Null, nullifying, or dummifying players: The difference between the Shapley value, the equal division value, and the equal surplus division value

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Abstract

We provide new characterizations of the equal surplus division value. This way, the difference between the Shapley value, the equal surplus division value, and the equal division value is pinpointed to one axiom.

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1. Introduction

Modern societies and organizations base the allocation of wealth among their members not only on individual productivities, but also on egalitarian or solidarity principles. Cooperative game theory provides versatile and simple tools to model the generation of worth in a society and to study the "fair" or "reasonable" distribution of this worth. In particular, a cooperative game with transferable utility (*TU-game*) specifies a player set and the worth that can be generated by any subcoalition of this player set. In his seminal paper, Shapley (1953) introduced a solution for this setup, nowadays called the *Shapley value*, which assigns to every player a payoff that measures his productivity within such a TU-game.

While the Shapley value probably is the most eminent performance oriented point-valued solution concept, there are (at least) two rival concepts that rely on solidarity considerations to some or more extent. The equal surplus division value (*ES-value*) is egalitarian only with respect to the genuine gains from cooperation, i.e., the players first obtain what they can achieve for themselves alone and, then, gains from cooperation within the whole society are divided equally among them.[1] In contrast, the equal division value (*ED-value*) reflects

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[1] The ES-value is also known as the Centre-of-gravity of the Imputation-Set value, shortly CIS-value (Driessen and Funaki, 1991).

solidarity in the most radical manner—the overall worth generated is distributed evenly among the players.

In this paper, the difference between the Shapley value, the ES-value, and the ED-value is pinpointed to one axiom. A player is said to be dummifying if his presence in a coalition prevents cooperation. That is, the worth generated by a coalition containing him is just the sum of worths generated by its members as singletons. According to the ES-value, a dummifying player obtains what he can generate alone. We show that this property, which we call the dummifying player property, is characteristic for the ES-value. Our result resembles an insight due to van den Brink (2007), who identifies a similar property for the ED-value. His nullifying player property assigns zero payoffs to players whose presence in a coalition not only blocks cooperation but prevents any production at all. Both properties correspond to the null player property, which requires a completely unproductive player to earn a zero payoff. Combined with the standard axioms of efficiency, additivity, and symmetry, the dummifying player property, the nullifying player property, or the null player property characterize the ES-value, the ED-value, and the Shapley value, respectively. Further, we modify two other axioms employed by van den Brink (2007) in order to characterize the ED-value and show that any of the resulting properties together with efficiency and symmetry characterizes the ES-value.

Other characterization of the ES-value and the ED-value are provided by Chun and Park (2012) and van den Brink et al. (2012). While our axiomatizations work on a fixed player set, their characterizations employ properties that consider the implication of shrinking player sets on the remaining players' payoffs. Kamijo and Kongo (2010, 2012) provide characterizations of the Shapley value, the ED-value, and the solidarity value due to Nowak and Radzik (1994) that differ in one axiom only. Béal et al. (2012) characterize the ED-value and the ES-value for graph games.

This paper is organized as follows. Basic definitions and notation are given in Section 2. Section 3 provides the new characterizations of the ES-value.

2. Basic definitions and notation

A **(TU-)game** is a pair (N, v) consisting of a non-empty and finite set of players N and a **coalition function** $v \in \mathbb{V}(N) := \{f : 2^N \to \mathbb{R} \mid f(\emptyset) = 0\}$. Since we work within a fixed player set, we frequently drop the player set as an argument. In particular, we address $v \in \mathbb{V}$ as a game. Subsets of N are called **coalitions**; $v(S)$ is called the **worth** of coalition S. For $v, w \in \mathbb{V}$, $\lambda \in \mathbb{R}$, the coalition functions $v + w \in \mathbb{V}$ and $\lambda \cdot v \in \mathbb{V}$ are given by $(v + w)(S) = v(S) + w(S)$ and $(\lambda \cdot v)(S) = \lambda \cdot v(S)$ for all $S \subseteq N$. A game $v \in \mathbb{V}$ is called **zero-normalized** if $v(\{i\}) = 0$ for all $i \in N$; for $v \in \mathbb{V}$, the **associated zero-normalized game** $v^0 \in \mathbb{V}$ is given by

$$v^0(S) := v(S) - \sum_{i \in S} v(\{i\}) \qquad \text{for all } S \subseteq N. \tag{1}$$

Player $i \in N$ is called a **dummy player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) - v(S) = v(\{i\})$ for all $S \subseteq N \setminus i$; player $i \in N$ is called a **null player** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S)$ for all

$S \subseteq N \setminus \{i\}$. Players $i, j \in N$ are called **symmetric** in $v \in \mathbb{V}$ if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$.

A **(TU-)value** on N is an function φ that assigns a payoff vector $\varphi(v) \in \mathbb{R}^N$ to any $v \in \mathbb{V}$. The **Shapley value** (Shapley, 1953) is given by

$$\mathrm{Sh}_i(v) := \sum_{S \subseteq N : S \ni i} \binom{|N|}{|S|}^{-1} \cdot \frac{1}{|S|} \cdot (v(S) - v(S \setminus \{i\})) \qquad \text{for all } i \in N.$$

The **equal division value (ED-value)** is given by

$$\mathrm{ED}_i(v) := \frac{v(N)}{|N|} \qquad \text{for all } i \in N. \tag{2}$$

The **equal surplus division value (ES-value)** (Driessen and Funaki, 1991) is given by

$$\mathrm{ES}_i(v) := v(\{i\}) + \frac{v^0(N)}{|N|} \qquad \text{for all } i \in N. \tag{3}$$

Later on, we employ the following standard axioms for values on N.
Efficiency, E. For all $v \in \mathbb{V}$, $\sum_{i \in N} \varphi_i(v) = v(N)$.
Null player, N. For all $v \in \mathbb{V}$ and every $i \in N$, who is a null player in v, $\varphi_i(v) = 0$.
Additivity, A. For all $v, w \in \mathbb{V}$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.
Equal treatment, ET. For all $v \in \mathbb{V}$ and $i, j \in N$, who are symmetric in v, $\varphi_i(v) = \varphi_j(v)$.

3. Dummifying players

The standard characterization of the Shapley value (Shapley, 1953) employs efficiency, additivity, the equal treatment property, and the null player property. The null player property indicates that the Shapley value particularly reflects a player's own productivity. Within the null player property, van den Brink (2007) replaces null players by nullifying players. Player $i \in N$ is **nullifying** in $v \in \mathbb{V}$ if $v(S) = 0$ for all $S \subseteq N$ such that $i \in S$. This yields the following axiom.
Nullifying player, Ng. For all $v \in \mathbb{V}$ and $i \in N$ such that i is nullifying in v, we have $\varphi_i(v) = 0$.
According to the nullifying player property, a zero payoff is assigned to nullifying players, i.e., to players whose presence in any coalition renders its worth zero. Replacing the null player property in the standard characterization of the Shapley value by the nullifying player property, one obtains a characterization of the ED-value.

Theorem 1 (van den Brink 2007). *The ED-value is the unique TU-value that satisfies efficiency (**E**), additivity (**A**), the equal treatment property (**ET**), and the nullifying player property (**Ng**).*

van den Brink (2007, Theorem 4.1) also provides a characterization of the ES-value, which deviates from his characterization of the ED-value above by restricting the nullifying player property to zero-normalized games and adding the invariance property. However, we feel that making use of the restricted nullifying property somewhat blurs the characteristic property of the equal surplus division rule.

A nullifying player does not only obstruct cooperation within any coalition containing him, but also neutralizes the productive potential of such a coalition. Dropping the latter feature of a nullifying player leads to the notion of a dummifying player, i.e., a player whose presence rules out any cooperation but does not neutralize the stand-alone productivities of the players in his coalition. Formally, a player $i \in N$ is **dummifying** in $v \in \mathbb{V}$ if $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subseteq N$ such that $i \in S$. Analogously to the nullifying player property one obtains the dummifying player property below.

Dummifying player, Dg. For all $v \in \mathbb{V}$ and $i \in N$ such that i is dummifying in v, we have $\varphi_i(v) = v(\{i\})$.

According to the dummifying player property, the singleton worth is assigned to dummifying players. When there is a dummifying player in a game, then the grand coalition creates worth amounting to the sum of the singleton worths. Assuming that any player claims at least his stand-alone worth, a plausible distribution of the grand coalition's worth would be to give any player his singleton worth. The dummifying player property requires this result for the dummifying player. Another way to justify this property is based on the idea that a dummifying player always "separates" the players. In particular, he himself can be considered to be separated from the other players. Therefore, it can be expected that he ends up with his singleton payoff.

Replacing the null player property in the standard characterization of the Shapley value by the dummifying player property, one obtains a characterization of the ES-value.

Theorem 2. *The ES-value is the unique TU-value that satisfies efficiency (**E**), additivity (**A**), the equal treatment property (**ET**), and the dummifying player property (**Dg**).*

Proof. By (3), it is clear that ES obeys **E**, **A**, and **ET**. Let $i \in N$ be dummifying in $v \in \mathbb{V}$. Hence, $v(N) = \sum_{j \in N} v(\{j\})$. By (3), we have $\mathrm{ES}_i(v) = v(\{i\})$. Thus, ES meets **Dg**.

Now, let φ be a TU-value on N that satisfies **E**, **A**, **ET**, and **Dg**. Clearly, **Dg** implies the restriction of **Ng** to zero-normalized games. In view of (the proof of) van den Brink (2007, Theorem 3.1), it suffices to show that **A** and **Dg** imply $\varphi(v + m_b) = \varphi(v) + b$ for all $v \in \mathbb{V}$ and $b \in \mathbb{R}^N$, where $m_b \in \mathbb{V}$ is given by $m_b(S) := \sum_{i \in S} b_i$ for all $S \subseteq N$. By **A**, we have $\varphi(v + m_b) = \varphi(v) + \varphi(m_b)$. Since all players are dummifying in m_b, **Dg** implies $\varphi(m_b) = b$ and we are done. $\square$

van den Brink (2007) suggests two other axioms in order to characterize the ED-value. The first axiom, coalitional standard equivalence[2] is related to van den Brink's (2007, Foot-

[2] A solution φ satisfies coalitional standard equivalence if for all $v, w \in \mathbb{V}$ and $i \in N$ such that i is a nullifying player in w, we have $\varphi_i(v + w) = \varphi_i(v)$.

note 3) version of Chun's (1989) coalitional strategic equivalence[3]. The second one, coalitional monotonicity[4], is related to Young's (1985) strong monotonicity[5]. In coalitional standard equivalence, we replace nullifying players by dummifying players and obtain coalitional surplus equivalence below. In order to account for the ES-value, we further modify coalitional monotonicity yielding coalitional surplus monotonicity below.

Coalitional surplus equivalence, CSE. For all $v, w \in \mathbb{V}$ and $i \in N$ such that i is dummifying in w, we have $\varphi_i(v+w) = \varphi_i(v) + w(\{i\})$.

Coalitional surplus monotonicity, CSM. For all $v, w \in \mathbb{V}$ and $i \in N$ such that $v^0(S) \geq w^0(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(v) - v(\{i\}) \geq \varphi_i(w) - w(\{i\})$.

Replacing coalitional standard equivalence in van den Brink's (2007) characterization of the ED-value by coalitional surplus equivalence, one obtains a characterization of the ES-value.

Theorem 3. *The ES-value is the unique TU-value that satisfies efficiency (**E**), the equal treatment property (**ET**), and either (i) coalitional surplus equivalence (**CSE**) or (ii) coalitional surplus monotonicity (**CSM**).*

By (3), it is clear that ES obeys **E**, **ET**, **CSE**, and **CSM**. Now, let φ be a TU-value on N that satisfies **E**, **ET**, and **CSE**.

For all $v \in \mathbb{V}$, all players are dummifying in $v - v^0$. By **CSE**, $\varphi_i(v) = \varphi_i(v^0) + v(\{i\})$ for all $i \in N$. Therefore, it suffices to restrict attention to the class of zero-normalized games. For this class, **CSE** becomes coalitional standard equivalence. Moreover, the proof of van den Brink (2007, Theorem 3.2) works within the class of zero-normalized games. Hence, $\varphi = \mathrm{ES}$ for zero-normalized games, what establishes uniqueness for (i). Part (ii) follows from the observation that **CSM** implies **CSE**. □

The theorems in this section show that the equal surplus division rule treats dummifying players as the Shapley value treats dummy players, while the equal division rule handles nullifying players as the Shapley value treats null players.

Remark 4. Our characterizations are non-redundant. Theorem 2: The Shapley value meets all the axioms but **Dg**. The null value Null given by $\mathrm{Null}_i(v) = 0$ for all $v \in \mathbb{V}$ and $i \in N$ meets all axioms but **E**. The value $\varphi^{\mathbf{A}}$, given by $\varphi^{\mathbf{A}}(v) = \mathrm{Sh}(v)$ if $v^0(N) \leq 0$ and $\varphi^{\mathbf{A}}(v) = \mathrm{ES}(v)$ if $v^0(N) > 0$, satisfies all axioms but **A**. Fix a non-constant mapping $w : N \to \mathbb{R}_+^N$ such that $\sum_{i \in N} w_i = 1$, where $w_i := w(i)$, $i \in N$. The value ES^w is given by $\mathrm{ES}_i^w(v) = v(\{i\}) + w_i \cdot v^0(N)$ for all $v \in \mathbb{V}$ and $i \in N$. The value ES^w meets all axioms but the **ET**.

Theorem 3: The Shapley value meets all the axioms but **CSE** and **CSM**. The null value meets all axioms but **E**. The value ES^w meets all axioms but the **ET**.

[3] A solution φ satisfies van den Brink's version of coalitional strategic equivalence if for all $v, w \in \mathbb{V}$ and $i \in N$ such that i is a null player in w, we have $\varphi_i(v+w) = \varphi_i(v)$.

[4] A solution φ satisfies coalitional monotonicity if for all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S) \geq w(S)$ for all $S \subseteq N$, $S \ni i$, we have $\varphi_i(v) \geq \varphi_i(w)$.

[5] A solution φ satisfies strong monotonicity if for all $v, w \in \mathbb{V}$ and $i \in N$ such that $v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S)$ for all $S \subseteq N \setminus \{i\}$, we have $\varphi_i(v) \geq \varphi_i(w)$.

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