \{ij\})|} \\ = &\sum_{k \in C_j(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \varphi_k(N, v, L \setminus \{ij\})}{|C_j(N, L \setminus \{ij\})|}. \end{aligned} \tag{1}$$

Since $\{C_i(N, L \setminus \{ij\}), C_j(N, L \setminus \{ij\})\} \subseteq \mathcal{C}(N, L \setminus \{ij\})$, Lemma 1 implies that both

$$\sum_{k \in C_i(N, L \setminus \{ij\})} \varphi_k(N, v, L \setminus \{ij\}) = \sum_{k \in C_i(N, L \setminus \{ij\})} \mathrm{EAT}_k(N, v, L \setminus \{ij\})$$

and

$$\sum_{k \in C_j(N, L \setminus \{ij\})} \varphi_k(N, v, L \setminus \{ij\}) = \sum_{k \in C_j(N, L \setminus \{ij\})} \mathrm{EAT}_k(N, v, L \setminus \{ij\}).$$

Therefore, (1) rewrites:

$$\begin{aligned} &\sum_{k \in C_i(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \mathrm{EAT}_k(N, v, L \setminus \{ij\})}{|C_i(N, L \setminus \{ij\})|} \\ = &\sum_{k \in C_j(N, L \setminus \{ij\})} \frac{\varphi_k(N, v, L) - \mathrm{EAT}_k(N, v, L \setminus \{ij\})}{|C_j(N, L \setminus \{ij\})|}. \end{aligned}$$

It is useful to express this equality as:

$$\begin{aligned} &\frac{\sum_{k \in C_j(N, L \setminus \{ij\})} \varphi_k(N, v, L)}{|C_i(N, L \setminus \{ij\})|} - \frac{\sum_{k \in C_j(N, L \setminus \{ij\})} \varphi_k(N, v, L)}{|C_j(N, L \setminus \{ij\})|} \\ = &\frac{\sum_{k \in C_j(N, L \setminus \{ij\})} \mathrm{EAT}_k(N, v, L \setminus \{ij\})}{|C_i(N, L \setminus \{ij\})|} - \frac{\sum_{k \in C_j(N, L \setminus \{ij\})} \mathrm{EAT}_k(N, v, L \setminus \{ij\})}{|C_j(N, L \setminus \{ij\})|} \end{aligned} \tag{2}$$

There are $|L| = |N| - |\mathcal{C}(N, L)|$ equations of type (2). Furthermore, for each $C \in \mathcal{C}(N, L)$, we also know from Lemma 1 that

$$\sum_{i \in C} \varphi_i(N, v, L) = \sum_{i \in C} \mathrm{EAT}_i(N, v, L). \tag{3}$$

There are $|\mathcal{C}(N, L)|$ equations of type (3). All in all, we obtain a system of $|N|$ equations. The left-hand sides of this system and of the system with a unique solution obtained by Herings et al. (2008) (proof of Theorem 3.4) are identical. Since only the right-hand side is different in our system, it also has a unique solution.

(ii) One easily checks that EAT obeys connected component fairness. Uniqueness follows from Lemma 1, Theorem 5, and Proposition 3. □

5. Concluding remarks

We emphasize that the Myerson value is an efficient CO-value for connected CO-games by providing a characterization that employs efficiency instead of component efficiency, and that works within the class of connected CO-games. Further, we consider fair and efficient extensions to the class of all CO-games. It turns out that only one such extension exists. It assigns to every player an equal share of the surplus created between the components, $\left(v(N) - v^L(N)\right)/|N|$, plus his payoff according to the Myerson value. We provide a characterization of this CO-value that rests on fairness properties and on efficiency. Similar results are obtained for the efficient and component-fair extension of the Average tree solution.

Neither efficiency, fairness, nor connected fairness alone convey an egalitarian flavor. Surprisingly, their combination requires that the surplus created between the components is shared equally among the players. We will further discuss the converse, i.e., the universitality of the procedure employed to construct the unique (component-)fair and efficient extensions of the Myerson value and of the Average tree solution. The reader might wonder whether the allocation of an equal share of the surplus created between the components, $\left(v(N) - v^L(N)\right)/|N|$, in addition to the payoff according to a CO-value works for other CO-values that are not efficient. However, it turns out that this procedure is not universal.

The Position value (Borm et al., 1992) is characterized on zero-normalized games by component efficiency and balanced total threats.[3] Consider the CO-value defined as follows: every player obtains $\left(v(N) - v^L(N)\right)/|N|$ plus the payoff according to the Position value. This CO-value satisfies efficiency and coincides with the Position value whenever the underlying graph is connected. However, it does not satisfy balanced total threats.

On the other hand, considering the Compensation solution (Béal et al., 2012a) for cycle-free CO-games, one obtains comparable results to those obtained for the Myerson value and for the Average tree solution.[4] In particular, the compensation solution is characterized by component efficiency and relative fairness.[5] Define the efficient compensation solution by adding an equal share of the surplus created between the components, $\left(v(N) - v^L(N)\right)/|N|$, to the compensation solution. This efficient compensation solution satisfies efficiency, relative fairness, and coincides with the compensation solution on the class of connected and cycle-free CO-games.

There exist other CO-values in the literature, for example, CO-values for line-graph CO-games (van den Brink et al., 2007). An interesting extension of our study would be to

[3]**Balanced total threats, BTT**. For all zero-normalized $(N, v, L) \in \mathcal{G}$, and $ij \in L$,

$$\sum_{k \in N: jk \in L} \left(\varphi_i(N, v, L) - \varphi_i(N, v, L \backslash \{jk\})\right) = \sum_{k \in N: ik \in L} \left(\varphi_j(N, v, L) - \varphi_j(N, v, L \backslash \{ik\})\right).$$

[4]The formal statements and their proofs are available from the authors upon request.

[5]**Relative fairness, RF.** For all $(N, v, L) \in \mathcal{G}_{CF}$, and $ij \in L$,

$$\varphi_i(N, v, L) - \sum_{k \in C_i(N, L \backslash \{ij\})} \frac{\varphi_k(N, v, L \backslash \{ij\})}{|C_i(N, L \backslash \{ij\})|} = \varphi_j(N, v, L) - \sum_{k \in C_j(N, L \backslash \{ij\})} \frac{\varphi_k(N, v, L \backslash \{ij\})}{|C_j(N, L \backslash \{ij\})|}.$$

replicate our approach to such CO-values.

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