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List of Symbols

$\lhd$	predecessor relation	29
$\lhd^x$	restriction of $\lhd$ to T^x	30
0	chance player	53
2^N	power set of N	75
$\mathbf{a}$	a pure strategy profile	30
A	action partition	30
a	an action	30
$\mathbf{A}^+$	set of action profiles	30
$\mathbf{a}^+$	an action profile	30
$\mathbf{A}^+(h)$	set of action profiles leading to h	30
$\mathbf{A}^+(x)$	set of action profiles leading to x	30
A_-	set of the personal players' actions	53
$A_-([z])$	reduced terminal history of $[z]$	54
$A_-(z)$	reduced terminal history	53
AD	AD-value	92
A_h	actions at information set h	30
A_i	player i's actions set	30
$\mathbf{A}_i$	set of player i's pure strategies	30
$\mathbf{a}_i$	a pure strategy of player i	30
A_n	apex game with $n+1$ players	83
ANF(Γ)	agent normal form of Γ	31
$a(\psi(x))$	unordered history of node x	30
$a(\boldsymbol{\psi}(x))$	(ordered) history of node x	30
$a(x)$	action containing node x	30
A^x	action partition of Γ^x	30
A_i^x	set of player i's actions in Γ^x	30
$\mathbf{A}([z])$	set of strategy profiles leading to $[z]$	54
B	set of behavior-strategy profiles	30
B^0	set of completely mixed behavior-strategy profiles	30
B_h	set of local strategies at h	30
b_h	a local strategy at h	30

$b_h(a)$	probility of a under b_h	30
B_i	set of player i's behavior strategies	30
χ	χ-value	79
$C(N,L)$	set of connected components of L in N	92
$C_i(N,L)$	player i's component in $C(N,L)$	92
$\chi^{\#}$	(another) CO-value	98
$\chi^{\sharp}$	the $\chi^{\sharp}$-value	97
δ	an assignment	53
$\mathbb{D}(\gamma)$	set of assignments on γ	53
$\Delta(K)$	average excess of worth over Shapley payoffs for coalition K	81
$\mathcal{E}$	set/class of finite extensive games with perfect recall	29
$\mathcal{E}^*$	non-pathological subset/subclass of $\mathcal{E}$	30
$\mathcal{EF}$	set of (extensive) forms	53
$\mathcal{EF}^*$	non-pathological subset of $\mathcal{EF}$	53
$\mathcal{EF}^{nc}$	subset of $\mathcal{EF}$ without chance mechanism	53
$\mathcal{E}^{nc}$	subset/subclass of $\mathcal{E}$ without chance mechanism	53
$\mathrm{ER}^{\jmath}(G)$	ordered representation of G	31
$\mathbf{f}$	isomorphism of strategic games	29
φ	a value	75
f	bijection of pure strategy profiles	29
φ_K	sum of payoffs of coalition K under the value φ	75
$\varphi(N,v)$	payoffs of the TU game (N,v)	75
γ	a game form	31
$\mathcal{G}$	set/class of finite strategic games	29
G	a finite strategic game	29
$\gamma(\delta)$	extensive game based on γ and specified by δ	53
Γ^{φ}	network formation game under the CO-value φ	103
Γ^x	subgame of Γ rooted in x	30
H	information partition	29
h	an information set	30
H_-	set of the personal players' information sets	53
H_i	set of player$i$'s information sets	30
$\mathrm{hist}(h)$	Oh history of h	36
$h(x)$	information set containing node x	30
H^x	information partition of Γ^x	30
H_i^x	set of player i's information sets in Γ^x	30
I	set of players	29
i	a player	29

I_-	set of personal players	53
i_0	the chance player	29
$i(h)$	player controlling information set h	30
$\bar{\imath}\bar{\jmath}$	original link ij in the link agent form	127
$\{i,j\}$	link between i and j	92
ij	link between i and j	92
(i,λ)	player i's agent for link λ	128
$i\,(x)$	player who controls node x	29
I^x	set of players of Γ^x	30
$K_i\,(\sigma)$	set of players not after i in σ	75
$[K,K']$	set of all links between K and K'	92
$\bar{L}$	link set of the link agent form	127
L	cooperation structure, undirected graph, link set	92
L	solution concept for strategic games	29
$L+ij$	graph L plus link ij	92
$L-ij$	graph L minus link ij	92
$\mathrm{LAF}\,(N,v,L)$	link agent form of the CO-game (N,v,L)	127
$L(\Gamma)$	solution set of the extensive game Γ	31
$L(G)$	solution set of the strategic game G	29
$L\,(i)$	lower outside option graph	95
$L^+\,(i)$	upper outside option graph	95
$L\,(i,N)$	lower outside option graph	94
$L^+\,(i,N)$	upper outside option graph	94
L^N	complete graph on N	92
$\bar{L}^o$	set of original links in the link agent form	127
$L^{\mathcal{P}}$	graph induced by $\mathcal{P}$	92
$[\lambda,\rho]$	gloves game with λ left gloves and ρ right ones	84
$L\,(s)$	graph induced by strategy profile s	103
$\left(L,v^N\right)$	link game	127
μ	Myerson value	92
μ	system of beliefs	46
$MC_i\,(\sigma)$	marginal contribution of i in σ	75
$MC_i\,(\sigma,v)$	marginal contribution of i under v and σ	92
$\mu(x)$	belief on x	46
$\bar{N}$	player set of the link agent form	127
ν	bijection of information partitions	32
N	player set	75
N	successor correspondence	29

$\mathrm{NF}(\Gamma)$	normal form of Γ	31
$\bar{N}(i)$	set of player i's agents in the link agent form	127
$N_i(L, ij)$	a certain subset of N	113
$N(\bar{K})$	original players induced by $\bar{K}$ in the link agent form	127
(N, p)	partition function form game	85
(N, u_T)	unanimity game	75
(N, v)	a TU game	75
(N, v, L)	a CO-game	92
$(N, v, \mathcal{P})$	a CS-game	75
$N(x)$	successor set of node x	29
o	the root	29
Ow	the Owen value	111
$\mathrm{Ow}^\sharp$	the $\mathrm{Ow}^\sharp$-value	117
π	bijection of player sets	29
π	the position value	127
P	player partition	29
p	collection of chance probabilities	30
p_h	chance probabilities at chance information set h	30
P_i	cell of player i in P	29
$p(P, \mathcal{P})$	worth of the embedded coalition P	85
$\psi(x)$	unordered path of node x	30
$\boldsymbol{\psi}(x)$	(ordered) path of node x	30
$\mathcal{P}(T)$	set of components intersected by T	79
P^x	player partition of Γ^x	30
p^x	chance mechanism of Γ^x	30
P_i^x	set of decision nodes of player i in Γ^x	30
r	bijection of action partitions	32
$\mathbf{r}$	bijection of sets of pure strategy profiles	33
$\mathbf{r}^+$	bijection of sets of action profiles	32
r_i	bijection of pure strategy sets	29
$\mathrm{RNF}(\Gamma)$	reduced normal form of Γ	31
$\langle S \rangle$	atomistic partition on S	111
Σ	set of mixed strategy profiles	29
σ	a mixed strategy profile	29
σ	order on a set	75
S	set of pure strategy profiles	29
Sh	the Shapley value	75
$\sigma(i)$	position of player i in σ	75

Σ_i	set of player i's mixed strategies	29
σ_i	a mixed strategy of player i	29
$\sigma_i(s_i)$	probability of s_i in σ_i	29
S_i	set of player i's pure strategies	29
s_i	a pure strategy of player i	29
$\Sigma(N)$	set of orders on N	75
$\Sigma(N, \mathcal{P})$	set of orders compatible with $\mathcal{P}$	111
$\Sigma(N, \mathcal{P}, \sigma_i, \rho)$	set of certain orders on N	111
Θ	correspondence on sets of terminal nodes	53
θ	bijection of sets of terminal nodes	32
θ	bijection on sets of terminal cells	54
T	node set	29
T^x	node set of Γ^x	30
$\mathcal{U}$	the universe, large enough set containg all player names	91
$\mathbb{U}(\gamma)$	set of payoff functions on γ	53
$u_h(\mu, b)$	conditional payoff at h under μ and b	47
u_i	payoff function of player i	29
$u_i(\mathbf{a}^+)$	player i's payoff of $\mathbf{a}^+$	30
$u_i(b)$	player i's payoff of b	31
u^φ	payoff function under the CO-value φ	103
u_i^x	player i's payoff function in Γ^x	30
$\bar{v}$	payoff function of the link agent form	127
V	predecessor mapping	29
v	coalition function	75
$v \circ \cup$	restriction of v to $C(N, L)$	113
$V(a)$	information set action a belongs to	30
v_C^σ	a modification of v	116
v^L	graph restricted coalition function	92
v^N	modification of v in the link game	127
$v_C^{N,L}$	a modification of v	120
${}^{(k)}v_C^{N,L}$	a modification of v	121
$v_P^{N,P}$	a modification of v	119
${}^{(k)}v_P^{N,P}$	a modification of v	121
$V(x)$	predecessor of node x	29
W	the Wiese value	80
$\mathbb{W}$	set of winning coalitions	82
$\mathbb{W}(\gamma)$	set of chance probabilities of γ	53
$\mathbb{W}_{\min}$	set of minimal winning coalitions	82

X	set of decision nodes	29
X^x	set of decision nodes of Γ^x	30
$[Z]$	set of terminal cells	54
$[z]$	terminal cell containing z	54
Z	set of terminal nodes	29
$[Z]\,(\mathbf{a})$	set of terminal cells reachable by $\mathbf{a}$	54
$Z\,(\mathbf{a})$	subset of Z reachable by $\mathbf{a}$	53
$z(\mathbf{a}^+)$	terminal node reached by $\mathbf{a}^+$	30
$Z(x)$	set of terminal nodes succeeding x	47
Z^x	set of terminal nodes of Γ^x	30

CHAPTER I

Introduction

This thesis consists of seven papers/chapters on game theoretic issues, mostly published, accepted or resubmitted for publication. Their submission/publication status is indicated later on in this introduction; more detailed information is provided at the beginning of the single chapters. The thesis' title "Beyond basic structures in game theory " tries to capture the common theme of the papers, structures which transcend certain "basic" game theoretic structures. In the non-cooperative part (Part 1), we consider isomorphism of extensive games which preserves not only the standard form but also recognizes genuinely sequential features of the extensive form. In the cooperative part (Part 2), we consider TU games with cooperation or bargaining restrictions which are modelled by additional structures.

In the following, we first introduce basic game theoretic concepts in an informal way and then present the motivation and the main results of these papers.

1. Game theoretic structures and related solution concepts

In contrast to decision theory, game theory deals with situations where (in principle) more than one person is involved. There, in general, what one person can achieve depends on what other persons do. In game theory, such situations are modelled by formal structures called games which reflect in a more or less detailed manner those aspects of the underlying situation which are considered to be relevant. The detailedness distinguishes two of the major parts of game theory—non-cooperative game theory (NGT) and cooperative game theory (CGT). Roughly speaking, NGT is strategy-oriented, i.e., it models how players can achieve their objectives, while CGT is payoff-oriented, i.e., it models what players can achieve but not how they can do so (van Damme 1998, pp. 195) .

In this section, we first illustrate concepts from both parts of game theory in an informal way. From NGT, we consider extensive games and its derivatives, the normal form, the agent normal form, and the standard form, and solution concepts as Nash equilibrium and subgame perfect equilibrium. From CGT, we consider TU games which are enriched by coalition structures or cooperation structures and one-point solution concepts as the Shapley value, the Aumann-Drèze value, the Owen value, and the Myerson value.

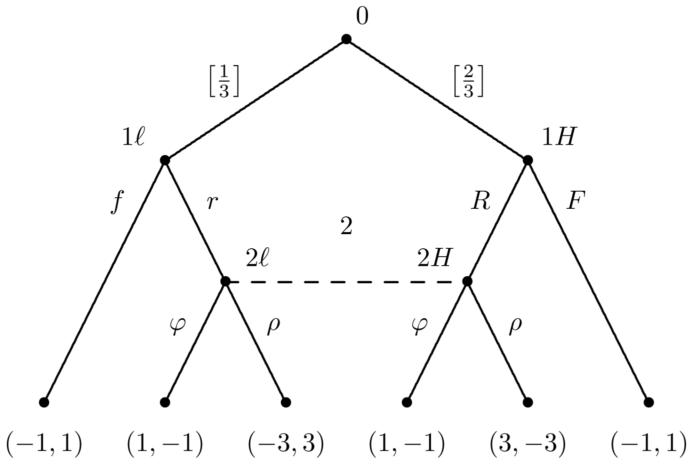


FIGURE 1.1. The extensive game Γ

1.1. Extensive games and its derivatives. The concept of *extensive games*, formally defined in Section II.2.2 (pp. 29), is best understood by an example. We consider an extremely simplified version of "poker". There are just two players, 1 and 2, who both pay an ante (participation fee) of \$1. Player 1 draws one card from a deck. With a probability of $\frac{1}{3}$ he obtains a "lower" card and with a probability of $\frac{2}{3}$ a "higher" one. Unlike player 2, of course, player 1 knows what kind of card he has drawn. Then, player 1 has two options, either to fold or to raise by \$2. If he folds, his ante is forfeited and the game ends. In case player 1 raises, player 2 has the same two options, to fold or to raise by \$2. If he folds, again, his ante is forfeited and the game ends. In the event of both players raising, player 1 takes the pot if he drew a high card, otherwise player 2 is lucky to do so.

This situation can be modelled by the extensive game Γ in Figure 1.1. Its basic ingredient is the *game tree* which models the sequence of possible choices. It consists of nodes (•) and top-down directed edges (——) which connect nodes (•——•). The node on top which has no predecessor is called *the root*; the nodes at the bottom line which have no successors are called *terminal nodes.* Nodes that are not terminal indicate points at which decisions are made, *decision nodes.* The edges pointing downwards from such a decision node stand for the options available to the *player* who decides at this node. The latter is indicated by the player's name placed top left or top right (1ℓ or $1H$ for player 1 and 2ℓ or $2H$ for player 2).

The root models the chance event of player 1 drawing a low or a high card. One could think of a *chance player* 0 who selects a low card (left edge) or a high card (right edge) with the probabilities in brackets. The fact that player 1 knows which kind of card he has drawn is modelled by two decision nodes, 1ℓ and $1H$, where 1ℓ stands for having drawn a low card and $1H$ stands for having drawn a high card. Depending on this information, player 1 can either fold or raise. While option f at 1ℓ stands for folding with a low card, F stands for folding with a high card, for example. If player 1 decides to fold, actions f and F, a terminal node is reached and the game is over. In this case, player 2 has not the opportunity to make any decision. But if player 1 raises at one of his decision nodes, then the information set of player 2 is reached. However, player 2 has no information on the kind of card player 1 has drawn. This is indicated by a dashed line connecting the decision nodes 2ℓ and $2H$.

Collections of decision nodes that cannot be distinguished by the player who controls them are called *information sets*. Whenever a player's information set contains a single node only, this player knows what happened in the course of the game up to this information set. A game in which all information sets are singletons is called to exhibit *complete information*, otherwise the information is *incomplete*. As we have seen, information sets may contain more than one decision node. In this case, of course, at any of these nodes, the player must have the same number of options; otherwise he could distinguish them by these numbers. Options that correspond to the same act are called *actions* and are indicated by labels at the respective edges. For example, the right options at 2ℓ and $2H$ stand for the same act of player 2, to raise, and are both marked as action ρ. Compare this with player 1.

Finally, the terminal nodes represent the results of playing the game. The players' preferences over results are indicated by *payoffs* which usually are viewed as von Neumann-Morgenstern utilities. In our example, the respective net gains are put below the nodes; the left number is for player 1 and the right one for player 2. If, for example, player 1 draws a low card and folds, he looses his ante of \$1, net payoff -1, and player 2 takes the pot of \$2, net payoff 1. Or, if player 1 draws a high card and raises, and player 2 raises too, then player 1 takes the pot of \$6, net payoff 3, and player 2 looses his ante of \$1 and the raise of \$2, net payoff -3.

A (pure) *strategy* of a player is a comprehensive plan for playing the game. Even though player 2 may not have to make any decision, he has two strategies, (φ) and (ρ). In contrast, player 1 has two make plans for two contingencies, drawing a low or a high card, and these plans can be independent. Hence, he has four pure strategies, (f, F), (f, R), (r, F), and (r, R). Any pair of strategies (*strategy profile*) induces a probability distribution on terminal nodes. For example, the strategies (f, R) and (φ) lead to the first and the fourth terminal node from the left with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Or, (r, R) and (ρ) lead to the third and fifth terminal node, again, with probabilities $\frac{1}{3}$ and $\frac{2}{3}$, respectively. Combining these probability distributions with the payoffs at the terminal nodes gives the (*expected*) *payoffs* of strategy profiles. For example, the strategy profile

		2 (φ)	2 (ρ)
1	(f,F)	$-1,1$	$-1,1$
1	(f,R)	$\frac{1}{3},-\frac{1}{3}$	$\frac{5}{3},-\frac{5}{3}$
1	(r,F)	$-\frac{1}{3},\frac{1}{3}$	$-\frac{5}{3},\frac{5}{3}$
1	(r,R)	$1,-1$	$1,-1$

FIGURE 1.2. The normal form NF(Γ)

φ

	F	R
f	$-1,-1,1$	$\frac{1}{3},\frac{1}{3},-\frac{1}{3}$
r	$-\frac{1}{3},-\frac{1}{3},\frac{1}{3}$	$1,1,-1$

ρ

	F	R
f	$-1,-1,1$	$\frac{5}{3},\frac{5}{3},-\frac{5}{3}$
r	$-\frac{5}{3},-\frac{5}{3},\frac{5}{3}$	$1,1,-1$

FIGURE 1.3. The agent normal form ANF(Γ)

$((f,R),(\varphi))$ yields the payoffs

$$\frac{1}{3}\cdot(-1)+\frac{2}{3}\cdot 1=\frac{1}{3} \qquad \text{and} \qquad \frac{1}{3}\cdot 1+\frac{2}{3}\cdot(-1)=-\frac{1}{3}$$

to players 1 and 2, respectively. Analogously, the strategy profile $((r,R),(\rho))$ gives the payoffs

$$\frac{1}{3}\cdot(-3)+\frac{2}{3}\cdot 3=1 \qquad \text{and} \qquad \frac{1}{3}\cdot 3+\frac{2}{3}\cdot(-3)=-1.$$

The assignment of payoffs to strategy profiles suggests a much less complex representation of Γ as in Figure 1.2 called the *normal form* of Γ, NF (Γ) (von Neumann & Morgenstern 1944) . The normal form of Γ just comprises the players, their strategies, and all players' payoffs for all strategy profiles. Transforming Γ into NF (Γ) some information is lost, in particular on the structure of information sets and on the sequence of actions. Nevertheless, one could argue that in the course of playing an extensive game no new information emerges. Hence, the players could foresee all contingencies and independently/simultaneously plan their behavior in advance. Since in real life people frequently make simultaneous decisions, normal form games, also called *strategic games* and formally defined in Section II.2.1 (pp. 29), are of interest independent of an underlying extensive game.

Another representation of Γ is presented in Figure 1.3, the *agent normal form* ANF (Γ) due to Selten (1975) , formally defined in Sections IV.2 (pp. 64) and II.2.1 (pp. 29). As the normal form, ANF (Γ) is a strategic game. In contrast to NF (Γ) , there are separate players for each information set, the information set agents. The agent for player 1's information set at 1ℓ chooses the row, the agent at $1H$ the column, and the agent for player 2's single information set chooses the matrix. These agents are assigned the same payoffs as the player to whom they belong; the payoffs are listed from left to right for the agents at 1ℓ, $1H$, and 2, respectively.

At first glance, ANF (Γ) may seem to contain more information than NF (Γ) . This is not the case. On the one hand, ANF (Γ) contains more information on the information sets than NF (Γ) , on the other hand, ANF (Γ) does not say anything about the assignment of agents to players, i.e., in this respect, ANF (Γ) contains less information than NF (Γ) . Adding this information to ANF (Γ) , one obtains the standard form SF (Γ) of Γ due to Harsanyi & Selten (1988) , formally defined in Sections II.2.1 (pp. 29) and IV.2 (pp. 64), which is more complex than both ANF (Γ) and NF (Γ) .

So far, we have represented aspects of real-life situations as games. In particular, we have formal expressions for the choices available to the persons involved. Although that may not be completely without interest in itself, one certainly would like have some predictions on the players' behavior or one would like to give them advice on how to behave, i.e., one would like to have some device to single out one or some courses of action in a game. One such device are equilibrium concepts.

The most basic equilibrium concept is the Nash (1950) equilibrium. A *Nash equilibrium* (equilibrium, for short) is a strategy profile, i.e., a comprehensive plan of action for all players, where no player wishes to deviate from his plan unilaterally. Again, this is best explained by examples.

Consider the extensive game Γ in Figure 1.4 and its normal form NF (Γ) in Figure 1.5 where player 1's payoff is the left one. These games have two equilibria, the strategy profile (L, r) and the strategy profile (R, ℓ) . Let us check this. If just player 1 deviates from L to R in (L, r), his payoff decreases, $-1 < 1$; and if player 2 deviates from r to ℓ, his payoff remains unchanged, $1 = 1$. Hence, no player gains by unilaterally deviating from (L, r) , i.e., (L, r) is an equilibrium. Similarly, the players cannot gain by unilaterally deviating from (R, ℓ) , $2 > 1$ and $0 > -1$. Hence, (R, ℓ) also is an equilibrium. In contrast, the strategy profiles (L, ℓ) and (R, r) are not so. For example, player 1 gains by choosing R in (L, ℓ) , $2 > 1$, and player 2 gains by choosing ℓ in (R, r) , $0 > -1$.

Yet in Γ, the equilibrium (L, r) is much less convincing than (R, ℓ): In (L, r) , player 2 intends to choose r which—if he were to choose at all—would be less favorable than ℓ, $0 > -1$. Then, of course, player 1 prefers L which is the best of all worlds for player 2,

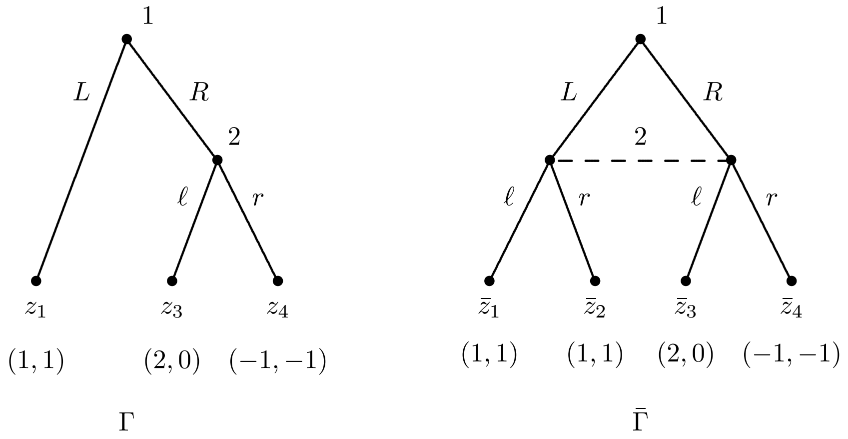


FIGURE 1.4. Two extensive games

	ℓ	r
L	$1, 1$	$1, 1$
R	$2, 0$	$-1, -1$

FIGURE 1.5. The normal form $\text{NF}(\Gamma) = \text{NF}(\bar{\Gamma})$

i.e., which gives him the highest possible payoff, 1. Usually, this is paraphrased by saying that (L, r) rests upon an incredible threat or that (L, r) involves irrational behavior at unreached information sets. If player 1 chooses L, then player 2's information set is not reached. In this case, the decision of player 2 does not affect the outcome. Hence, he is indifferent between his actions and thus may plan to take the "bad" action r.

In order to preclude sequentially irrational behavior as incredible threats, several so-called refinements of Nash equilibrium have been introduced, both for extensive games and for strategic games. The most basic refinement for extensive games is *subgame perfect equilibrium* (SPE) which—as its name suggests—involves substructures of extensive games called *subgames,* formally defined in Section II.2.2 (pp. 29). Roughly speaking, a subgame is a part of a game which is strategically independent of the rest of the game, i.e., the choices outside the subgame do not interfere with the choices inside. Any extensive game is a (trivial) subgame of itself because there is no outside. All other subgames are called proper. More precisely, a subgame consists of some node and all of its successors, and it inherits the structure of the original game on this node set. Once more, this is illustrated by an example. Intuitively, it is clear that the game Γ in Figure 1.4 has one proper subgame, namely, the one starting at the unique decision node of player 2. This subgame consists of three nodes, player 2's decision node which is the unique decision node and the two terminal nodes z_3 and z_4.

A SPE is a strategy profile which induces equilibria in any subgame. Since this includes the whole game, any SPE also is an equilibrium, i.e., SPE refines the Nash equilibrium concept. In Γ, there are two candidates for a SPE, (L, r) and (R, ℓ), the equilibria of Γ itself. Yet, in the only non-trivial subgame as identified above, just player 2 has to make a decision, to choose r or ℓ. Since ℓ gives the higher payoff, $0 > -1$, choosing ℓ is the unique equilibrium in this subgame. Hence, (R, ℓ) is the unique SPE in Γ. This shows that SPE may indeed sort out sequentially irrational behavior as incredible threats, for example.

In more complex games exhibiting incomplete information, SPE may be much less useful. For example, the game $\bar{\Gamma}$ in Figure 1.4 has no proper subgame implying that any equilibrium is a SPE. Since $\bar{\Gamma}$ has the same normal form as Γ, (L, r) is an equilibrium of $\bar{\Gamma}$ which is not discarded by SPE. However, (L, r) is implausible in $\bar{\Gamma}$—if player 2 is slightly uncertain that player 1 chooses L, then he prefers action ℓ. In order to avoid such equilibria, more powerful equilibrium concepts have been introduced, perfect equilibrium (Selten 1975) , sequential equilibrium (Kreps & Wilson 1982) , defined in the proof of Theorems II.5.2 (pp. 46) or III.3.7 (pp. 60), or quasi-perfect equilibrium (van Damme 1984) , defined in Remark IV.3.7 (pp. 67).

In general, the standard form contains much less information than the underlying extensive game. In a sense, an extensive game can be viewed as an extension of the derived strategic games, the normal form, the agent normal form, and the standard form. Two interesting questions arise. Do these derivatives comprise all of the strategically relevant information contained in an extensive game, for example, the information required to determine equilibria of some kind? How can this strategically relevant part be characterized? The papers in Part 1 attempt to answer these questions.

1.2. TU games, coalition structures, and cooperation structures. Again, the concept of a coalitional game with transferable utility (TU game), formally defined in Section V.2 (pp. 75), is best understood by an example. We consider an entrepreneur, E, who owns the capital, and two workers, $W1$ and $W2$. Neither the entrepreneur alone nor one or both of the workers alone are productive. If the entrepreneur and worker 1 cooperate, they can produce a net gain of \$3. Worker 2 is less productive; together with the entrepreneur, a net gain of \$2 can be achieved. If all of them cooperate, they can create a net gain of \$6.

Such a situation can be modelled by a TU game (N, v) as follows. There is a player set $N = \{E, W1, W2\}$ which contains the "names" of all players. The *coalition function* v assigns a worth $v(S)$ to any *coalition* S (subset of N) where the empty coalition $\emptyset$ obtains a zero worth, $v(\emptyset) = 0$. In our example, we have

$$v(S) = \begin{cases} 0, & |S| \leq 1 \text{ or } S = \{W1, W2\}, \\ 2, & S = \{E, W2\}, \\ 3, & S = \{E, W1\}, \\ 6, & S = N. \end{cases}$$

Imagine two constellations: (A) The entrepreneur E employs the more productive worker $W1$ while worker $W2$ remains unemployed. Then, E and $W1$ create a worth of $v(\{E, W1\}) = 3$ and $W2$ creates $v(\{W2\}) = 0$. (B) Both workers are employed, i.e., a worth of $v(N) = 6$ is produced, but they are organized within a trade union. It is clear that such constellations transcend the structural features embodied in TU games.

In order to model these circumstances, TU games are enriched by so-called *coalition structures*, i.e., partitions of the player set N, formally introduced in Section V.2 (pp. 75). The so-called *components* of a partition exhaust the player set and are mutually disjoint, i.e., any player is a member of exactly one component. Constellation (A) corresponds to the coalition structure

$$\mathcal{P} = \{\{E, W1\}, \{W2\}\}.$$

The cooperating players E and $W1$ are gathered in one component, $\{E, W1\}$, while the excluded player $W2$ forms a singleton component, $\{W2\}$. Similarly, constellation (B) is reflected by the coalition structure

$$\mathcal{B} = \{\{E\}, \{W1, W2\}\}.$$

In a sense, the component $\{W1, W2\}$ stands for the trade union, which E is not a member of.

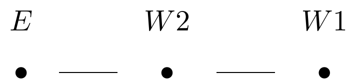
Obviously, the coalition structures $\mathcal{P}$ and $\mathcal{B}$ have different interpretations. The components in $\mathcal{P}$ are productive units, i.e., the players in any component $P \in \mathcal{P}$ pool their resources and cooperate in order to create the worth $v(P)$, excluding outsiders from doing so. In contrast, the components of $B \in \mathcal{B}$ can best be viewed as bargaining blocs within the grand coalition N: All players cooperate in the production of worth, i.e., the worth $v(N)$ is created, but they form alliances in order to strengthen (hopefully) their position in bargaining on the distribution of that worth.

Finally, consider a third constellation (C). Now, worker $W2$ is unproductive, i.e., concerning the net gain, it does not matter whether $W2$ is present or not. This can be modelled by the coalition function w as follows:

$$w(S) = \begin{cases} 0, & |S| \leq 1 \text{ or } S = \{W1, W2\} \text{ or } S = \{E, W2\}, \\ 3, & S = \{E, W1\} \text{ or } S = N. \end{cases} \tag{1.1}$$

Further, the entrepreneur E and the productive worker $W1$ can cooperate only because they communicate via worker $W2$ or because both of them have bilateral cooperation agreements with player $W2$ but not with each other. In both cases, E and $W1$ need $W2$ in order to cooperate and to produce the worth of $w(\{E, W1\})$, even though worker $W2$ is not productive, i.e., $w(\{E, W1\}) = w(N)$. Again, this constellation transcends the world of TU games, even if they are enriched by coalition structures.

In order to model such circumstances, TU games are extended by so-called *cooperation structures*, i.e., undirected graphs on the player set N, formally defined in Section VI.2 (pp. 91). Remember, an undirected graph on a set N consists of a set of links which connect different players. Constellation (C) can be modelled by the following graph:



(1.2)

The link set L consists of the links $\{E, W2\}$ and $\{W1, W2\}$ which stand for communication channels or bilateral agreements that enable cooperation. Players who are not connected via (a chain of) links cannot cooperate, neither can they create worth together nor can they bargain together.

So far, again, we represented aspects of real-life situations as games and additional structures. In particular, we know what any coalition can achieve if its members cooperate and, possibly, we know which players actually do so. Now, one certainly would like to have some predictions on the single players' proceeds, i.e., one would like to have some device to determine them. One such device are values which assign to every player a unique payoff.

The best-known and widely employed such value for TU games is the Shapley (1953) value which also is the point of departure of a number of value concepts for TU games with additional structures. The intuition behind his value is the following: Suppose the players enter a room in some order. Each player is payed the difference of worth of the coalition in the room after he entered the room and before he did so, which is called his *marginal contribution* under the order under consideration, formally defined in Section V.2 (pp. 75). The Shapley value, Sh, formally defined in Section V.2 (pp. 75), assigns the average of these marginal contributions over all orderings. This is illustrated with our example.

σ	$\begin{bmatrix} E \\ W1 \\ W2 \end{bmatrix}$	$\begin{bmatrix} E \\ W2 \\ W1 \end{bmatrix}$	$\begin{bmatrix} W1 \\ E \\ W2 \end{bmatrix}$	$\begin{bmatrix} W1 \\ W2 \\ E \end{bmatrix}$	$\begin{bmatrix} W2 \\ E \\ W1 \end{bmatrix}$	$\begin{bmatrix} W2 \\ W1 \\ E \end{bmatrix}$	Sh_i
MC_E^σ	0	0	3	6	2	6	$\frac{17}{6}$
MC_{W1}^σ	3	4	0	0	4	0	$\frac{11}{6}$
MC_{W2}^σ	3	2	3	0	0	0	$\frac{8}{6}$

TABLE 1.1. Marginal contributions and Shapley payoffs

Consider the order $\sigma = [E, W1, W2]$ where the left player is the first one and the right player the last one. The marginal contributions $MC(\sigma)$ are given as follows:

$$\begin{aligned} MC_E^\sigma(\sigma) &= v(\{E\}) - v(\emptyset) = 0 - 0 = 0 \\ MC_{W1}^\sigma(\sigma) &= v(\{E, W1\}) - v(\{E\}) = 3 - 0 = 3 \\ MC_{W2}^\sigma(\sigma) &= v(\{E, W1, W2\}) - v(\{E, W1\}) = 6 - 3 = 3 \end{aligned}$$

In a similar fashion, one easily calculates the marginal contributions for the remaining five orders. Table 1.1 lists the results, where the orders are to be read top-down. Averaging over all orders finally yields the Shapley payoffs listed in the last column.

One easily checks that the Shapley value is efficient, i.e., the players' payoffs add up to the worth of the grand coalition $v(N) = 6$. This corresponds to the interpretation of a value as modelling production and bargaining on the distribution of worth in the grand coalition.

Departing from the Shapley value, there is a number of values that apply to situations where production or bargaining is restricted by some structure on the player set. In the following, we briefly illustrate three such concepts by examples: the Aumann & Drèze (1974) value (henceforth AD-value), the Owen (1977) value, and the Myerson (1977) value. The first two apply to coalition structures (CS-values) and the last one to cooperation structures (CO-value). While the second value is efficient, the other two values are component efficient, i.e., under these values, the players of a component obtain the component's worth.

The idea of the AD-value, formally defined in Section VI.2 (pp. 92), is to apply the Shapley value to the restriction of the original TU game to the single components of the coalition structure. Reconsider the TU game (N, v) and the coalition structure $\mathcal{P}$

above. The coalition structure $\mathcal{P}$ contains two components, $\{E, W1\}$ and $\{W2\}$. The game restricted to $\{W2\}$ contains the player $W2$ only who is assigned the payoff $v(\{W2\}) = 0$ by component efficiency of the AD-value. In the game restricted to $\{E, W1\}$, the players are symmetric which here means that they create the same worth when they stand alone. Together with component efficiency, this implies that the Shapley value splits the worth of $v(\{E, W1\}) = 3$ equally between them. Altogether, we thus have

$$\mathrm{AD}_E(N, v, \mathcal{P}) = \mathrm{AD}_{W1}(N, v, \mathcal{P}) = \frac{3}{2}, \qquad \text{and} \qquad \mathrm{AD}_{W2}(N, v, \mathcal{P}) = 0.$$

Indeed, E and $W1$ together earn $v(\{E, W1\}) = 3$ and $W2$ earns $v(\{W2\}) = 0$. In $(N, v, \mathcal{P})$, the equal split between E and $W1$ indicates that the AD-value considers the components of $\mathcal{P}$ to be "islands" which do not interact with the other ones.

The Owen value, Ow, formally defined in Section VII.2 (pp. 111), results from another kind of restriction of the Shapley value. Remember, the Shapley payoffs are calculated as the average of the marginal contributions over all orderings of players. Instead, the Owen value just considers those orders where the players within a component are kept together, i.e. in terms of our visualization, the players of a component immediately follow each other when they arrive at the "room" evoked above.

In our example, there are six orders indicated in Table 1.1. But only four of them are compatible with $\mathcal{B}$, $[E, W1, W2]$, $[E, W2, W1]$, $[W1, W2, E]$, and $[W2, W1, E]$. The order $[W1, W2, E]$, for example, does not respect $\mathcal{B}$—E and $W1$ inhabit the same component but are separated by $W2$. Averaging the marginal contributions from Table 1.1 over the orders compatible with $\mathcal{B}$ gives

$$\mathrm{Ow}_E(N, v, \mathcal{B}) = 3, \qquad \mathrm{Ow}_{W1}(N, v, \mathcal{B}) = \frac{7}{4}, \qquad \text{and} \qquad \mathrm{Ow}_{W2}(N, v, \mathcal{B}) = \frac{5}{4}.$$

Compared with unrestricted bargaining modelled by the Shapley value, player E gains while $W1$ and $W2$ loose. This seems to be quite plausible: Under $\mathcal{B}$, the workers loose flexibility in bargaining which hurts because their marginal contributions increase with the coalition size.

We now turn to the Myerson value, μ, formally defined in Section VI.2 (pp. 92). In a sense, μ also results from some kind of restriction concerning the Shapley value. Roughly speaking, the ability of the players to cooperate productively is restricted by their ability to communicate via links. Players who are connected via a chain of links can cooperate while player who are separated cannot. This induces a graph restricted version of the original coalition function.

We explain this with constellation (C) of our example above. One obtains the graph restricted payoff function w^L from w in (1.1) as follows: It is easy to see from the graph L in (1.2) that all coalitions—with exception of coalition $\{E, W1\}$—are connected by internal links. For example, coalition $\{E, W2\}$ is connected via the link $\{E, W2\}$ which runs completely within $\{E, W2\}$. Also, the players in N are connected: E and $W2$ as well as $W1$ and $W2$ are linked directly while E and $W1$ are connected via the chain of links $(\{E, W2\}, \{W2, W1\})$. Of course, all these links run within N. Since the chain $(\{E, W2\}, \{W2, W1\})$ does not run entirely within the coalition $\{E, W1\}$, E and $W1$ are not connected internally. Hence, the players in all coalitions, except of $\{E, W1\}$, can cooperate. We set $w^L(S) = w(S)$ if $S \neq \{E, W1\}$ and $w^L(\{E, W1\}) = w(\{E\}) + w(\{W1\})$, i.e. we have

$$w^L(S) = \begin{cases} 0, & |S| \leq 2, \\ 3, & S = N. \end{cases} \tag{1.3}$$

The Myerson payoffs of (N, v, L) are the Shapley payoffs for (N, v^L). Calculations as in Table 1.1 yield

$$\mu_E(N, v, L) = \mu_{W1}(N, v, L) = \mu_{W2}(N, v, L) = 1,$$

i.e., the grand coalition's worth $v(N) = 3$ is split equally. Since all players are needed in order to create a non-zero worth, this result is quite intuitive.

The AD-value and the Myerson value share one deficiency: They do not account for outside options, i.e., the players' productive and linking potential outside their own component; in order to determine a player's payoff one can restrict attention to his component. Further, there is no efficient value for games with a cooperation structure. The papers in Part 2 attempt to overcome this deficiency and to fill the gap. Finally, the last paper establishes a relation between two values for games with a cooperation structure.

2. Isomorphism of extensive games

In this section, we motivate the papers in Part 1 and indicate their main results where the subsections refer to the single papers.

Roughly speaking, an isomorphism from one game to another is a bijective mapping of the players' actions or strategies that respects those aspects of a game which are considered to be relevant. This is explained with an example. Reconsider the games Γ and $\bar{\Gamma}$ in Figure 1.4. Are these games isomorphic? Should we consider them to be so?

Obviously, the games are different. First, their game trees differ concerning the number of nodes and edges. Less technically, Γ exhibits complete information, while in $\bar{\Gamma}$, player 2 does not know which action player 1 has chosen. Nevertheless, both games have much in common: There are two players who both have two actions. Even stronger, the games share the same standard form depicted in Figure 1.5. Hence, the identity mapping on the actions establishes an isomorphism of the standard forms (henceforth SFI), formally defined in Section IV.2 (pp. 64), i.e., the games are standard-form isomorphic. Yet, as we have already seen in Section 1.1 (pp. 6), the strategy profile (L, r) is a SPE in $\bar{\Gamma}$, but it is not so in Γ. So, one could argue that the structural features reflected by SFI, in general,

	t	f
T	$3,1$	$0,0$
F	$0,0$	$1,3$

FIGURE 2.1. The battle-of-the-sexes game G

do not suffice to apply the sequential rationality considerations embodied in the concept of subgame perfect equilibrium.

SPE shares this property with sequential equilibrium (henceforth SEQ) and quasi-proper equilibrium (henceforth QPE). In contrast, the agent normal form already contains the information needed to determine all perfect equilibria. Yet, perfect equilibrium shares some major drawback with SEQ. Moreover, Mertens (1995) argues that QPE seems to exhibit the right combination of desirable properties as for example reflecting sequential rationality considerations. Therefore, one might be interested in concepts of isomorphism for extensive games under which QPE is invariant. As we have seen, such concepts have to preserve structural features of extensive games beyond the standard form. The papers in Part 1 answer this challenge by successively weakening the concept of strong isomorphism (Elmes & Reny 1994, Peleg, Rosenmüller & Sudhölter 1999) .

2.1. Weak isomorphism of extensive games. This chapter has been published as "André Casajus (2003): Weak isomorphism of extensive games, in: *Mathematical Social Sciences* 46, 267–290", henceforth CA03.

Consider the Battle-of-the-Sexes game G in Figure 2.1 where *he* chooses the row and *she* chooses the column. The strategies F and f stand for going to a football match, and T and t stand for going to the theatre. The payoffs indicate that both prefer to be together but even more prefer to be together at the favored event, *he* at the theatre and *she* at the football match.

In a sense, the players and the strategies T and f on the one hand, and F and t on the other hand are symmetric—one could interchange them without changing the game. Traditionally, G is represented by the extensive games Γ or $\bar{\Gamma}$ in Figure 2.2. So one could argue that the players and the corresponding actions should also be symmetric in Γ and in $\bar{\Gamma}$.

Peleg et al. (1999) (henceforth PRS) introduce strong isomorphisms of extensive games that preserve the tree structure underlying the games. This implies that a strong isomorphism cannot interchange the order of actions within a path from the root to a terminal node. Hence, the identity mapping on actions does not constitute a strong isomorphism from Γ to $\bar{\Gamma}$. For example, the left terminal of Γ is reached via the sequence

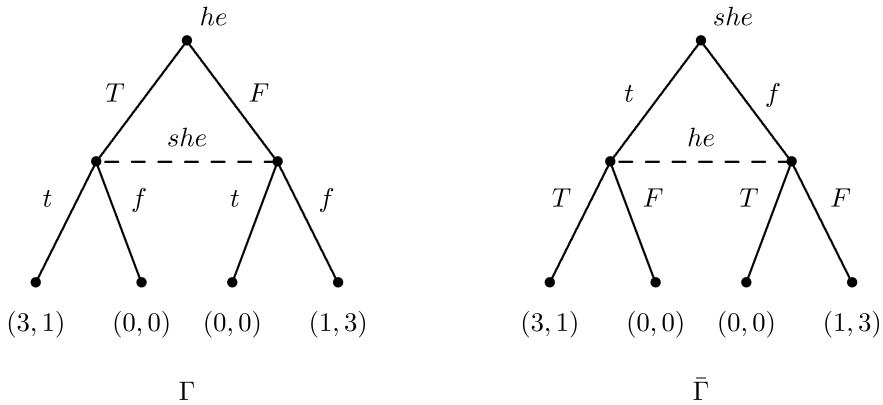


FIGURE 2.2. Traditional extensive representations of G

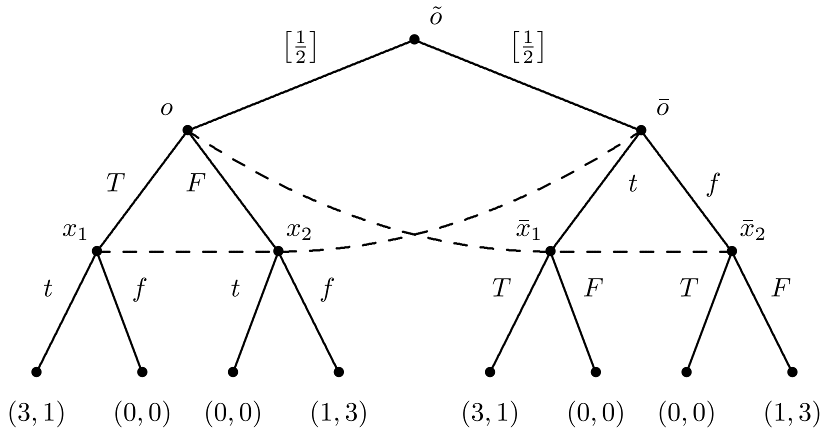


FIGURE 2.3. The PRS representation $\hat{\Gamma}$ of G

(T, t) of actions. Yet, there is no such terminal node in $\bar{\Gamma}$. Note, that the left terminal node in $\bar{\Gamma}$ is reached via (t, T), i.e. the order of T and t is reversed. Analogously, a strong isomorphism cannot interchange the players in Γ or in $\bar{\Gamma}$, i.e. the players are not symmetric under strong isomorphism.

In order to resolve this peculiarity, Sudhölter, Rosenmüller & Peleg (2000) (also PRS) introduce an alternative extensive representation for which both notions of symmetry coincide. Figure 2.3 gives the PRS representation $\hat{\Gamma}$ of G where $\tilde{o}$ is a chance node with the respective probabilities in brackets, *he* controls o, $\bar{x}_1$, and $\bar{x}_2$, and *she* controls $\bar{o}$, x_1, and x_2. Obviously, the PRS representation is more complex and more difficult to deal with than the traditional one. So, a concept of isomorphism of extensive games that fits the traditional representation of strategic games seems to be desirable.

To achieve this, we extend the Selten (1983) and Oh (1995) symmetries into weak isomorphisms of extensive games. In contrast to strong isomorphism, we do not require that the order of actions on a path from the root to a terminal is respected. What we require is that the *set* of actions on a path is preserved. This way, the identity mapping on actions constitutes a weak isomorphism. For example the set of actions $\{T, t\}$ leading to the left terminal node in Γ is mapped onto $\{T, t\}$ which also is the set of actions leading to the left terminal node in $\bar{\Gamma}$. Analogously for the other terminal nodes. Hence, as desired, in Γ and in $\bar{\Gamma}$, the players are symmetric with respect to weak isomorphism.

Weak isomorphism may interchange the order of actions in a game tree. Hence, one may wonder whether perfect equilibrium or sequential equilibrium are invariant under weak isomorphism, i.e., whether equilibria of the same type are mapped onto each other. Fortunately, this is the case.

Weak isomorphism exhibits another interesting property for games without chance player and where all information sets have at least two actions. Even though the agent normal form is less complex than the underlying extensive game, in generic cases, i.e. almost always, extensive games are weakly isomorphic if and only if their agent normal forms are isomorphic.

2.2. Super weak isomorphism of extensive games. This chapter has been published as "André Casajus (2006): Super weak isomorphism of extensive games, in: *Mathematical Social Sciences* 51, 107–116", henceforth CA06.

This paper departs from two results of CA03: (i) Sequential equilibrium is invariant under weak isomorphism. (ii) Weak isomorphism is generically equivalent to isomorphism of the agent normal form for the games without chance player and where all information sets have at least two actions. Now, one may wonder whether there is some relaxation of weak isomorphism which is (a) equivalent to isomorphism of the agent normal form for a larger class of games, but (b) such that sequential equilibrium remains invariant under this relaxation of weak isomorphism. Since the agent normal form does not contain any information on the chance mechanism besides its effect on the payoffs, such a relaxation would have to ignore the structure and the embedding of the chance mechanism which are preserved by weak isomorphism to a large extent.

We relax weak isomorphism into a concept called super weak isomorphism, basically by dropping its requirements related to the structure and the embedding of the chance mechanism. Since all paths to a terminal node in $\hat{\Gamma}$ in Figure 2.3 contain one chance node, the game Γ in Figure 2.2 cannot be weakly isomorphic to its PRS representation $\hat{\Gamma}$. Since super weak isomorphism neglects chance actions, the identity mapping on actions establishes such an isomorphism from Γ to $\hat{\Gamma}$. The set $\{T, t\}$ of non-chance actions leading to the left terminal node in Γ is mapped onto the set $\{T, t\}$ in $\hat{\Gamma}$ which actually is the set of non-chance actions leading to the left terminal node and to the fourth from right terminal node in $\hat{\Gamma}$. Analogously for the other terminal nodes.

It turns out that perfect equilibrium and sequential equilibrium remain invariant under super weak isomorphism. However, super weak isomorphism and isomorphism of the agent normal form do not coincide generically on the class of games where all information sets have at least two actions, i.e., there are great many cases where both concepts diverge. At least, some progress is made in this respect. There is a (small) class of games with a chance mechanism for which equivalence in the above sense holds. For example, the game $\hat{\Gamma}$ is contained in this class.

2.3. Strong agent normal form isomorphism. This paper takes up the question concluding CA06 whether super weak isomorphism can be further relaxed towards generic equivalence to isomorphism of the agent normal form without loosing the invariance of sequential equilibrium.

As an answer, we introduce the concept of strong agent normal form isomorphism which strengthens isomorphism of the agent normal form. The point of departure is the following observation. Though the games Γ and $\bar{\Gamma}$ in Figure 1.4 have the same (agent) normal form (Figure 1.5), they have different sets of sequential equilibria. While the strategy profile (L, r) is a sequential equilibrium in $\bar{\Gamma}$, it is not so in Γ. Yet, the coincidence of the agent normal forms of Γ and $\bar{\Gamma}$ crucially rests upon the fact that two of the terminal nodes in $\bar{\Gamma}$, $\bar{z}_1$ and $\bar{z}_2$, are assigned the same payoffs. Under slight perturbations of these payoffs, Γ and $\bar{\Gamma}$ no longer have the same/isomorphic agent normal form.

This gives rise to our notion of a strong agent normal form isomorphism. Basically, a strong agent normal form isomorphism is an agent normal form isomorphism which remains an agent normal form isomorphism under (slight) perturbations of payoffs. Note that, since payoff are assigned to terminal nodes, strong agent normal form isomorphism is a genuine extensive form concept.

It turns out that strong agent normal form isomorphism does its intended job: While being generically equivalent to its weaker cousin, sequential, perfect, and quasi-perfect equilibrium remain invariant. Moreover, super weak isomorphism and strong agent normal form isomorphism do not coincide generically, i.e., there are great many cases where they diverge.

3. Outside options and communication restrictions in TU games

In this section, we motivate the papers in Part 2 and indicate their main results where the subsections refer to the single papers.

	AD-value Owen value	Wiese value	Shapley value	χ-value	core
left with right	.5000	.7167	.7333	.8000	1
right with left	.5000	.2833	.1333	.2000	0
single right	0	0	.1333	0	0

TABLE 3.1. Payoffs for the gloves game

3.1. Outside options, component efficiency, and stability. This chapter has been published "André Casajus (2009): Outside options, component efficiency, and stability, in: *Games and Economic Behavior* 65 (1), 49–61".

Consider a gloves game (Shapley & Shubik 1969) with two left-glove holders and four right-glove holders where the worth of a coalition is the number of matching pairs it contains. Further, let there be two matching-pair coalitions whereas the remaining right-glove holders stand alone. This situation corresponds to a TU game with the player set $N = R \cup L$, where the sets $R := \{R1, R2, R3, R4\}$ and $L := \{L1, L2\}$ represent the right-glove holders and the left-glove holders, respectively. The coalition function v and the coalition structure $\mathcal{P}$ are given by

$$v(K) = \min\{|K \cap R|, |K \cap L|\}, \qquad K \subseteq N$$

and

$$\mathcal{P} = \{\{R1, L1\}, \{R2, L2\}, \{R3\}, \{R3\}\}.$$

How should the players in a matching-pair coalition split the worth of 1? In order to answer such questions, several values for TU games with a given coalition structure have been introduced. Table 3.1 lists their payoffs. Interestingly, both the AD-value and the Owen value split the worth of 1 equally between the members of a matching-pair coalition. I.e., these values are insensitive to outside options which in the present context means that they do not respond to the relative scarcity of the left gloves.

In contrast to the AD- and the Owen value, the unique core payoffs give the whole worth of 1 to the left-glove holders. I.e., the core neglects the productive role of a right-glove holder within a *given* matching-pair coalition. Only recently, Wiese (2007) suggested another component efficient CS-value which steers a course between these extreme positions. This can be seen from the Wiese payoffs listed in Table 3.1. On the one hand, the payoff of a left-glove holder is higher than that of the right-glove holder in his coalition—the Wiese value accounts for outside options. On the other hand, a right-glove holder in a matching-pair coalition obtains a higher payoff than the single right-glove holders—the Wiese value recognizes the productive role of right-glove holders in the matching-pair coalitions.

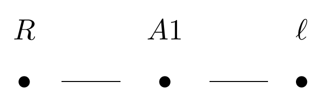
Nevertheless, the Wiese value has some drawbacks. Most notably, it lacks a "nice" axiomatization. In essence, there is a not too intuitive ad-hoc specification of the payoffs for unanimity games which is expanded by linearity to the whole class of games. Further, it is not yet clear whether there are stable coalition structures (in the sense of Hart & Kurz 1983) with respect to the Wiese value for all TU games.

In order to remedy these deficiencies, we introduce a component efficient CS-value—the χ-value. The main idea underlying the χ-value is that splitting a structural coalition affects players who remain together in the same component in the same way. Together with some other axioms, this property characterizes the χ-value which easily can be computed from the Shapley value. Further, it turns out that stable coalition structures with respect to the χ-value exist for all TU games. The χ-payoffs for our example in Table 3.1 indicate that the χ-value balances outside options and the contribution to ones own coalition.

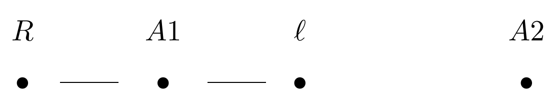
3.2. Outside options in TU games with a cooperation structure. A revised version of this chapter has been published as "André Casajus (2009): Networks and outside options, in: *Social Choice and Welfare* 32 (1), 1–13".

One right-glove holder, R, and one left-glove holder, ℓ, actually sell their pair of gloves which is worth 1 via some agent $A1$. How should R, ℓ, and $A1$ split the proceeds? Would this split change if there were a second agent $A2$? In order to answer this kind of questions, one can employ the Myerson value.

Our example corresponds to a TU game with 3 (or 4) players, R, ℓ, $A1$, (and $A2$), where the worth of a coalition is 1 if it contains a matching pair, i.e. the players R and ℓ, and is 0 if it does not so. The fact that R and ℓ sell their pair via $A1$ can be modelled by the following graphs:



(3.1)



(3.2)

In both cases, the Myerson value μ assigns the same payoffs to R, ℓ, and $A1$, $\mu_R = \mu_\ell = \mu_{A1} = \frac{1}{3}$. Though $A1$ is not productive, he obtains a positive payoff for his intermediation.

Yet, a bit unintuitively, the share of $A1$ is not affected by the presence of the potential competitor $A2$. Thus, the Myerson value does not account for the outside option of R and ℓ to sell their pair of gloves via $A2$. Once more we emphasize the importance of outside options.

The Myerson value shares this neglect of outside options with the AD-value. In order to remedy this peculiarity of the AD-value, Wiese (2007) and Casajus (2009)[1] introduce the outside-option value and the χ-value, respectively. Hence, it seems to be worthwhile to look for a CO-value which generalizes these concepts.

As an attempt, we introduce and axiomatize the "graph-χ-value", $\chi^{\#}$, which extends the χ-value to CO-games and thus accounts for outside options. To achieve this, we restrict the crucial axiom of the original characterization of the Myerson value, the fairness axiom, to situations without outside options or where outside options are not affected. An outside-option consistency axiom determines how players within the same component assess their outside options and restores the uniqueness lost by relaxing the fairness axiom. It turns out that the $\chi^{\#}$-value generalizes the χ-value—if all possible links within the components are present, then the $\chi^{\sharp}$-value and the χ-value coincide. For our example, we obtain the following payoffs: If $A2$ is not present, then the payoffs are as for the Myerson value. But in presence of $A2$, the payoff of $A1$ decreases. In particular, we have $\chi^{\#}_R = \chi^{\#}_\ell = \frac{4}{9}$ and $\chi^{\#}_{A1} = \frac{1}{9}$ which shows that the $\chi^{\#}$-value rewards outside options without neglecting the role of player $A1$ as intermediary.

Further, we explore some properties of this CO-value. In particular, we further clarify its relation to the χ-value and demonstrate the difference to the Myerson value concerning stability issues.

3.3. An efficient value for TU games with a cooperation structure. This chapter was submitted for publication in the *International Journal of Game Theory* in September 2006 and resubmitted in October 2007.

Consider the TU game with the player set $N = \{P1, P2, P3, A\}$ and the coalition function given by

$$v(K) = \begin{cases} 1 & , |K \cap \{P1, P2, P3\}| > 1, \\ 0 & , |K \cap \{P1, P2, P3\}| \leq 1, \end{cases} \qquad , K \subseteq N.$$

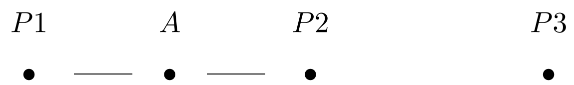
A is not productive and the presence of any two of the productive players $P1$, $P2$, and $P3$ already suffices to produce the worth of 1. Suppose all these players cooperate in order to create the grand coalition's worth of $v(N) = 1$. If the players do not form any coalitions when bargaining on the distribution of $v(N)$, then, for symmetry reasons, one would expect an equal split between the three productive players. Would/should this

[1] Also Chapter V of this thesis.

split change if $P1$ and $P2$ formed a bargaining bloc? What if these players need the unproductive A in order to form this bloc?

The first type of questions is answered by the Owen value which assigns the payoff $\frac{1}{2}$ to $P1$ and to $P2$ while $P3$ and A get nothing. Since the players $P1$ and $P2$ already produce the grand coalition's worth and since they bargain as one person as well as for symmetry reasons, this fits nicely with our intuitions.

Yet, the Owen value cannot give an adequate answer to the second type of questions. If $P1$ and $P2$ need A in order to form a bargaining bloc, then one may argue that—despite being not productive—A should obtain a positive payoff. However, adding A to the bloc formed by $P1$ and $P2$ does not affect the Owen payoffs. One reason for this is that coalition structures are too coarse structures. From the coalition $\{P1, P2, A\}$ alone one cannot infer whether A is necessary to connect the productive players $P1$ and $P2$ or not. The necessity of A can be modelled by the undirected graph



(3.3)

where $P1$ and $P2$ are connected only via a chain of links involving A. Yet, this transcends the world of coalition structures and leads into the realm of cooperation structures.

Hence, one would like to have an efficient CO-value which recognizes, for example, the coordinating role of player A in the situation above. As an attempt, we introduce and axiomatize a CO-value that generalizes the Owen value to the class of CO-games and which, in a sense, does not deviate too much from the Myerson value. More specifically, our CO-value coincides with the Owen value if all possible links within the components are present and it coincides with the Myerson value if there is just one component. For the graph (3.3) in our example, that CO-value assigns the payoffs $\varphi_{P1} = \varphi_{P2} = \frac{5}{12}$, $\varphi_A = \frac{1}{6}$, and $\varphi_{P3} = 0$ which meet our intuitions concerning player A.

The axiomatization involves four axioms. Besides efficiency, we require our CO-value to assign the same payoffs for the complete graph as for the empty graph which, in fact, represents some mild version of the requirement to treat similar players in a similar way. Further, the internal organization of the components should not affect the components' payoffs. Finally, we modify the crucial axiom of the original characterization of the Myerson value, the fairness axiom, such that the number of components involved is not affected by removing a link. Though, the player set involved may shrink. Further, we explore the relation of our CO-value to the Myerson value and to the Owen value as well as consistency properties. Finally, we touch stability issues.

3.4. On a relation between the Myerson value and the position value. An extended version of this chapter has been published as "André Casajus (2007): The position value ist the Myerson value, in a sense, in: *International Journal of Game Theory* 36 (1), 47-55".

The Myerson value is calculated as the Shapley value for the original player set and the graph-restricted payoff function (see Section 1.2, pp. 11). Hence, the Myerson value emphasizes the role of the players in creating cooperation structures.

As an alternative to the Myerson value, Meessen (1988) suggests the position value, π, which was popularized by Borm, Owen & Tijs (1992) . Only recently, Slikker (2005) gave a general characterization. In contrast to the Myerson value, the focus of the position value is on the links. It is calculated in two steps.

First, the payoffs of the links are determined as the Shapley payoffs in the so-called link game where the player set is the set of links and the payoff of any subset of links is given by the sum of the worths of the components induced by that subset of links. Reconsider constellation (C) in Section 1.2 which was represented by the player set $\{E, W1, W2\}$, the coalition function w in (1.1), and the cooperation structure in (1.2). In the link game, we have the player set $L = \{\{E, W2\}, \{W1, W2\}\}$. The coalition function w^N is given by

$$w^N(S) = \begin{cases} 0, & S \neq L, \\ 3, & S = L. \end{cases} \tag{3.4}$$

This can be seen as follows: If S does not contain both links, N splits into components the worth of which is 0; otherwise there is one component containing all players whose worth is $v(N) = 3$. The Shapley payoffs for (L, v^N) are $\mathrm{Sh}_{\{E,W2\}} = \mathrm{Sh}_{\{W1,W2\}} = \frac{3}{2}$, i.e., the grand coalition's worth is split equally among the links. In the second step, every player obtains half of the payoffs of his links. This yields the position value payoffs

$$\pi_E(N, w, L) = \pi_{W1}(N, w, L) = \frac{3}{4} \qquad \text{and} \qquad \pi_{W2}(N, w, L) = \frac{3}{2}$$

which illustrates the position value's focus on the links. Though all players are indispensable to create a positive worth, player $W2$'s central position with more links secures a payoff higher than that of the other players.

We suggest another way to determine the position value. In particular, we split the players into separate agents, one for each link, and directly connect any two of a player's agents. Further, any of a player's link agents can play his productive role, but one of them suffices to do so. Based on this idea, we introduce the link agent form (LAF) of a CO-game. It turns out that the sum of the Myerson payoffs of a player's agents in the LAF equals his position value payoffs in the original game. We extend this construction to TU games with a conference structure and obtain an analogous the result.

In our example above, player $W2$ has two links. Hence in the link agent form, $W2$ is split into two link agents, $W2.E$ and $W2.W1$, and these agents are linked with each other. The link agent form consists of the player set $M = \{E, W1, W2.E, W2.W1\}$, the payoff function u,

$$u(S) = \begin{cases} 0, & S \neq M, \\ 3, & S = M, \end{cases}$$

and of the cooperation structure

which gives the Myerson payoffs $\mu_E = \mu_{W2.E} = \mu_{W2.W1} = \mu_{W1} = \frac{3}{4}$. Summing up over a player's link agents yields the position value payoffs as above.

4. Technicalities

The papers in this thesis appear in the form they have been published or (re)submitted. In particular, there is an abstract and a separate bibliography for any of these. Within the published papers, margin notes indicate the original page numbers. Except for the heading line, the chapter number is suppressed within a chapter; references within the same chapter appear without chapter number. In the index, italicized page numbers indicate where a concept is defined.

5. Acknowledgments

I am indebted to Harald Wiese for granting the freedom to do the research on exotic themes which led to the papers in this thesis. Acknowledgments related to the single papers can be found in the respective chapters of this thesis. Besides, I wish to thank Franziska Beltz, Tobias Hiller, Lothar Tröger, Andreas Tutic, and Harald Wiese for their critical comments on the introduction.

Last but not least, I deeply thank Franzissikilein who cheerfully attended this endeavor for quite a long time ;-)

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Part 1

Isomorphism of extensive games

CHAPTER II

Weak isomorphism of extensive games

This Chapter has been published as "André Casajus (2003): Weak isomorphism of extensive games, in: *Mathematical Social Sciences* 46, 267–290". The margin notes indicate the first lines of the respective pages in the published version.

An early version appeared as "André Casajus (2000): Weak isomorphism of extensive games, Diskussionsbeitrag 185/2000, Institut für Volkswirtschaftslehre, Universität Hohenheim, Germany" and was presented in a poster session at the First World Congress of the Game Theory Society (Games 2000, July 24–28, 2000, Bilbao, Spain). Moreover, it is based on Chapter 3 of my doctoral dissertation at the University of Leipzig, Germany published as "André Casajus (2001): Focal Points in Framed Games: Breaking the Symmetry, Vol. 499 of *Lecture Notes in Economics and Mathematical Systems*, Springer, Berlin".

The reasons for putting this paper into this thesis are the following: Sections 4.1, 4.2, and 4.4 contain new results which triggered the research underlying Chapters III and IV. Further, Chapters III and IV are addenda to this paper which rely on its basic definitions and notation.

Abstract [267]

Based on the Selten (1983) and Oh (1995) symmetries, we introduce weak isomorphism of extensive games that, in contrast to the Peleg et al. (1999) isomorphism, is compatible with the traditional extensive representation of strategic games. While being sufficiently "weak" to ignore the order of moves to some extent, weak isomorphism is "strong" enough not to violate sequential rationality considerations as incorporated in the concept of sequential equilibrium. In addition, there is some generic equivalence between weak isomorphism and isomorphism of the agent normal form.

Key Words: Symmetry, Representation, Equivalence, Transformation, Sequential rationality

JEL classification: C72.

1. Introduction

Intuitively, within the Battle-of-the-Sexes game G

	s_{21}	s_{22}
s_{11}	$3, 1$	$0, 0$
s_{12}	$0, 0$	$1, 3$

the players and their strategies s_{11} and s_{22} as well as s_{12} and s_{21} are symmetric. Harsanyi & Selten (1988) formalize this intuition with their isomorphism of strategic games. [268] Traditionally, G is represented by the extensive game Γ or $\bar{\Gamma}$ in Figure 1 (information sets henceforth indicated by dashed lines). So one could argue that the players and the corresponding actions should also be symmetric in Γ and $\bar{\Gamma}$.

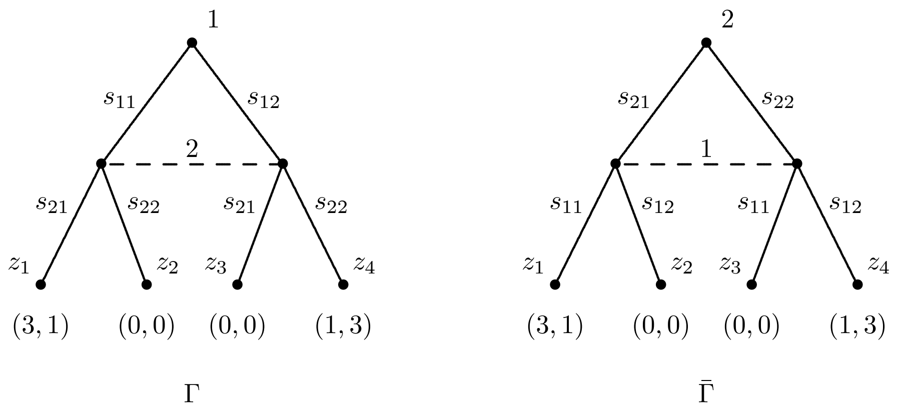

FIGURE 1. Traditional extensive representations of G

Similar to Elmes & Reny (1994) (henceforth E&R), Peleg et al. (1999) (henceforth PRS) introduce isomorphisms of extensive games that preserve the structure of extensive games beyond purely strategical considerations. In particular, these isomorphisms preserve the (strict) order of moves. Then, PRS observe that the induced symmetry of players in the traditional extensive game representation is incompatible with their counterpart for strategic games based on the Harsanyi & Selten (1988) isomorphism: Since the players move in some order in Γ, they cannot be symmetric. In order to remedy this shortcoming, Sudhölter et al. (2000) (also PRS) introduce and axiomatize an alternative extensive representation for which both notions of symmetry coincide. Figure 2 gives the PRS representation of G where $\tilde{o}$ is a chance node with the respective probabilities in brackets, player 1 controls o, $\bar{x}_1$, and $\bar{x}_2$, and player 2 controls $\bar{o}$, x_1, and x_2. Obviously, the PRS representation is more complex and more difficult to deal with than the traditional one. So a concept of isomorphism of extensive games that fits the traditional representation of strategic games seems to be desirable.

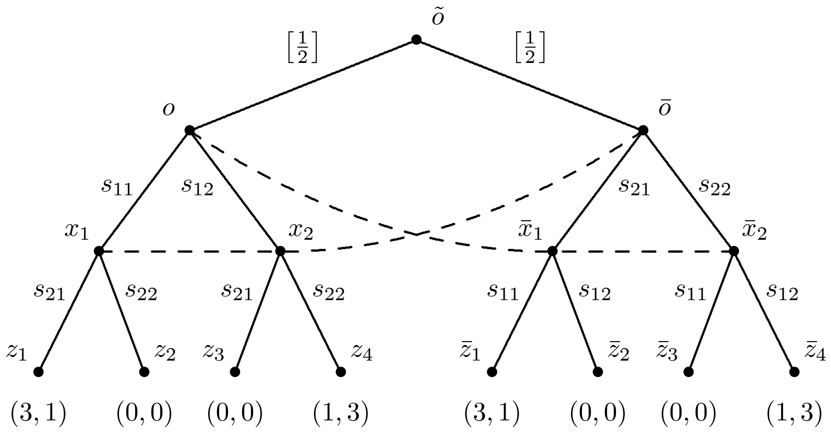

FIGURE 2. The PRS representation of G

While it is clear that these isomorphisms cannot respect the order of moves in the strong sense of E&R and PRS, we cannot dispense with them preserving the "essential" part of the order of moves. This poses the delicate question what this essence is. Without trying to give a comprehensive answer, the following property seems to be necessary for such an essential order preservation: All "reasonable" equilibrium concepts—especially equilibrium concepts that explicitly refer to the sequential nature of moves as the subgame perfect equilibrium (Selten 1975) or the sequential equilibrium (Kreps & Wilson 1982) should be invariant with respect to this isomorphism, i.e. equilibria of the types under consideration should be mapped onto each other, respectively. Note that this criterion reverses the relation between equilibrium concepts and invariance with respect to isomorphism: Usually, this invariance serves as a requirement on solution concepts. Here, in contrast, solution concepts are exploited to assess isomorphism. Of course, this reversal should be founded on some agreement that the equilibrium concepts used are sound in some sense. [269]

In view of the sufficiency of the normal form to preclude sequentially irrational behavior as "incredible threats" (Kohlberg & Mertens 1986) and that extensive games can be transformed into their reduced normal form by strategically "inessential" transformations (E&R), one might question the need for preserving structural features beyond the normal form. However, it is not yet clear whether the (reduced) normal form contains all of the strategically relevant information (see Fudenberg & Tirole 1991, ch. 11). In addition, strategically irrelevant features may trigger the focal point effect (Schelling 1960).

The latter is explained with some examples: Consider the following two-player one-shot matching game—both players independently have to name one of the numbers "1", "2", or "3", and both get a prize only if they choose the same one. Then, symmetry invariance prescribes them to randomize over the numbers leading to a probability of one

third of winning the prize. Consider now a modification of this game: If the players coordinate at "1", then they win a lottery that gives them a bigger prize with probability $\frac{1}{2}$ and otherwise nothing; and the players are indifferent between this lottery and the original prize. From a purely strategic point of view, both games are equivalent. However, the lottery makes the "1" focal, and the players can coordinate for sure by naming the "1". Hence, the structure of the chance mechanism itself may create a focal point. In addition, actions, even strategically irrelevant ones, may bear labels that constitute a focal point: Let the game above lead to identical lotteries in case of coordination. Again, symmetry invariance then prescribes uniform randomization. From a strategic point of view these lotteries could be replaced by their expected payoffs. But if the lottery resulting from coordinating at "1" is effected, say, by a Wheel of Fortune which is colored black while the others are effected by a red one, naming the "1" is focal. Of course, deriving this focal point requires some formal labelling of information sets or—more general—of actions and some solution concept that exploits this labelling. In our example, chance actions could be assigned labels that indicate the color of the Wheel of Fortune. Hence, a concept of isomorphism that keeps much of the structure of extensive games would be a good candidate for transferring the framed strategic games approach of focal points to extensive games (see Casajus 2001, ch. 4). Note that this approach is restricted to focal points as symmetry breaking devices. Indeed, it cannot discriminate between the two [270]
pure-strategy equilibria in the Battle-of-the-Sexes game.

The plan of the paper is as follows: Basic definitions and notation are given in the next section. The third one extends the Selten (1983) and Oh (1995) symmetries into weak isomorphisms of extensive games. Particularly, we discus the path condition which makes these isomorphisms "weak". It is shown that—in non-pathological cases—this condition cannot be weakened without loosing too much of the games' structure (Theorem 3.5). In the fourth section, we explore the relation between weak isomorphism and other concepts of equivalence of extensive games—the Oh history preservation, (reduced) normal form equivalence, E&R transformations, and agent normal form equivalence. Interestingly, we can show the generic equivalence of weak isomorphism and agent normal form isomorphism for the class of non-pathological games without chance player (Theorem 4.8). We also establish the desired equivalence of strategy symmetry in strategic games and their traditional representation (Corollary 4.6). The invariance of equilibria under weak isomorphism and the existence of symmetry invariant equilibria is shown in the fifth section (especially Theorem 5.2). The final section makes some concluding remarks. All proofs and some technical lemmas are referred to the appendix.

2. Basic definitions and notation

2.1. Strategic games. We consider the class $\mathcal{G}$ of finite strategic games where the constituents of $G = (I, (S_i)_{i \in I}, (u_i)_{i \in I}) \in \mathcal{G}$ are defined as usual: I is the non-empty and finite set of players, S_i the non-empty and finite set of player i's pure strategies s_i, and u_i player i's payoff function $u_i : S \to \mathbb{R}$ where $S := \prod_{i \in I} S_i$. Σ_i denotes player i's set of mixed strategies σ_i, where $\sigma_i(s_i)$ is the probability of s_i, and $\Sigma := \prod_{i \in I} \Sigma_i$ is the set of mixed-strategy profiles σ.

An isomorphism (Harsanyi & Selten 1988) from $G \in \mathcal{G}$ onto $\bar{G} \in \mathcal{G}$ is a system of bijective mappings $\mathbf{f} = (\pi, (r_i)_{i \in I})$, $\pi : I \to \bar{I}$, and $r_i : S_i \to \bar{S}_{\pi(i)}$, with the following property: For all $i \in I$, there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that

$$\bar{u}_{\pi(i)} \left(f \left(s \right) \right) = \alpha_i u_i \left(s \right) + \beta_i \tag{2.1}$$

for all $s \in S$ where $f = (f_{\bar{\imath}})_{\bar{\imath} \in \bar{I}} : S \to \bar{S}$, $s \mapsto f(s)$ with

$$f_{\pi(i)}(s) := r_i(s_i) \tag{2.2}$$

for all $i \in I$ and $s \in S$. G and $\bar{G}$ are called isomorphic ($G \cong \bar{G}$) if there is an isomorphism $\mathbf{f}$ from G onto $\bar{G}$ ($G \xrightarrow{\mathbf{f}} \bar{G}$).

A solution concept L for $\mathcal{G}$ assigns a set of strategy profiles $L(G) \subseteq \Sigma$—the solutions—to any $G \in \mathcal{G}$. L is invariant with respect to isomorphism if for all $G, \bar{G} \in \mathcal{G}$ and all isomorphisms $G \xrightarrow{\mathbf{f}} \bar{G}$ we have $f(L(G)) = L(\bar{G})$ for f given by $\mathbf{f}$ where $f : S \to \bar{S}$ is extended to Σ by $f_{\pi(i)} \left(\sigma \right) \left(r_i(s_i) \right) := \sigma_i(s_i)$ for all $i \in I$, $s_i \in S_i$, and $\sigma \in \Sigma$. Automorphisms are called [271]
symmetries. A strategy combination σ is called symmetry invariant if $f \left(\sigma \right) = \sigma$ for all f given by symmetries of G. Players (strategies) are called symmetric if they are mapped onto each other by some symmetry. By (2.2), symmetry-invariant strategy profiles can be characterized by symmetric strategies being assigned the same probabilities.

2.2. Extensive games. We consider the class $\mathcal{E}$ of finite extensive games with perfect recall, where the constituents of $\Gamma = (T, \lhd, I, P, H, A, p, u) \in \mathcal{E}$ are defined as usual (e.g. Selten 1975):

The finite node set T and the predecessor relation $\lhd$ constitute a game tree: T contains at least two nodes; $\lhd$ is transitive and asymmetric; there is a root $o \in T$ such that $o \lhd x$ for all $x \neq o$; and for all $x, x', x'' \in T$, $x' \lhd x$, $x'' \lhd x$ and $x' \neq x''$ imply $x' \lhd x''$ or $x'' \lhd x'$; Z denotes the set of terminal nodes; $X := T \backslash Z$ denotes set of decision nodes. $V(x)$ denotes the unique immediate predecessor of $x \neq o$, and $N(x)$ denotes the set of x' immediate successors.

The player set I contains the personal players i and the chance player i_0. For all $i \in I$, the player partition P divides X into player cells P_i where P_{i_0} may be empty; node x is controlled by player $i\left(x\right)$. The information partition H is a subpartition of P satisfying

$|N(x)| = |N(x')|$ for all information sets $h \in H$ and $x, x' \in h$, and $h(x) = \{x\}$ for all $x \in P_{i_0}$. Node x is contained in $h(x)$, $H_i := h(P_i)$ denotes the set of player i's information sets, and $i\,(h)$ is the player who controls h. The action partition A divides $T \backslash \{o\}$ into actions a, sets of nodes that are reached by the same act, such that $V(a) \in H$ and $|a| = |V(a)|$ for all $a \in A$. Node x is contained in $a(x)$; $A_h := a(N(h))$ denotes the actions at h, i.e. $a \in A_{V(a)}$, and $A_i := a(N(P_i))$ is the set of player i's actions. The chance probabilities are collected in $p = \{p_h | h \in H_{i_0}\}$ where p_h is a probability distribution over A_h such that $p_{V(a)}(a) > 0$ for all $a \in A_{i_0}$. The payoff structure $u := (u_i)_{i \in I \backslash \{i_0\}}$ contains the payoff function $u_i : Z \to \mathbb{R}$.

The subgame of $\Gamma \in \mathcal{G}$ rooted in $x \in X$ is denoted $\Gamma^x = (T^x, \lhd^x, I^x, P^x, H^x, A^x, p^x, u^x) \in \mathcal{E}$ where $\lhd^x$ is the restriction of $\lhd$ to T^x, T^x just contains x itself and all nodes succeeding x, i.e. $Z^x := T^x \cap Z$, $X^x := T^x \cap X$; I^x contains i_0 and all i for which $P_i^x := P_i \cap T^x \neq \emptyset$; $P^x := \{P_i^x | i \in I^x\}$; $H \supseteq H^x := \{h(x') | x' \in X^x\}$; $H^x \supseteq H_i^x := \{h(x') | x' \in P_i^x\}$ for all $i \in I^x$; $A^x := \bigcup_{h \in H^x} A_h$; $A_i^x := A_i \cap A^x$ for all $i \in I^x$; $p^x := \{p_h | h \in H_{i_0}^x\}$; $u_i^x := u_i|_{Z^x}$ for all $i \in I^x \backslash \{i_0^x\}$, and $h\,(x) \subset X^x$ for all $x \in X^x$.

The unordered path $\psi(x) \subset T$ contains x itself and its predecessors with exception of the root, $\psi\,(o) = \emptyset$. The path $\boldsymbol{\psi}(x)$ is the sequence $(\boldsymbol{\psi}_k(x))_{k \in \{1,\ldots,|\psi(x)|\}}$ such that $\boldsymbol{\psi}_k(x) \in T$, $\boldsymbol{\psi}_{|\psi(x)|}(x) = x$, and $\boldsymbol{\psi}_{k-1}(x) = V(\boldsymbol{\psi}_k(x))$ for $1 < k \leq |\psi(x)|$. The history of x is the sequence $a(\boldsymbol{\psi}(x)) = (a(\boldsymbol{\psi}_k(x)))_{k \in \{1,\ldots,|\psi(x)|\}}$, and the unordered history of x is the set $a(\psi(x)) \subset A$. (Unordered) histories of terminal nodes are called terminal.

An extensive game exhibits *perfect recall*, if for all $i \in I \backslash \{i_0\}$, $a \in A_i$, $x \in P_i$, and $x' \in h\,(x)$ $\psi(x) \cap a \neq \emptyset$ implies $\psi(x') \cap a \neq \emptyset$. Note that perfect recall implies that different nodes have different (unordered) histories. An extensive game is called *non-pathological* if every decision node is followed by at least two nodes; otherwise it is called *pathological*. The subclass of non-pathological games is denoted $\mathcal{E}^*$.

We denote the set of player i's pure strategies $\mathbf{a}_i$ by $\mathbf{A}_i = \prod_{h \in H_i} A_h$ and the set [272]
of pure-strategy profiles $\mathbf{a}$ by $\mathbf{A} := \prod_{i \in I} \mathbf{A}_i$. An action profile is a vector $\mathbf{a}^+ \in \mathbf{A}^+ := \mathbf{A} \times \mathbf{A}_{i_0}$. By perfect recall, each $\mathbf{a}^+ \in \mathbf{A}^+$ is assigned the unique terminal node $z(\mathbf{a}^+)$ satisfying

$$a(\psi(z(\mathbf{a}^+))) \subseteq \{\mathbf{a}_h^+, h \in H\}. \tag{2.3}$$

We set $u_i(\mathbf{a}^+) := u_i(z(\mathbf{a}^+))$ for all $i \in I \backslash \{i_0\}$ and $\mathbf{a}^+ \in \mathbf{A}^+$. Let $\mathbf{A}^+(h)$ and $\mathbf{A}^+(x)$ denote the set of action profiles leading to terminal nodes succeeding $h \in H$ and $x \in X$, respectively. I.e., $\mathbf{A}^+(o) = \mathbf{A}^+$ and

$$\mathbf{A}^+(h) = \{\mathbf{a}^+ | a(\psi(z(\mathbf{a}^+))) \cap A_h \neq \emptyset\}, \quad \mathbf{A}^+(x) = \{\mathbf{a}^+ | x \in \psi(z(\mathbf{a}^+)))\}. \tag{2.4}$$

Personal player $i\,(h)$'s local strategies $b_h \in B_h$are the probability distributions on A_h where $b_h(a)$ is the probability of a; $B_i = \prod_{h \in H_i} B_h$ denotes the set of i's behavior strategies b_i. The set of behavior-strategy profiles b is denoted $B := \prod_{i \in I \backslash \{i_0\}} B_i$; $B^0 \subseteq B$ denotes the subset of completely mixed behavior strategy profiles. Each $b \in B$ constitutes a probability distribution on $\mathbf{A}^+$ with $\text{prob}(\mathbf{a}^+|b) := \prod_{h \in H_{i_0}} p_h(\mathbf{a}_h^+) \cdot \prod_{h \in H \backslash H_{i_0}} b_h(\mathbf{a}_h^+)$ for all $\mathbf{a}^+ \in \mathbf{A}^+$, and the payoff functions are extended to B and $\mathbf{A}$ by $u_i(b) := \sum_{\mathbf{a}^+ \in \mathbf{A}^+} \text{prob}(\mathbf{a}^+|b) u_i(\mathbf{a}^+)$ for all $b \in B$ and $i \in I \backslash \{i_0\}$. A solution concept L for $\mathcal{E}$ assigns a set of behavior-strategy profiles $L(\Gamma) \subseteq B$—the solutions—to any $\Gamma \in \mathcal{E}$.

2.3. Representations. The agent normal form of $\Gamma \in \mathcal{E}$ is the game $\text{ANF}(\Gamma) = (H \backslash H_{i_0}, (A_h)_{h \in H \backslash H_{i_0}}, (u_h)_{h \in H \backslash H_{i_0}}) \in \mathcal{G}$ with $u_h = u_{i(h)}$ for all $h \in H \backslash H_{i_0}$, and with $\mathbf{A}$ (B) as the set of pure-strategy (mixed-strategy) profiles. The normal form of Γ is the game $\text{NF}(\Gamma) = (I \backslash i_0, (\mathbf{A}_i)_{i \in I \backslash i_0}, (u_i)_{i \in I \backslash i_0}) \in \mathcal{G}$ with $\mathbf{A}$ as the set of pure-strategy profiles. For all $i \in I \backslash i_0$ we define an equivalence relation $\sim_i$ on $\mathbf{A}_i$: $\mathbf{a}_i \sim_i \mathbf{a}'_i$ iff $u_{i'}(\mathbf{a}_i, \mathbf{a}_{-i}) = u_{i'}(\mathbf{a}'_i, \mathbf{a}_{-i})$ for all $i' \in I \backslash i_0$ and $\mathbf{a}_{-i} \in \mathbf{A}_{-i} := \prod_{j \in I \backslash \{i, i_0\}} \mathbf{A}_j$. Let $[\mathbf{A}_i]$ denote the set of equivalence classes of $\mathbf{A}_i$ and $[\mathbf{a}_i]$ the equivalence class containing $\mathbf{a}_i$. The reduced normal form of Γ is the game $\text{RNF}(\Gamma) = (I \backslash \{i_0\}, ([\mathbf{A}_i])_{i \in I \backslash \{i_0\}}, (u_i)_{i \in I \backslash \{i_0\}}) \in \mathcal{G}$ with $[\mathbf{A}] := \prod_{i \in I \backslash \{i_0\}} [\mathbf{A}_i]$ as the set of pure-strategy profiles $[\mathbf{a}] := ([\mathbf{a}_i])_{i \in I \backslash \{i_0\}}$ and the u_i extended to $[\mathbf{A}]$ by $u_i([\mathbf{a}]) := u_i(\mathbf{a})$.

While players are considered to "move" simultaneously in strategic games, players move in some order in the traditional extensive representations. Consider some $G \in \mathcal{G}$ and some order of players given by a bijection $\jmath : \{1, 2, \dots |I|\} \to I$ where player $\jmath(n)$ moves as the nth one. The $\jmath$-ordered representation of $G \in \mathcal{G}$ is the game $\text{ER}^\jmath(G) = (T^\jmath, \lhd^\jmath, I^\jmath, P^\jmath, H^\jmath, A^\jmath, p^\jmath, w^\jmath) \in \mathcal{E}$ where $Z^\jmath = \{z^\jmath[s] | s \in S\}$, $I = I^\jmath$, $P^\jmath = H^\jmath$, $A_i^\jmath = \{a^\jmath[s_i] | s_i \in S_i\}$ for all i, $p^\jmath = \emptyset$, $u_i^\jmath(z^\jmath[s]) = u_i(s)$ for all $s \in S$, $a^\jmath(\psi^\jmath(z^\jmath[s])) = \{a^\jmath[s_i] | i \in I\}$, where $a^\jmath[s_i]$ represents player i's pure strategy s_i and $z^\jmath[s]$ represents outcome of strategy combination s. The full definition is straightforward.

2.4. Genericity. A game form γ is a tuple $(T, \lhd, I, P, H, A, p)$, i.e. an extensive game without payoff functions. Let $\deg(\gamma) := |Z \times I \backslash \{i_0\}|$. Any extensive game Γ based on a fixed game form γ can be represented by a vector $\upsilon \in \mathbb{R}^{\deg(\gamma)}$; we then write $\Gamma = \gamma(\upsilon)$. A proposition involving pairs of extensive games (as isomorphism is) holds for generic payoffs iff for all game forms γ' there are closed Null sets $N_{\gamma'} \subset \mathbb{R}^{\deg(\gamma')}$ with respect to the $\deg(\gamma')$-dimensional Lebesgue measure such that the proposition holds for all pairs of game forms $(\gamma, \bar{\gamma})$ and all pairs of games $(\gamma(\upsilon), \bar{\gamma}(\bar{\upsilon}))$, $\upsilon \in \mathbb{R}^{\deg(\gamma)} \backslash N_\gamma$, $\bar{\upsilon} \in \mathbb{R}^{\deg(\bar{\gamma})} \backslash N_{\bar{\gamma}}$. A pair of games $(\gamma(\upsilon), \bar{\gamma}(\bar{\upsilon}))$ is called generic with respect to some property if there is some neighborhood U of υ and $\bar{U}$ of $\bar{\upsilon}$ such that the property holds for all pairs $(\gamma(\omega), \bar{\gamma}(\bar{\omega}))$, $\omega \in U$, $\bar{\omega} \in \bar{U}$. [273]

3. Weak isomorphism

In order to identify the players' corresponding actions in two-player extensive games, Selten (1983) introduces symmetries that are based on bijective mappings of the action partition onto itself. Oh (1995) extends these symmetries to extensive games in general and adds a condition of history (other than ours) preservation in order to exploit history as a coordination device. Our definition straightforwardly extends Oh symmetries into weak isomorphisms. Yet, we drop the preservation of the Oh histories. In Section 4.1, we show that the preservation of these histories already is embodied in weak isomorphism in non-pathological cases.

DEFINITION 3.1. *A weak isomorphism from* $\Gamma \in \mathcal{E}$ *onto* $\bar{\Gamma} \in \mathcal{E}$ *is a bijection* $r : A \to \bar{A}$ *with the following properties: There are bijections* $\nu : H \to \bar{H}$, $\pi : I \to \bar{I}$, *and* $\theta : Z \to \bar{Z}$ *such that*

ISA $r(A_h) = \bar{A}_{\nu(h)}$ *for all* $h \in H$,

PL $r(A_i) = \bar{A}_{\pi(i)}$ *for all* $i \in I$,

CPL $r(A_{i_0}) = \bar{A}_{\bar{\imath}_0}$,

CPR $p_h(a) = \bar{p}_{\nu(h)}(r(a))$ *for all* $h \in H_{i_0}$ *and* $a \in A_h$,

PTH $r(a(\psi(z))) = \bar{a}(\bar{\psi}(\theta(z)))$ *for all* $z \in Z$,

PY *for all* $i \in I\backslash i_0$, *there are* $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ *such that* $\bar{u}_{\pi(i)}(\theta(z)) = \alpha_i u_i(z) + \beta_i$ *for all* $z \in Z$.

In the following, the single conditions of the definition are explained in more detail and some implications are explored. Note that **ISA** and **PL** are implicit in the E&R and PRS isomorphism. Since E&R focus on games without chance player, **CPL** and **CPR** are not part of the E&R isomorphism, but are embodied in the PSR one. Condition **PY** allows for positive affine transformations of the players' payoffs, i.e. transformations that preserve the players' preferences over outcomes. In contrast to (2.1) and Oh (1995), Selten (1983), E&R, and PSR do not allow for these transformations. Yet, this is rather inessential as [274]
one could restrict attention to games with payoffs normed on the unit interval. The main difference between our weak isomorphism based on Selten (1983) and Oh (1995) on the one hand, and the E&R and the PRS one on the other lies in condition **PTH**: While **PTH** respects unordered terminal histories only, the E&R and the PRS isomorphism, in fact, respects (ordered) histories, even non-terminal ones.

3.1. Conditions ISA and PL. Condition **ISA** secures that weak isomorphisms respect the assignment of actions to information sets. Actions that belong to the same information set are mapped onto actions of one information set. The bijection ν is determined uniquely by r. In addition, any bijection $r : A \to \bar{A}$ satisfying **ISA** induces a unique bijection $\mathbf{r}^+ = (\mathbf{r}^+_{\bar{h}})_{\bar{h}\in\bar{H}} : \mathbf{A}^+ \to \bar{\mathbf{A}}^+$ with

$$\mathbf{r}^+_{\nu(h)}(\mathbf{a}^+) := r(\mathbf{a}^+_h) \tag{3.1}$$

for all $\mathbf{a}^+ \in \mathbf{A}^+$ and $h \in H$ where ν is determined by r according to **ISA**.

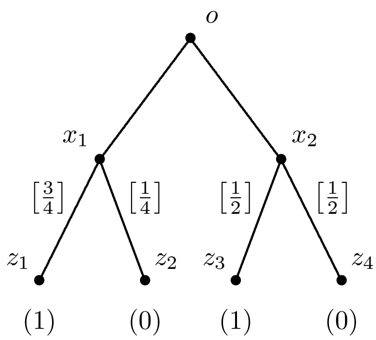

FIGURE 3. Condition **CPR** is indispensable

Condition **PL** secures that weak isomorphisms respect the assignment of actions to players, and together with **ISA** also the assignment of information sets to players, $\nu(H_i) = \bar{H}_{\pi(i)}$ for all $i \in I$, i.e., information sets that belong to the same player are mapped onto information sets of one player. The bijection π is uniquely determined by r.

3.2. Preservation of the structure of the chance mechanism. Together with **ISA**, **CPL** and **CPR** secure that weak isomorphisms respect the structure of the chance mechanism. I.e., alternative decompositions of compound lotteries do matter, even if the resulting expected payoffs are the same. As explained in the Introduction, such differences might create focal points.

CPL preserves the external structure, i.e. the assignment of actions and information sets to the chance mechanism. In view of **PL**, **CPL** implies $\pi(i_0) = \bar{\imath}_0$ and $\nu\left(H_{i_0}\right) = \bar{H}_{\bar{\imath}_0}$. In addition, any bijection $r : A \to \bar{A}$ satisfying **ISA** and **CPL** induces a unique bijection $\mathbf{r} = (\mathbf{r}_{\bar{h}})_{\bar{h} \in \bar{H} \setminus \bar{H}_{\bar{\imath}_0}} : \mathbf{A} \to \bar{\mathbf{A}}$ and its extension to B and $\bar{B}$ with

$$\mathbf{r}_{\nu(h)}(\mathbf{a}) := r(\mathbf{a}_h) \qquad \text{and} \qquad \mathbf{r}_{\nu(h)}(b)(r(a)) := b_h(a) \tag{3.2}$$

for all $h \in H \setminus H_{i_0}$, $\mathbf{a} \in \mathbf{A}$, $b \in B$, and $a \in A_h$.

Obviously, **CPR** makes sense only together with **ISA** and **CPL**. Then, **ISA** and **CPR** preserve the internal structure of the chance mechanism. Actions of one chance information set are kept together and are mapped onto chance actions with the same probabilities. As the following example shows, we hardly can do without **CPR**.

EXAMPLE 3.2. *Consider the game in Figure 3 where player* 1 *controls node* o; *and* x_1 *and* x_2 *are controlled by the chance mechanism with the probabilities in brackets. Let* $r : A \to A$ *be such that* $r(\{x_1\}) = \{x_2\}$, $r(\{z_1\}) = \{z_3\}$, $r(\{z_2\}) = \{z_4\}$ *and vice versa. While* r *violates **CPR**, it satisfies **PTH**, **ISA**, **PL**, **CPL**, and **PY**. But choosing* $\{x_1\}$ *gives player* 1 *the payoff* $\frac{3}{4}$, *while choosing* $\{x_2\}$ *gives* $\frac{1}{2}$. *Therefore, action* $\{x_1\}$ *should not be* [275]
mapped onto $\{x_2\}$.

3.3. Preservation of unordered terminal histories. Condition **PTH** preserves the sequential structure to some extent: *Unordered terminal* histories are required to be mapped onto each other. By perfect recall, this determines the unique bijection $\theta : Z \to \bar{Z}$. Instead, one could think of the possibly more intuitive requirement of mapping terminal histories onto each other. Within Definition 3.1, this is equivalent to θ satisfying

PTH$^+$ $\quad r(a(\boldsymbol{\psi}(z))) = \bar{a}(\bar{\boldsymbol{\psi}}(\theta(z)))$ for all $z \in Z$,

where r is extended to histories by $r(a(\boldsymbol{\psi}(x))) := (r(a(\boldsymbol{\psi}_k(x))))_{k\in\{1,\dots,|\psi(x)|\}}$ for all $x \in T$. This implies the preservation of histories in general. Practically, this was done by PRS which can be seen from the fact that histories in the sequence representation (see e.g. Osborne & Rubinstein 1994) can be identified with the nodes in the tree representation. Since **PTH**$^+$ implies **PTH**, the PRS isomorphisms (extended by positive affine transformations of the payoff functions) are the weak isomorphisms that satisfy **PTH**$^+$. Our leading example shows that the converse does not hold.

While in the following it is argued that **PTH** is not to weak a requirement, firstly, we show that **PTH** is not too strong.

LEMMA 3.3. *Let $\Gamma, \bar{\Gamma} \in \mathcal{E}$; and let r be a bijection $r : A \to \bar{A}$ that satisfies **ISA** and **PTH**. Then,*

PTH$^-$ $\quad \theta(z(\mathbf{a}^+)) = \bar{z}(\mathbf{r}^+(\mathbf{a}^+))$ *for all* $\mathbf{a}^+ \in \mathbf{A}^+$,

*where $\mathbf{r}^+$ is given by r via (3.1), and θ is determined by **PTH**.*

In a sense, **PTH**$^-$ (replacing **PTH)** seems to be the weakest conceivable requirement that preserves the sequential structure of extensive games: Action profiles that lead to the same terminal node should be mapped onto each other. By perfect recall, this determines a unique bijection θ, and by Lemma 3.3, **PTH**$^-$ is implied by **PTH** and **ISA**. A counterexample reveals that the converse may not hold in pathological cases. [276]

EXAMPLE 3.4. *Consider the games in Figure 4 where $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in]0, 1[$ are pairwise different, i.e. payoffs are generic. Let $r : A \to \bar{A}$ map any action (A, B, a, b, c, d) on the action with the same label. Obviously, r satisfies **ISA** and **PTH**$^-$, and induces the bijection $\theta : Z \to \bar{Z}$, $\theta(z_k) = z_k$ for $k = 1, 2, 3$. But $r(a(\psi(z_2))) = \{A, b, c\} \neq \{A, b, d\} = \bar{a}(\bar{\psi}(\theta(z_2)))$, i.e., **PTH** does not hold. Note that r also satisfies **PL**, **CPL**, **CPR**, and **PY**.*

One could argue that **PTH**$^-$ is to weak. Yet, for non-pathological games, **PTH**$^-$ and **PTH** coincide in presence of **ISA**. Hence, **PTH**$^-$ does not weaken **PTH** substantially.

THEOREM 3.5. *Let $\Gamma, \bar{\Gamma} \in \mathcal{E}^*$ and $r : A \to \bar{A}$ be a bijection that satisfies **ISA** and **PTH**$^-$. Then, r also satisfies **PTH**.*

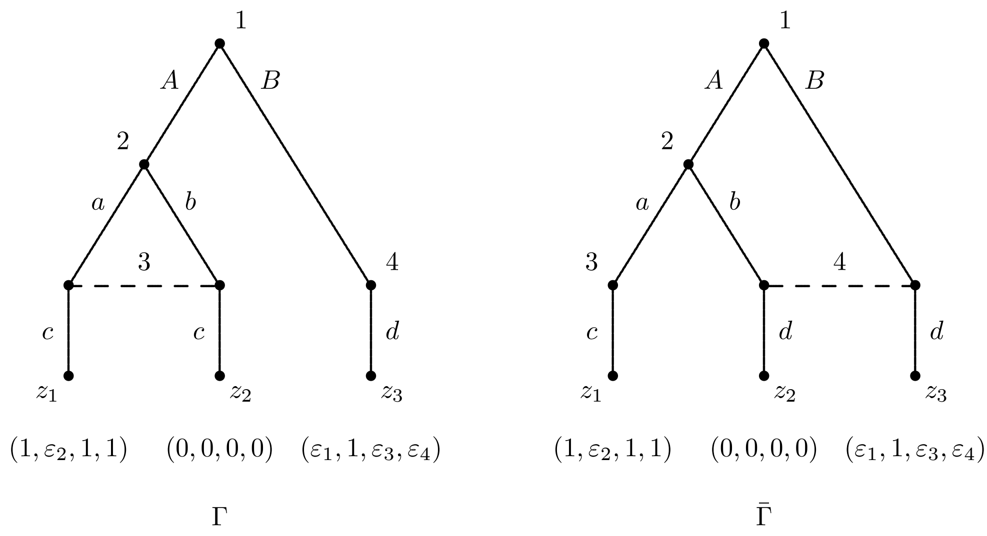

FIGURE 4. Pathological games

3.4. Preservation of preferences. In view of **PL**, **CPL**, and **PTH**, condition **PY** simply requires the usual preservation of the players' preferences over outcomes.

3.5. Invariance under weak isomorphism. Obviously, weak isomorphism constitutes an equivalence relation on $\mathcal{E}$. Two games $\Gamma, \bar{\Gamma} \in \mathcal{E}$ are called weakly isomorphic ($\Gamma \cong \bar{\Gamma}$) if there is a weak isomorphism r from Γ onto $\bar{\Gamma}$ ($\Gamma \xrightarrow{r} \bar{\Gamma}$). A solution concept L is invariant under weak isomorphism if for all $\Gamma, \bar{\Gamma} \in \mathcal{E}$ and all weak isomorphisms $\Gamma \xrightarrow{r} \bar{\Gamma}$ we have $\mathbf{r}(L(\Gamma)) = (\bar{\Gamma})$ for $\mathbf{r}$ given by r via (3.2).

Weak automorphisms are called weak symmetries. A behavior-strategy profile b is called weakly symmetry invariant if $\mathbf{r}(b) = b$ for all mappings $\mathbf{r}$ given by weak symmetries r of Γ via (3.2). Actions (players) are called weakly symmetric if they are mapped onto each other by some weak symmetry. By (3.2), weakly symmetry-invariant behavior-strategy profiles can be characterized by symmetric actions being assigned the same probabilities. [277]
Clearly, a one-point solution concept that is invariant under weak isomorphism has to select a weakly symmetry invariant behavior strategy.

4. Equivalence

4.1. Oh histories. In order to refine the Selten (1983) symmetry, Oh (1995) introduces a structure called *history* (other than ours) and then sharpens symmetry by a requirement of history preservation. In non-pathological cases, however, history preservation already is embodied in the original symmetry. It seems as if Selten's (1983, p. 287) 'warning' that "… the pure strategy sets of both players never coincide even if the game is obviously symmetric in any reasonable sense" has not been taken seriously enough: Not even actions

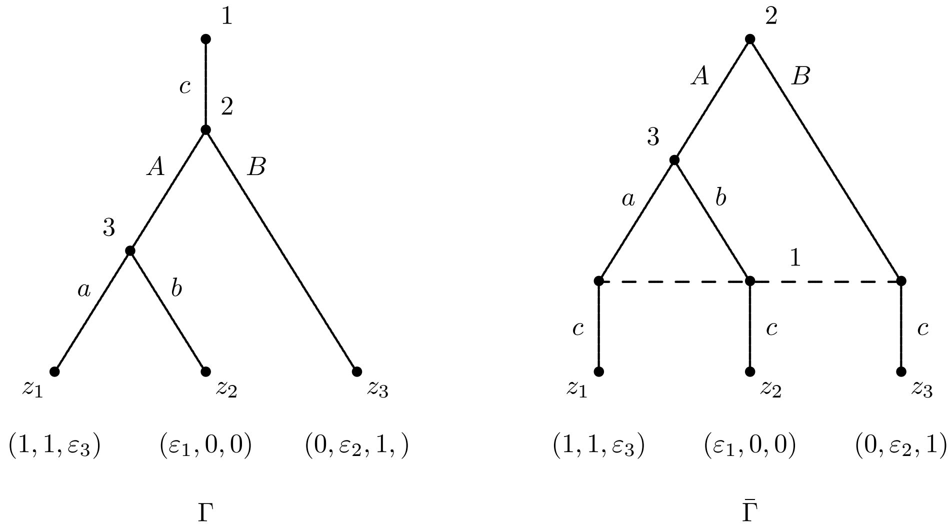

FIGURE 5. Subgame preservation

of one player can be the same at different information sets—an action is just a cell of the action partition.

According to Oh, the history hist(h) of player $i\,(h)$'s information set h is the collection of all players' past action choices, from the root to h, which player $i\,(h)$ can identify. Yet, the phrase "which player i can identify" is not made more explicit. Applications suggest that a history is a correspondence hist : $H \rightrightarrows A$,

$$\text{hist}(h) := \{a : \forall x \in h : \psi(x) \cap a \neq \emptyset\}\,. \tag{4.1}$$

So a history hist(h) comprises the actions of which player $i(h)$ knows for sure at h that they actually have been taken. Histories are exploited in analogy to the following extension of weak isomorphisms $\Gamma \xrightarrow{r} \bar{\Gamma}$ by a condition of history preservation:

HIS $r(\text{hist}(h)) = \overline{\text{hist}}(\nu(h))$ for all $h \in H$.

THEOREM 4.1. *For $\mathcal{E}^*$, **HIS** is implied by weak isomorphism.*

The following example reveals that the Theorem may fail for pathological games.

EXAMPLE 4.2. *Consider the games in Figure 5 where $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \,]0, 1[$, $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \neq \varepsilon_1$, i.e. payoffs are generic. Let the weak isomorphism $\bar{\Gamma} \xrightarrow{r} \Gamma$ map any action on the action with the same label. The Oh history of player 2's information set is empty in $\bar{\Gamma}$. Yet in Γ, the history of player $r\,(2)$'s information set is $\{c\}$. Hence, **HIS** does not hold.*

4.2. Reduced normal form. Weak isomorphism largely respects the structure of a game. So it is not too astonishing that arguments similar to those in the proof of Theorem 4.7 below, together with **PL**, show that weak isomorphism implies isomorphism of the

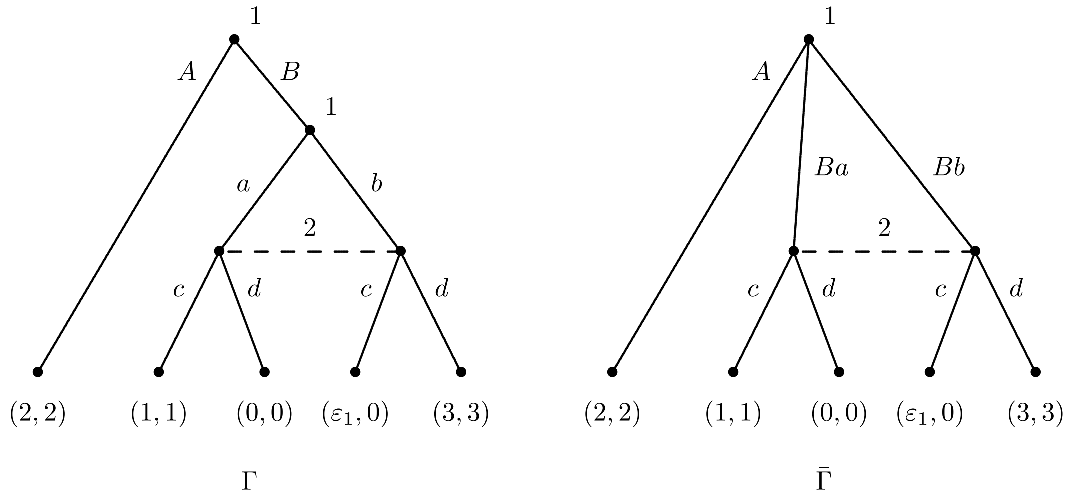

FIGURE 6. The COA transformation

(reduced) normal form. Since the (reduced) normal form contains less structure than the original game, the converse does not hold in general.

THEOREM 4.3. *For* $\Gamma, \bar{\Gamma} \in \mathcal{E}$, $\Gamma \cong \bar{\Gamma}$ *implies* $\mathrm{NF}(\Gamma) \cong \mathrm{NF}(\bar{\Gamma})$ *and* $\mathrm{RNF}(\Gamma) \cong \mathrm{RNF}(\bar{\Gamma})$. [278]

Considering extensive games with perfect recall without chance player, E&R show that extensive games with isomorphic reduced normal forms can be transformed into each other via a finite chain of games that differ by one of three transformations, Addition of Decision Nodes (ADD), Coalescing of Information Sets (COA), and Interchange of Decision Nodes (INT), which all preserve perfect recall. Kohlberg & Mertens (1986, pp. 1008) show how these results can be generalized to games with a chance player. Thompson (1952) obtains a similar result with four transformations, one of them not preserving perfect recall. Hence, weakly isomorphic games are E&R and Thompson equivalent.

EXAMPLE 4.4. *Consider the games in Figure 6 due to Kohlberg & Mertens (1986) which both have isomorphic reduced normal forms and therefore are E&R and Thompson equivalent:*

$\mathrm{RNF}(\Gamma) \cong \mathrm{RNF}(\bar{\Gamma})$ [279]

	c	d
A	$2, 2$	$2, 2$
Ba	$1, 1$	$0, 0$
Bb	$1, \varepsilon_1$	$3, 3$

Yet, the games itself are not weakly isomorphic, even generically: Γ *has more information sets than* $\bar{\Gamma}$.

In view of Example 4.4, at least one of the E&R transformations must violate weak isomorphism. While ADD increases the number of terminal nodes, COA reduces the

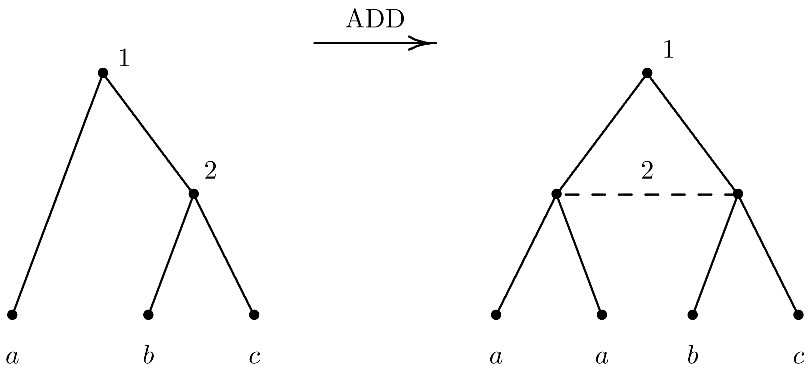

FIGURE 7. The ADD transformation

number of information sets. Hence, both transformations violate weak isomorphism. Note that in Example 4.4, the game Γ is transformed into $\bar{\Gamma}$ by the COA transformation. It reduces two moves in a row by a single player (B and a, B and b) into a single one (Ba, Bb). Figure 7 presents an example of the ADD transformation.

In contrast, INT transforms a game into a weakly isomorphic one. Since INT changes the order of players, it does not preserve the PSR and the E&R isomorphism in general. Figure 8 provides an example that illustrates the original definition of INT where $a\,(w) = a\,(v)\,,\ a\,(\hat{w}) = a\,(\hat{v})\,,\ \bar{a}\,(w) = \bar{a}\,(\hat{w})\,,$ and $\bar{a}\,(v) = \bar{a}\,(\hat{v})\,.$ The game remains unchanged except for the changes in the figure. There are no new actions (information sets) and no new terminal nodes in $\bar{\Gamma}$. Consider the bijection $r : A \to \bar{A}$, $r\,(a\,(t)) = \bar{a}\,(t)$ for $t \in T \backslash \{y, \hat{y}, v, \hat{v}, w, \bar{w}\}\,,$ $r\,(a\,(y)) = \bar{a}\,(w)$, $r\,(a\,(\hat{y})) = \bar{a}\,(v)$, $r\,(a\,(w)) = \bar{a}\,(y)\,,$ and $r\,(a\,(\hat{w})) = \bar{a}\,(\hat{y})\,.$ Then, r satisfies **PTH** and induces the identity mapping on the set of terminal nodes. [280]
This can be seen from the fact that unordered histories of terminal nodes succeeding $\hat{w}$ contain the actions $a\,(\hat{w})$ and $a\,(y)$ which are mapped onto $\bar{a}\,(\hat{y})$ and $\bar{a}\,(w) = \bar{a}\,(\hat{w})$, respectively, which are contained in the unordered histories of terminal nodes succeeding $\hat{w}$; the same holds for w, v, and $\hat{v}$. Since the other properties are quite immediate, this establishes r to be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$.

4.3. Normal form. The representations $\mathrm{ER}^{\jmath}(G)$ contain just as much structure as the games G itself with respect to weak isomorphism. Therefore, we have some (very limited) converse of Theorem 4.3. For the class of traditional extensive game representations of strategic games weak isomorphism and isomorphism of the normal form coincide. Note that $\mathrm{NF}(\mathrm{ER}^{\jmath}(G)) \cong G$.

THEOREM 4.5. *Let $G, \bar{G} \in \mathcal{G}$ be ordered by $\jmath$ and $\bar{\jmath}$, respectively. Then, $G \cong \bar{G}$ iff* $\mathrm{ER}^{\jmath}(G) \cong \mathrm{ER}^{\bar{\jmath}}(\bar{G})$.

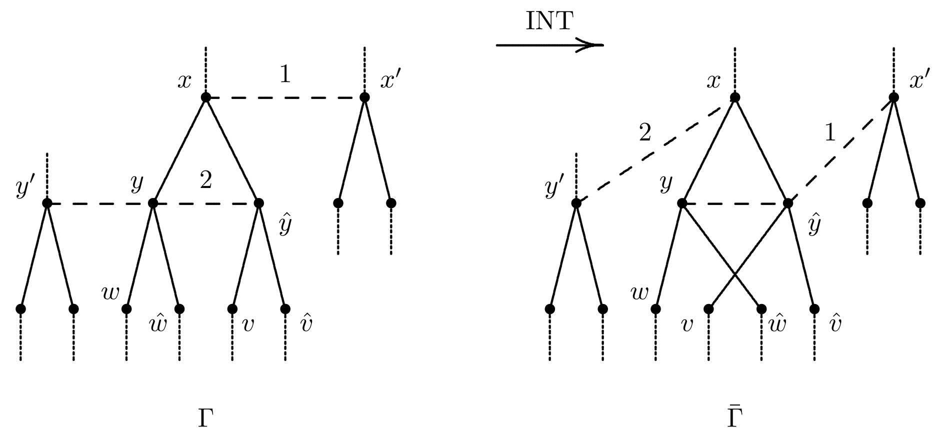

FIGURE 8. The INT transformation

This implies weak isomorphism of an extensive game's ordered representations. E.g., the ordered representations in Figure 1 are weakly isomorphic. In contrast, though having the same normal form, the ordered representations and the PSR ones (Figure 2) are not weakly isomorphic. Another immediate consequence is the desired equivalence of symmetry in strategic games and their traditional representation.

COROLLARY 4.6. *Strategies and players in* $G \in \mathcal{G}$ *are symmetric iff their counterparts in some* $\mathrm{ER}^J(G)$ *are weakly symmetric, respectively.*

4.4. Agent normal form. The strong structure preservation property of weak isomorphism spreads to the agent normal forms.

THEOREM 4.7. *For* $\Gamma, \bar{\Gamma} \in \mathcal{E}$, $\Gamma \cong \bar{\Gamma}$ *implies* $\mathrm{ANF}(\Gamma) \cong \mathrm{ANF}(\bar{\Gamma})$.

Even though the agent normal form contains more structure than the (reduced) normal form, some structure of the original game is lost under transformation. Therefore, the converse of Theorem 4.7 does not hold in general. Instead, we have some weaker generic result for non-pathological games without chance player.

THEOREM 4.8. *Let* $\Gamma, \bar{\Gamma} \in \mathcal{E}^*$ *be without chance player. For generic payoffs,* $\mathrm{ANF}(\Gamma) \cong \mathrm{ANF}(\bar{\Gamma})$ *implies* $\Gamma \cong \bar{\Gamma}$.

The following examples show that we cannot do without the restrictions in Theorem 4.8: By **ISA**, **CPL**, and **CPR**, weak isomorphism is sensitive to alternative decompositions of compound lotteries. For the strategic game representations, of course, this makes no [281] difference. Instead, one could think about restricting oneself to the class of games where the chance player moves (at most) once only at the beginning of the game. Yet, even in this simple case, the converse does not hold: Consider a generic, non-pathological game Γ without chance player and the game Γ̄ which is the same as Γ except for that the root of

Γ has one chance node as immediate predecessor, and the single other successor of that chance node is a terminal node. Then, the agents' payoffs in the agent normal forms differ by positive affine transformations only. Hence, the agent normal forms are isomorphic, while the games itself are not. The ordered representations in Figure 1 and the PRS representations in Figure 2 are another example. While being not weakly isomorphic, the agent normal forms, i.e. the underlying Battle-of-the-Sexes game in strategic form are identical.

Besides the structure of the chance mechanism, the agent normal form disregards the assignment of information sets/agents to players as well as the relation between strategies and outcomes (terminal nodes). Note that in the proof of Theorem 4.8, we account for this by twice employing a genericity argument. As the ADD transformation (see Figure 7) adds some terminal nodes which bear the same payoff vectors as some of the original ones, the transformed game is "non-generic".

Consider the game Γ from Figure 9 and a modification $\bar{\Gamma}$ where the root also is controlled by player 1. Since player 1 and 3 have the same preferences, on the one hand, this example is non-generic. On the other hand, this enables the agent normal forms both of Γ and $\bar{\Gamma}$ to be isomorphic to the strategic game below. In contrast, the games itself are not weakly isomorphic—there are more players in Γ than in $\bar{\Gamma}$. This non-equivalence of Γ and $\bar{\Gamma}$, however, seems to be desirable: While some forward induction argument applies to $\bar{\Gamma}$, there is no such argument for Γ. Whenever player 2 is to move in Γ, he knows that player 1 gave up the sure payoff 2 which only makes sense if she aims at getting the higher payoff 3 by choosing a. So, player 2 should take action C and player 1 take action B. Formally, (Ba, c) is the unique equilibrium surviving iterated deletion of weakly dominated strategies in RNF($\bar{\Gamma}$). Since no strategy is weakly dominated in RNF (Γ), the other equilibrium (A, b, D) survives for Γ.

[282]

	A		B	
	C	D	C	D
a	$2,2,2$	$2,2,2$	$3,1,3$	$0,0,0$
b	$2,2,2$	$2,2,2$	$0,0,0$	$1,3,1$

ANF (Γ) , ANF($\bar{\Gamma}$), NF (Γ) , RNF (Γ)

	C	D
A	$2,2$	$2,2$
Ba	$3,1$	$0,0$
Bb	$0,0$	$1,3$

RNF($\bar{\Gamma}$)

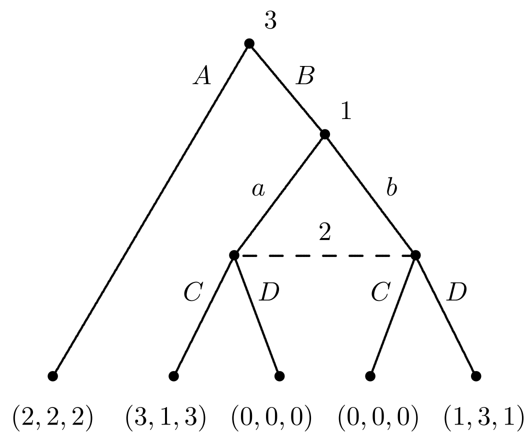

FIGURE 9. Assignment of information sets to players matters

Reconsider the pathological games Γ and $\bar{\Gamma}$ from Example 3.4 which have the same agent normal form:

$\mathrm{ANF}(\Gamma), \mathrm{ANF}(\bar{\Gamma})$

c, d	a	b
A	$1, \varepsilon_2, 1, 1$	$0, 0, 0, 0$
B	$\varepsilon_1, 1, \varepsilon_3, \varepsilon_4$	$\varepsilon_1, 1, \varepsilon_3, \varepsilon_4$

Yet, generically, the games itself cannot be weakly isomorphic: In generic cases, the preferences in Γ and $\bar{\Gamma}$ differ for all players. Hence, any weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$ would induce the mapping θ as in the Example. By **PTH**, we had $r(a(\psi(z_2)) \cup a(\psi(z_3))) = r(\{A, B, b, c, d\}) = \{A, B, b, d\} = \bar{a}(\bar{\psi}(\theta(z_2)) \cup \bar{a}(\bar{\psi}(\theta(z_3))))$, contradicting r being bijective.

5. Invariance of equilibria under weak isomorphism

In order to fit the traditional strategic representations, weak isomorphism sacrifices some structure of extensive games—the strict order of moves. Hence, invariance of solution concepts under weak isomorphism is stronger then the PSR equivalent. Nevertheless, Nash equilibrium, perfect equilibrium, and sequential equilibrium show this invariance in general, and subgame perfect equilibrium in non-pathological cases. Since the Nash and the perfect equilibrium can be defined via the normal form or the agent normal form, respectively, this is not too astonishing. In contrast, subgame perfect equilibrium and sequential equilibrium more directly refer to the sequence of moves. Interestingly, sequential equilibrium is not invariant under the E&R transformations: Kohlberg & Mertens (1986) present an example where the COA transformation produces a new sequential equilibrium. Reconsider Example 4.4: The game Γ is transformed into $\bar{\Gamma}$ by COA. While (A, c) is a sequential equilibrium in $\bar{\Gamma}$ for $\varepsilon_1 \leq 2$, (B, b, d) is the only sequential equilibrium in Γ for $1 < \varepsilon_1 \leq 2$. Of course, Γ and $\bar{\Gamma}$ are not weakly isomorphic.

Together with the invariance of Nash equilibrium (in $\mathcal{G}$) under isomorphism (Harsanyi & Selten 1988), Theorem 4.3 implies the invariance of Nash equilibrium under weak isomorphism. Analogously, Theorem 4.7 implies

THEOREM 5.1. *Perfect equilibrium is invariant with respect to weak isomorphism.*

The same holds true for sequential equilibrium. [283]

THEOREM 5.2. *Sequential equilibrium is invariant with respect weak isomorphism.*

In contrast, subgame perfect equilibrium is not invariant with respect to weak isomorphism, not even generically. Reconsider Example 4.2. Since there is no non-trivial subgame in $\bar{\Gamma}$, the behavior-strategy profile (B, b, c) is a subgame perfect equilibrium. In contrast, the game Γ has a subgame rooted in the decision node of player 3. Since (a) is the unique Nash equilibrium of this subgame, the unique subgame perfect equilibrium of the whole game is (A, a, c). Yet, the weak isomorphism r does not map (B, b, c) onto (A, a, c). Note that this invariance is caused in that r does not respect the subgame structure: The inverse $\Gamma \xrightarrow{r^{-1}} \bar{\Gamma}$ maps the action partition $\{a, b\}$ of the subgame in Γ onto $\{a, b\}$ which is not the action partition of some subgame of $\bar{\Gamma}$. The games in the example, however, are pathological. Without this peculiarity we have

THEOREM 5.3. *For $\mathcal{E}^*$, subgame perfect equilibrium is invariant with respect to weak isomorphism.*

Since one-point solution concepts that are invariant under weak isomorphism select a unique weakly symmetry-invariant behavior-strategy profile, one might be concerned about the existence of weakly symmetry invariant equilibria. With our definition, we allow for a wider range of mappings to be isomorphisms than PSR. So our weak symmetry invariance is more restrictive. Nevertheless, even symmetry invariant perfect equilibria do exist for every extensive game. The existence of weakly symmetry invariant Nash equilibria, sequential equilibria, and subgame perfect equilibria then directly follows from Selten (1975) and Kreps & Wilson (1982).

THEOREM 5.4. *Every extensive game has a weakly symmetry-invariant perfect equilibrium.*

6. Conclusion

In this paper, we introduced and advocated weak isomorphism of extensive games. As the Harsanyi & Selten (1988) isomorphisms of strategic games, isomorphisms of extensive games can be viewed as a means to identify *structurally* similar extensive games and to identify corresponding structural elements of these games—players, information sets, actions, and nodes. And it is this emphasis of structural features that distinguishes isomorphisms from considerations of strategic equivalence as the Kohlberg & Mertens (1986) invariance requirement or the Thompson (1952) and the E&R transformations.

In order to make our isomorphism fit strategic game isomorphism, we had to give up the strong preservation of the order of moves within a game—in contrast to the E&R and PSR ones. Nevertheless, the invariance of equilibrium concepts under weak isomorphism—as a [284]
necessary property of preserving the essence of the order of moves—remains untouched. So weak isomorphism can be viewed as an adequate means to describe structural similarities of extensive games *if* one considers the sequential nature of moves as a technical peculiarity of the extensive game formalism and not necessarily as a representation of sequential choices. In addition, it seems to be the weakest conceivable such a concept of isomorphism. After all, we provide some more justification for the use of these isomorphisms by Selten (1983) and Oh (1995).

Acknowledgements

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Appendix A

Lemma A.1. *Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$. For all $a^+ \in A^+$ and $b \in B$, we have $\text{prob}(a^+|b) = \text{prob}(r^+(a^+)|r(b))$ where r and r^+ are given by r via (3.2) and (3.1), respectively.*

Proof. For all $\mathbf{a}^+ \in \mathbf{A}^+$ and $b \in B$ we have

$$\begin{aligned}
\text{prob}(\mathbf{r}^+(\mathbf{a}^+)|\mathbf{r}(b)) &:= \prod_{\bar{h}\in\bar{H}_{\bar{\imath}_0}} \bar{p}_{\bar{h}}(\mathbf{r}^+_{\bar{h}}(\mathbf{a}^+)) \prod_{\bar{h}\in\bar{H}\setminus\bar{H}_{\bar{\imath}_0}} \mathbf{r}_{\bar{h}}(b)(\mathbf{r}^+_{\bar{h}}(\mathbf{a}^+)) \\
&= \prod_{h\in H_{i_0}} \bar{p}_{\nu(h)}(\mathbf{r}^+_{\nu(h)}(\mathbf{a}^+)) \prod_{h\in H\setminus H_{i_0}} \mathbf{r}_{\nu(h)}(b)(\mathbf{r}^+_{\nu(h)}(\mathbf{a}^+)) \\
&= \prod_{h\in H_{i_0}} \bar{p}_{\nu(h)}(r(\mathbf{a}^+_h)) \prod_{h\in H\setminus H_{i_0}} \mathbf{r}_{\nu(h)}(b)(r(\mathbf{a}^+_h)) \\
&= \prod_{h\in H_{i_0}} p_h(\mathbf{a}^+_h) \prod_{h\in H\setminus H_{i_0}} b_h(\mathbf{a}^+_h) \\
&=: \text{prob}(\mathbf{a}^+|b)
\end{aligned}$$

from (3.2), and (3.1); ν being bijective and **CPL**; (3.2) and (3.1); **CPR** and (3.2), respectively. $\square$

Proof of Lemma 3.3. For all $\mathbf{a}^+ \in \mathbf{A}^+$, we have $\{\mathbf{r}^+_{\bar{h}}(\mathbf{a}^+)|\bar{h} \in \bar{H}\} = \{\mathbf{r}^+_{\nu(h)}(\mathbf{a}^+)|h \in H\}$ $= \{r(\mathbf{a}^+_h), h \in H\} \supseteq r(a(\psi(z(\mathbf{a}^+)))) \supseteq \bar{a}(\bar{\psi}(\theta(z(\mathbf{a}^+))))$ from ν being bijective, (3.1), (2.3), and **PTH**. Hence, $\bar{z}(\mathbf{r}^+(\mathbf{a}^+)) = \theta(z(\mathbf{a}^+))$ by (2.3) and (3.1). $\square$

Proof of Theorem 3.5. Let r be a bijection $r : A \to \bar{A}$ that satisfies **ISA** and **PTH**$^-$. By definition, **PTH**$^-$ induces a bijection $\theta : Z \to \bar{Z}$ such that $\theta(z(\mathbf{a}^+)) = \bar{z}(\mathbf{r}^+(\mathbf{a}^+))$ [285] for all $\mathbf{a}^+ \in \mathbf{A}^+$ where $\mathbf{r}^+$ is determined by r and **ISA** via (3.1). Since $z(\mathbf{A}^+) = Z$, it is sufficient to show that we have $r(a(\psi(z(\mathbf{a}^+)))) = \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+))))$ for all $\mathbf{a}^+ \in \mathbf{A}^+$. Suppose, there were some $\mathbf{a}^+ \in \mathbf{A}^+$ and $h \in H$ such that $\mathbf{a}^+_h \in a(\psi(z(\mathbf{a}^+)))$ but $r(\mathbf{a}^+_h) \notin \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+))))$. Since $\Gamma \in \mathcal{E}^*$, some $\mathbf{a}^{+'} \in \mathbf{A}^+$ existed such that $\mathbf{a}^+_{h'} = \mathbf{a}^{+'}_{h'}$ for all $h' \in H \setminus \{h\}$ and $\mathbf{a}^+_h \neq \mathbf{a}^{+'}_h$. By **ISA** and (2.3), we have $r(\mathbf{a}^{+'}_h) \notin \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+))))$ and therefore $\bar{z}(\mathbf{r}^+(\mathbf{a}^+)) = \bar{z}(\mathbf{r}^+(\mathbf{a}^{+'}))$. Yet, by (2.3), we have $z(\mathbf{a}^+) \neq z(\mathbf{a}^{+'})$, contradicting θ being bijective. Thus, $r(a(\psi(z(\mathbf{a}^+)))) \subseteq \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+))))$. Analogously, one can show that $r(a(\psi(z(\mathbf{a}^+)))) \supseteq \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+))))$. Therefore, $r(a(\psi(z(\mathbf{a}^+)))) = \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+)))) =$ $\bar{a}(\bar{\psi}(\theta(z(\mathbf{a}^+))))$ for all $\mathbf{a}^+ \in \mathbf{A}^+$, i.e., **PTH** holds.

Proof of Theorem 3.5, improved version. Let r be a bijection $r : A \to \bar{A}$ that satisfies **ISA** and **PTH**$^-$. By definition, **PTH**$^-$ induces a bijection $\theta : Z \to \bar{Z}$ such that $\theta(z(\mathbf{a}^+)) = \bar{z}(\mathbf{r}^+(\mathbf{a}^+))$ for all $\mathbf{a}^+ \in \mathbf{A}^+$ where $\mathbf{r}^+$ is determined by r and **sISA** via (3.1). Consider $z \in Z$ and $a \in a(\psi(z))$. Non-pathologically, there is some $a' \in A_{V(a)}$, $a' \neq a$. Let $\mathbf{a}^{+'} \in \mathbf{A}^+$ be such that $\mathbf{a}^{+'}_{V(a)} = a'$ and $\mathbf{a}^{+'}_h = \mathbf{a}_h$ for $h \neq V(a)$. Obviously, $z \neq z\left(\mathbf{a}^{+'}\right)$. Suppose, $r(a) \notin \bar{a}\left(\bar{\psi}(\theta(z))\right)$. Then, we had

$$\bar{a}\left(\bar{\psi}(\theta(z))\right) \subset \left\{\mathbf{r}^+_{\bar{h}}(\mathbf{a})\,|\bar{h} \in \bar{H}\right\} \setminus \{r(a)\} \subset \left\{\mathbf{r}^+_{\bar{h}}\left(\mathbf{a}^{+'}\right)|\bar{h} \in \bar{H}\right\}$$

i.e. $\theta(z) = z\left(\mathbf{r}\left(\mathbf{a}^{+'}\right)\right)$, contradicting **PTH**$^-$. Henze, $r(a(\psi(z))) \subset \bar{a}\left(\bar{\psi}(\theta(z))\right)$. Since the inverse of r satisfies **ISA** and **PTH**$^-$, the converse inclusion is immediate. Hence, **PTH** holds.

The following Lemma gives another implication of **PTH** which is used in some proofs below.

Lemma A.2. *Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$. For all $h \in H$, we have $\mathbf{r}^+(\mathbf{A}^+(h)) =$ $\bar{\mathbf{A}}^+(\nu(h))$ for $\mathbf{r}^+$ determined by r via (3.1).*

Proof. By (2.4), this can be seen from $r(a(\psi(z(\mathbf{a}^+))) \cap A_h) = r(a(\psi(z(\mathbf{a}^+)))) \cap r(A_h) =$ $\bar{a}(\bar{\psi}(\theta(z(\mathbf{a}^+)))) \cap \bar{A}_{\nu(h)} = \bar{a}(\bar{\psi}(\bar{z}(\mathbf{r}^+(\mathbf{a}^+)))) \cap \bar{A}_{\nu(h)}$, where the single equations follow from r being bijective, **PTH** and **ISA**, and Lemma 3.3, respectively. $\square$

Lemma A.3. *Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$. For all $\mathbf{a} \in \mathbf{A}$ and $i \in I \backslash \{i_0\}$, we have $\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i(\mathbf{a}) + \beta_i$ where $\mathbf{r}$ is given by r via (3.2).*

Proof. For all $b \in B$ and $i \in I \backslash \{i_0\}$ we have

$$\begin{aligned} \bar{u}_{\pi(i)}(\mathbf{r}(b)) &:= \sum_{\bar{\mathbf{a}}^+ \in \bar{\mathbf{A}}^+} \mathrm{prob}(\bar{\mathbf{a}}^+ | \mathbf{r}(b)) \cdot \bar{u}_{\pi(i)}(\bar{\mathbf{a}}^+) \\ &= \sum_{\mathbf{a}^+ \in \mathbf{A}^+} \mathrm{prob}(\mathbf{r}^+(\mathbf{a}^+) | \mathbf{r}(b)) \cdot \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^+)) \\ &= \sum_{\mathbf{a}^+ \in \mathbf{A}^+} \mathrm{prob}(\mathbf{a}^+ | b) \cdot \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^+)) \\ &= \alpha_i \left(\sum_{\mathbf{a}^+ \in \mathbf{A}^+} \mathrm{prob}(\mathbf{a}^+ | b) \cdot u_i(\mathbf{a}^+) \right) + \beta_i \\ &=: \alpha_i u_i(b) + \beta_i \end{aligned}$$

from $\mathbf{r}^+$ being bijective, Lemma A.1, (2.3) and **PY**, respectively. □

Proof of Theorem 4.1. Let Γ,$\bar{\Gamma} \in \mathcal{E}^*$, and let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$ inducing $\nu : H \to \bar{H}$ via **ISA**. We first provide a characterization of hist: For $\Gamma \in \mathcal{E}^*$, $a \in \mathrm{hist}(h)$ if and only if $\mathbf{a}^+_{V(a)} = a$ for all $\mathbf{a}^+ \in \mathbf{A}^+(h)$. The if part directly follows from the definition. Suppose on the contrary there were some $a \in A$, $h \in H$ such that $\mathbf{a}^+_{V(a)} = a$ for all $\mathbf{a}^+ \in \mathbf{A}^+(h)$ $\mathbf{a}^+_{V(a)} = a$ and $a \notin \mathrm{hist}(h)$ for some a. Then, there were some $x \in h$ such that $\psi(x) \cap a = \emptyset$. By assumption, there were some $\bar{\mathbf{a}}^+$ such that $x \in \psi(z(\mathbf{a}^+))$ and $\psi(z(\mathbf{a}^+)) \cap a \neq \emptyset$. Let x' be unique element of $\psi(z(\mathbf{a}^+)) \cap a$. So we have $x \lhd x'$. Since the game [286]
is non-pathological, $N(V(x))$ contains some $x'' \neq x'$. Clearly, $a(x'') \neq a(x') = a$. Hence, there were some $\mathbf{a}^{+'} \in \mathbf{A}^+(h)$ with $\mathbf{a}^{+'}_{V(a)} = a(x'') \neq a$. A contradiction.

Let $a \in \mathrm{hist}(h)$. By the characterization, $\mathbf{a}^+ \in \mathbf{A}^+(h)$ implies $\mathbf{a}^+_{V(a)} = a$. By Lemma A.2, we have $\bar{\mathbf{a}}^+_{\nu(V(a))} = r(a)$ for all $\bar{\mathbf{a}}^+ \in \bar{\mathbf{A}}^+(\nu(h))$. Again by the characterization, we have $r(a) \in \overline{\mathrm{hist}}(\nu(h))$ and therefore, $r(\mathrm{hist}(h)) \subseteq \overline{\mathrm{hist}}(\nu(h))$. Since the inverse r^{-1} is a weak isomorphism, the converse inclusion is immediate. This establishes the claim.

Proof of Theorem 4.5. Let $\mathbf{f} = (\pi, (r_i)_{i \in I})$ be some isomorphism $G \xrightarrow{\mathbf{f}} \bar{G}$. It is easy to see that the mapping $r : A^{\jmath} \to \bar{A}^{\bar{\jmath}}$, $r(a^{\jmath}[s_i]) = a^{\bar{\jmath}}[r_i(s_i)]$ for all $i \in I$ and $s_i \in S_i$ is a weak isomorphism $\mathrm{ER}^{\jmath}(G) \xrightarrow{r} \mathrm{ER}^{\bar{\jmath}}(\bar{G})$. Let now r be a weak isomorphism $\mathrm{ER}^{\jmath}(G) \xrightarrow{r} \mathrm{ER}^{\bar{\jmath}}(\bar{G})$ inducing π by **PL**. The system $\mathbf{f} = (\pi, (r_i)_{i \in I})$ with $\pi(i) = \pi^{\jmath}(i)$ for all $i \in I$ and $r_i(s_i) = s_{i'}$ iff $r(a^{\jmath}[s_i]) = a^{\bar{\jmath}}[s_{i'}]$ for all $i, i' \in I$, $s_i \in S_i$, and $s_{i'} \in S_{i'}$ obviously is an isomorphism $G \xrightarrow{\mathbf{f}} \bar{G}$.

Proof of Theorem 4.7. Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$, and let π, ν, and θ be the bijections induced by r according to Definition 3.1. By **ISA** and **CPL**, the restriction

$\bar{\nu} := \nu|_{H \backslash H_{i_0}}$ of ν on $H \backslash H_{i_0}$ is a bijection of $H \backslash H_{i_0}$ onto $\bar{H} \backslash \bar{H}_{\bar{\imath}_0}$. By **ISA**, r can be split into bijections $r_h : A_h \to A_{\bar{\nu}(h)}$ with $r_h(a) := r(a)$ for all $h \in H \backslash H_{i_0}$ and $a \in A$. By Lemma A.3, we have $u_{\bar{\nu}(h)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_h(\mathbf{a}) + \beta_i$ for all $h \in H \backslash H_{i_0}$ and $\mathbf{a} \in \mathbf{A}$. Since by (3.2) $\mathbf{r}$ is induced by the r_h in accordance with (2.2), $\mathbf{f} =(\bar{\nu}, (r_h)_{h \in H \backslash H_{i_0}})$ is an isomorphism ANF $(\Gamma) \xrightarrow{\mathbf{f}}$ ANF$(\bar{\Gamma})$.

Proof of Theorem 4.8. Let $\mathbf{f} = (\nu, (r_h)_{h \in H})$ be an isomorphism ANF $(\Gamma) \xrightarrow{\mathbf{f}}$ ANF$(\bar{\Gamma})$. Since A_{i_0} and $\bar{A}_{\bar{\imath}_0}$ are empty, the mapping $r : A \to \bar{A}$, $r(a) = r_{V(a)}(a)$ for all $a \in A$ is well-defined and satisfies **CPL** and **CPR**. Since $\mathbf{f}$ is an isomorphism, r is bijective and induces the bijection $\nu : H \to \bar{H}$, $r(A_h) = \bar{A}_{\nu(h)}$ for all $h \in H$, and therefore complies **ISA**. For $|I| \leq 1$, r trivially satisfies **PL**. By (2.1), for all $h \in H$, there are $\alpha_h, \beta_h \in \mathbb{R}$, $\alpha_h > 0$, such that $\bar{u}_{\nu(h)}(\mathbf{r}(\mathbf{a})) = \alpha_h u_h(\mathbf{a}) + \beta_h$, i.e.

$$\bar{u}_{\bar{\imath}(\nu(h))}(\bar{z}(\mathbf{r}(\mathbf{a}))) = \alpha_h u_{i(h)}(z(\mathbf{a})) + \beta_h \tag{A.1}$$

for all $\mathbf{a} \in \mathbf{A} = \mathbf{A}^+$, where $\mathbf{r} = \mathbf{r}^+$ is induced by $\mathbf{f}$ and r via (2.1) or (3.2). For $|I| > 1$, we have $|Z| \geq 3$ by $\Gamma \in \mathcal{E}^*$. Hence, generically, the players' preferences are pairwise different. Therefore, $i(h) = i(h')$ and $\bar{\imath}(\nu(h)) = \bar{\imath}(\nu(h'))$ are equivalent for all $h, h' \in H$. Hence, r induces a bijection $\pi : I \to \bar{I}$, $r(A_i) = \bar{A}_{\pi(i)}$ for all $i \in I$, and therefore complies **ISA**. Fix some $i' \in I$. Generic payoffs imply that the $u_{i'}(z)$, $z \in Z$ and the $\bar{u}_{\pi(i')}(\bar{z})$, $\bar{z} \in \bar{Z}$ are pairwise different, respectively. Therefore, $z(\mathbf{a}) = z(\mathbf{a}')$ is equivalent to $\bar{z}(\mathbf{r}(\mathbf{a})) = \bar{z}(\mathbf{r}(\mathbf{a}'))$ for all $\mathbf{a}, \mathbf{a}' \in \mathbf{A}$. Hence, we have a bijection $\theta : Z \to \bar{Z}$ that satisfies **PTH**$^-$. Theorem 3.5 then implies **PTH**. As $z(\mathbf{A}) = Z$ and $\bar{z}(\mathbf{r}(\mathbf{A})) = \bar{z}(\bar{\mathbf{A}}) = \bar{Z}$, **PY** holds by (A.1). Hence, r is a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$.

Proof of Theorem 5.1. Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$, and let b be a perfect equilibrium of Γ. By Selten (1975), b is a perfect equilibrium of ANF (Γ). Let $\mathbf{r}$ be induced by r via (3.2). Since $\mathbf{r}$ also is the bijection induced by some isomorphism ANF$(\Gamma) \xrightarrow{\mathbf{f}}$ ANF$(\bar{\Gamma})$ via (2.2), $\mathbf{r}(b)$ is a perfect equilibrium of ANF$(\bar{\Gamma})$ (see the proof of Theorem 4.7), [287]
hence a perfect equilibrium of $\bar{\Gamma}$ (Selten 1975). Since r^{-1} also is a weak isomorphism, this proves the claim.

Proof of Theorem 5.2. Within our notation, sequential equilibria are defined as follows: For $\Gamma \in \mathcal{E}$, a system of beliefs is a mapping $\mu : X \to [0,1]$ satisfying $\sum_{x \in h} \mu(x) = 1$ for all $h \in H \backslash H_{i_0}$. Together with a behavior-strategy profile it is called an assessment. An assessment (μ, b) is a sequential equilibrium if it is consistent and sequentially rational.

The payoff $u_h(\mu, b)$ of a personal player $i(h)$ at $h \in H \backslash H_{i_0}$ with respect to μ is defined via

$$\begin{aligned}
\operatorname{prob}(z|b,x) &= \begin{cases} \prod\limits_{\substack{a \in a(\psi(z)\backslash\psi(x)) \\ a \notin A_{i_0}}} b_{V(a)}(a) \prod\limits_{\substack{a \in a(\psi(z)\backslash\psi(x)) \\ a \in A_{i_0}}} p_{V(a)}(a) & : \ z \in Z(x), \\ 0 & : \ z \notin Z(x), \end{cases} && \text{(A.2a)} \\
u_x(b) &= \sum_{z \in Z(x)} \frac{\operatorname{prob}(z|b,x)}{\sum\limits_{z' \in Z(x)} \operatorname{prob}(z'|b,x)} u_{i(x)}(z), && \text{(A.2b)} \\
u_h(\mu, b) &= \sum_{x \in h} \mu(x) u_x(b), && \text{(A.2c)}
\end{aligned}$$

where $Z(x)$ denotes the set of terminal nodes succeeding x. An assessment (μ, b) is called *sequentially rational* if $u_h(\mu, b) \geq u_h(\mu, b'_i b_{-i})$ for all $i \in I \backslash \{i_0\}$, $h \in H_i$, and $b'_i \in B_i$, where $b'_i b_{-i}$ denotes the behavior-strategy profile in which all personal players follow b, except for player i who follows b'_i.

For $b \in B^0$, let $\mu(b)$ denote the system of beliefs that is associated with b via Bayes' rule, i.e., we have

$$\mu(b)(x) := \frac{\operatorname{prob}(x|b)}{\operatorname{prob}(h(x)|b)} = \frac{\sum\limits_{\mathbf{a}^+ \in \mathbf{A}^+(x)} \operatorname{prob}(\mathbf{a}^+|b)}{\sum\limits_{\mathbf{a}^+ \in \mathbf{A}^+(h(x))} \operatorname{prob}(\mathbf{a}^+|b)}. \qquad \text{(A.3)}$$

An assessment (μ, b) is called *consistent* if there is some sequence $(b^n)_{n \in \mathbb{N}}$ in B_0 such that $\lim_{n \to \infty}(\mu(b^n), b^n) = (\mu, b)$.

Proof. Let r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$ with ν, $\mathbf{r}^+$, and $\mathbf{r}$ induced via **ISA**, (3.1), and (3.2), respectively. Let further (μ, b) be a sequential equilibrium of Γ. By definition, there is some sequence $(\mu(b^n), b^n)_{n \in \mathbb{N}}$, $b^n \in B^0$ with $\lim_{n \to \infty}(\mu(b^n), b^n) = (\mu, b)$. Consider the sequence $(\bar{\mu}(\mathbf{r}(b^n)), \mathbf{r}(b^n))_{n \in \mathbb{N}}$. By (3.2), $\mathbf{r}(b^n) \in \bar{B}^0$. Since the set of belief systems on $\bar{\Gamma}$ is compact, $(\bar{\mu}(\mathbf{r}(b^n)))_{n \in \mathbb{N}}$ contains a converging subsequence. For notational parsimony, let $(\bar{\mu}(\mathbf{r}(b^n)))_{n \in \mathbb{N}}$ itself be this subsequence. Since $\mathbf{r}$ is continuous, we have $\lim_{n \to \infty}(\bar{\mu}(\mathbf{r}(b^n)), \mathbf{r}(b^n)) = (\bar{\mu}, \mathbf{r}(b))$ for some system of beliefs $\bar{\mu}$. Hence, $(\bar{\mu}, \mathbf{r}(b))$ is consistent.

The beliefs $\mu(b)$ derived from b by Bayes' rule at some information set h are not affected by changing player $i(h)$'s part of b only: By perfect recall, any action of $i(h)$ that precedes [288]
some $x \in h$ also precedes all other nodes of h. Therefore, changing $i(h)$'s behavior strategy only changes $\operatorname{prob}(x|b)$ for all $x \in h$ by the same factor, and $\mu(b)(x)$ remains unchanged. So we have

$$u_h(\mu(b), b) = u_h(\mu(b'), b) \qquad \text{(A.4)}$$

for all $b, b' \in B^0$ with $b'_{i'} = b_{i'}$ for all $i' \neq i(h)$. For all $h \in H \backslash H_{i_0}$ and $b \in B^0$, we have

$$\begin{aligned}
\bar{u}_{\nu(h)}(\bar{\mu}(\mathbf{r}(b)), \mathbf{r}(b)) &= \sum_{\bar{\mathbf{a}}^+ \in \bar{\mathbf{A}}^+(\nu(h))} \frac{\text{prob}(\bar{\mathbf{a}}^+ | \mathbf{r}(b))}{\sum_{\bar{\mathbf{a}}^{+\prime} \in \bar{\mathbf{A}}^+(\nu(h))} \text{prob}(\bar{\mathbf{a}}^{+\prime} | \mathbf{r}(b))} \bar{u}_{\bar{\imath}(\nu(h))}(\bar{\mathbf{a}}^+) \\
&= \sum_{\mathbf{a}^+ \in \mathbf{A}^+(h)} \frac{\text{prob}(\mathbf{r}^+(\mathbf{a}^+) | \mathbf{r}(b))}{\sum_{\mathbf{a}^{+\prime} \in \mathbf{A}^+(h)} \text{prob}(\mathbf{r}^+(\mathbf{a}^{+\prime}) | \mathbf{r}(b))} \bar{u}_{\pi(i(h))}(\mathbf{r}^+(\mathbf{a}^+)) \\
&= \sum_{\mathbf{a}^+ \in \mathbf{A}^+(h)} \frac{\text{prob}(\mathbf{a}^+ | b)}{\sum_{\mathbf{a}^{+\prime} \in \mathbf{A}^+(h)} \text{prob}(\mathbf{a}^{+\prime} | b)} \bar{u}_{\pi(i(h))}(\mathbf{r}^+(\mathbf{a}^+)) \\
&= \sum_{\mathbf{a}^+ \in \mathbf{A}^+(h)} \frac{\text{prob}(\mathbf{a}^+ | b))}{\sum_{\mathbf{a}^{+\prime} \in \mathbf{A}^+(h)} \text{prob}(\mathbf{a}^{+\prime} | b)} \left(\alpha_{i(h)} u_{i(h)}(\mathbf{a}^+) + \beta_{i(h)}\right) \\
&= \alpha_{i(h)} u_h(\mu(b), b) + \beta_{i(h)} \qquad \text{(A.5)}
\end{aligned}$$

from (A.2), v being bijective and Lemma A.2, Lemma A.1, Lemma A.3, and (A.2), respectively.

Suppose $(\bar{\mu}, \mathbf{r}(b))$ were not sequentially rational. Then, some $\bar{\imath} \in \bar{I} \backslash \{\bar{\imath}_0\}$, $\bar{h} \in \bar{H}_{\bar{\imath}}$ and $\bar{b} \in \bar{B}$ existed such that $\bar{b}_{\bar{h}'} = \mathbf{r}_{\bar{h}'}(b)$ for all $\bar{h}' \notin \bar{H}_{\bar{\imath}}$ and $\bar{u}_{\bar{h}}(\bar{\mu}, \bar{b}) > \bar{u}_{\bar{h}}(\bar{\mu}, \mathbf{r}(b))$. Let $(\bar{b}^n)_{n \in \mathbb{N}}$ be some sequence such that $\lim_{n \to \infty} \bar{b}^n = \bar{b}$, $\bar{b}^n \in \bar{B}^0$, and $\bar{b}^n_{\bar{h}'} = \mathbf{r}_{\bar{h}'}(b^n)$ for all $\bar{h}' \notin \bar{H}_{\bar{\imath}}$. Since $\bar{u}_{\bar{h}}$ is continuous, there are $n_0 \in \mathbb{N}$ and $0 < \varepsilon \in \mathbb{R}$ such that

$$\bar{u}_{\bar{h}}(\bar{\mu}(\mathbf{r}(b^n)), \bar{b}^n) > \bar{u}_{\bar{h}}(\bar{\mu}(\mathbf{r}(b^n)), \mathbf{r}(b^n)) + \varepsilon$$

for all $n_0 < n$. Since $\bar{b}^n$ only differs from $\mathbf{r}(b^n)$ at information sets of player $\bar{\imath}(\bar{h})$, by (A.4), we have

$$\bar{u}_{\bar{h}}(\bar{\mu}(\bar{b}^n), \bar{b}^n) > \bar{u}_{\bar{h}}(\bar{\mu}(\mathbf{r}(b^n)), \mathbf{r}(b^n)) + \varepsilon$$

and, by (A.5),

$$u_{\nu^{-1}(\bar{h})}(\mu(\mathbf{r}^{-1}(\bar{b}^n)), \mathbf{r}^{-1}(\bar{b}^n)) > u_{\nu^{-1}(\bar{h})}(\mu(b^n), b^n) + \frac{\varepsilon}{\alpha_{i(\nu^{-1}(\bar{h}))}}.$$

Since by **PL** b^n and $\mathbf{r}^{-1}(\bar{b}^n)$ differ at information sets of player $i(\nu^{-1}(\bar{h}))$ only, by (A.4), we have

$$u_{\nu^{-1}(\bar{h})}(\mu(b^n), \mathbf{r}^{-1}(\bar{b}^n)) > u_{\nu^{-1}(\bar{h})}(\mu(b^n), b^n) + \frac{\varepsilon}{\alpha_{i(\nu^{-1}(\bar{h}))}}$$

and, since u_h, μ, and $\mathbf{r}^{-1}$ are continuous,

$$u_{\nu^{-1}(\bar{h})}(\mu, \mathbf{r}^{-1}(\bar{b})) > u_{\nu^{-1}(\bar{h})}(\mu, b),$$ [289]

where $\mathbf{r}^{-1}(\bar{b})$ differs from b at information sets of player $i(\nu^{-1}(\bar{h}))$ only. This contradicts (μ, b) being sequentially rational. Thus, $(\bar{\mu}, \mathbf{r}(b))$ is a sequential equilibrium. Since converses of weak isomorphisms are weak isomorphisms, this proves the claim. □

Proof of Theorem 5.3. The proof is prepared by a lemma. Extending Selten's (1983, Theorem 1) proof for symmetries to weak isomorphisms and the case of more than two players, we have

Lemma A.4. *Let* $\Gamma, \bar{\Gamma} \in \mathcal{E}^*$, *and let* $r : A \to \bar{A}$ *be a bijection that satisfies* ***ISA*** *and* ***PTH***. *Then, for any subgame* Γ^x *there is some subgame* $\bar{\Gamma}^{\bar{x}}$ *such that* $r(A^x) = \bar{A}^{\bar{x}}$.

Proof. (Theorem 5.3) Let $\Gamma, \bar{\Gamma} \in \mathcal{E}^*$, r be a weak isomorphism $\Gamma \xrightarrow{r} \bar{\Gamma}$ with $\mathbf{r} : B \to \bar{B}$ induced via (3.2), $\bar{\Gamma}^{\bar{x}}$ some subgame of $\bar{\Gamma}$, and b some subgame perfect equilibrium of Γ. We have to show that the restriction $\mathbf{r}(b)^{\bar{x}}$ of $\mathbf{r}(b)$ to $\bar{\Gamma}^{\bar{x}}$ is a Nash equilibrium of $\bar{\Gamma}^{\bar{x}}$. By Lemma A.5, there is some subgame Γ^x such that $r(A^x) = \bar{A}^{\bar{x}}$. Then, the restriction r^x of r to A^x is a weak isomorphism $\Gamma^x \xrightarrow{r^x} \bar{\Gamma}^{\bar{x}}$: Obviously, r^x is bijective and inherits the properties **ISA**, **CPL**, **CPR.** For all $i \in I^x$ we have $r(A_i^x) = r(A_i \cap A^x) = r(A_i) \cap r(A^x)$ $= A_{\pi(i)} \cap A^x = A^x_{\pi(i)}$, i.e. $\pi(I^x) = I^x$; r^x also satisfies **PL**. Since $a(\psi^x(z)) = a(\psi(z)) \cap A^x$ for all $z \in Z^x$, we have $r^x(a(\psi^x(z))) = r(a(\psi(z)) \cap A^x) = r(a(\psi(z))) \cap A^x = a(\psi(\theta(z))) \cap A^x = a(\psi^x(\theta(z)))$. Hence, **PTH** and **PY** hold. Since b is subgame perfect, the restriction b^x of b to Γ^x is a Nash equilibrium. By **ISA** and the invariance of Nash equilibrium under weak isomorphism $\mathbf{r}(b)^{\bar{x}} = \mathbf{r}^x(b^x)$ is a Nash equilibrium of $\bar{\Gamma}^{\bar{x}}$. □

Proof of Theorem 5.4. Symmetry invariant equilibria of finite strategic games do always exist (Nash 1951). Together with the continuity of f given by (2.2), applied within the usual existence proofs for perfect equilibria, this implies the existence of symmetry-invariant perfect equilibria (in $\mathcal{G}$). So any ANF(Γ) has a symmetry-invariant perfect equilibrium b, which also is a perfect equilibrium of Γ (Selten 1975). Remains to show that b is weakly symmetry invariant in Γ. This can be restated as follows: Weakly symmetric actions of personal players in Γ are symmetric in ANF(Γ). Let r be a weak symmetry of Γ and let $r(a) = \bar{a}$ for $a \in A_h$, $\bar{a} \in A_{\bar{h}}$, and $h, \bar{h} \in H \backslash H_{i_0}$. The symmetry $\mathbf{f}$ of ANF(Γ) from the proof of Theorem 4.7 then gives $\bar{\nu}(h) = \bar{h}$ and $r_h(a) = \bar{a}$.

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CHAPTER III

Super weak isomorphism of extensive games

This Chapter has been published as "André Casajus (2006): Super weak isomorphism of extensive games, in: *Mathematical Social Sciences* 51, 107–116". Unfortunately, the published version contains a lot of misprints which appeared *after* proof-reading. The margin notes indicate the first lines of the respective pages in the published version.

Abstract [107]

It is well-known that the normal form suffices to determine some but not to determine all sequential equilibria of a game in general. How much more structure does so? In this addendum to Casajus (2003), we suggest the concept of super weak isomorphism (SWI) as an attempt to answer this question. In contrast to weak isomorphism, SWI is not sensitive to the structure of the chance mechanism and the assignment of payoffs to the individual terminal nodes. Yet, sequential equilibrium remains invariant under SWI, i.e. the structural features preserved by SWI already determine sequential equilibrium. In addition, SWI is generically equivalent to isomorphism of the agent normal form for a larger set of games than weak isomorphism.

Key Words: Symmetry, Representation, Equivalence, Sequential equilibrium, Agent normal form.

JEL classification: C72.

1. Introduction

There are games with the same agent normal (ANF) form but different sets of sequential equilibria (e.g. Kreps & Wilson 1982, Figures 2 and 13). Hence in general, the ANF does not suffice the determine *all* sequential equilibria of an extensive game. Generically, however, it does so: Generically, sequential equilibrium coincides with perfect equilibrium which can be defined via the ANF (Selten 1975). Kohlberg & Mertens (1986) show that the normal form suffices to find *some* of the sequential equilibria of an extensive game: Proper [108]
equilibria (Myerson 1978) of strategic games can be extended into sequential equilibria of extensive games with that normal form.

Which part of the structure of extensive games suffices to determine sequential equilibrium? We employ isomorphism to characterize structural features: Isomorphic games share the features implicit in the concept of isomorphism under consideration. For extensive games, there are two such concepts, strong isomorphism (Elmes & Reny 1994, Peleg et al. 1999) and weak isomorphism (WI) (Casajus 2003, henceforth CA03[1]). In addition, the Harsanyi & Selten (1988) isomorphism of the ANF (*ANF isomorphism*) or of the (reduced) normal form can be regarded as such concepts. The question then is whether sequential equilibrium is invariant under the isomorphism under consideration. Our leading example reveals that ANF isomorphism is not such a concept.

Sequential equilibrium is invariant under strong isomorphism and WI. Yet, both concepts keep (most of) the structure of extensive games. Can we do with less? We can. In this paper, we relax WI into the concept of *super weak isomorphism* (SWI) which ignores the structure of the chance mechanism while preserving the payoffs of strategy profiles. This way, the generic equivalence of WI and ANF isomorphism extends to some subset of games with a chance mechanism (Theorem 3.6). Nevertheless, sequential equilibrium remains invariant under SWI (Theorem 3.7). To enable this, SWI must preserve the sequential structure beyond the ANF. This, however, seems to be in line with Govindan & Wilson (2004) who "accept the relevance of extensive form analysis" and weaken the reduced normal form invariance requirement of Kohlberg & Mertens (1986).

This note is organized as follows: Basic definitions and notation not found in CA03 are given in the next section. In the third one, we relax WI into SWI and explore its properties. Some remarks conclude the note. The appendix contains some proofs.

2. Basic definitions and notation

We only give the definitions and notation not given in or deviating from CA03. In order to avoid set theoretic complications, we assume that there is a set which contains all

[1]Also Chapter II of this thesis.

labels for players, pure strategies, and nodes. This way, the collections of all games and of all forms (strategic, extensive) are sets.

We set $i_0 = 0$, $I_- = I\backslash\{0\}$, $A_- := A\backslash A_0$, $H_- := H\backslash H_0$. The *reduced terminal history* of $z \in Z$ is the set $A_-(z) := A(\psi(z))\backslash A_0$. Further,

$$Z(\mathbf{a}) := \{z \in Z | \exists \mathbf{a}_0 \in \mathbf{A}_0 : z = z(\mathbf{a}, \mathbf{a}_0)\} = \{z \in Z | A_-(z) \subseteq \{\mathbf{a}_h | h \in H_-\}\} \tag{2.1}$$

denotes the subset of Z reachable by $\mathbf{a}$.

We denote by $\mathcal{E}^{\mathrm{nc}} \subset \mathcal{E}$ the set of games with $P_0 = \emptyset$. An *(extensive) form* γ is a tuple $(T, \lhd, I, P, H, A)$ where the constituents are defined as in $\mathcal{E}$. $\mathcal{EF}$ $(\mathcal{EF}^*, \mathcal{EF}^{\mathrm{nc}})$ denotes the set of forms corresponding to $\mathcal{E}$ $(\mathcal{E}^*, \mathcal{E}^{\mathrm{nc}})$. Any $\Gamma \in \mathcal{E}$ based on a fixed $\gamma \in \mathcal{EF}$ can be described by an *assignment* $\delta = (u, p) \in \mathbb{D}(\gamma) := \mathbb{U}(\gamma) \times \mathbb{W}(\gamma)$, where $\mathbb{U}(\gamma) := \mathbb{R}^{|Z||I_-|}$, $\mathbb{W}(\gamma) := \prod_{h \in H_0} \Delta_{|A_h|-1}$, $u = (u_i)_{i \in I_-}$, $u_i \in \mathbb{R}^{|Z|}$, $p = (p_h)_{h \in H_0}$, $p_h \in \Delta_{|A_h|-1}$, and where $\Delta_k \subseteq \mathbb{R}^{k+1}$, $k \in \mathbb{N}$ denotes the k-dimensional standard simplex. We then write $\Gamma = \gamma(\delta)$. A proposition on pairs of games from $\mathcal{E}' \subseteq \mathcal{E}$ based on $\mathcal{EF}' \subseteq \mathcal{EF}$ holds *generically* iff for all $\gamma' \in \mathcal{EF}'$ there is some open and dense subset $\mathcal{D}(\gamma') \subseteq \mathbb{D}(\gamma')$ such that for all $(\gamma, \bar{\gamma}) \in \mathcal{EF}' \times \mathcal{EF}'$ the proposition holds for all $(\gamma(\delta), \bar{\gamma}(\bar{\delta}))$, $\delta \in \mathcal{D}(\gamma)$, $\bar{\delta} \in \mathcal{D}(\bar{\gamma})$.

3. Super weak isomorphism [109]

3.1. Definitions. The following definition relaxes weak isomorphism by dropping its conditions related to the chance mechanism (**CPL**, **CPR**) and by weakening the other conditions accordingly. The latter is indicated by the prefix "**s**" which should be read as "superweak version of". Non-technically, a super weak isomorphism is an isomorphism of the ANF (**sISA**, **sPY**) that respects the assignment of information sets to players (**sPL**) and therefore also is an isomorphism of the normal form, for example. In addition, it preserves the RTH structure (**sPTH**).

Definition 3.1. *A* super weak isomorphism (SWI) *from $\gamma \in \mathcal{EF}$ to $\bar{\gamma} \in \mathcal{EF}$ is a bijection $r : A_- \to \bar{A}_-$ with the following properties: There are bijections $\nu : H_- \to \bar{H}_-$, $\pi : I_- \to \bar{I}_-$, and a surjective and nowhere empty correspondence $\Theta : Z \rightrightarrows \bar{Z}$ such that*

sISA $r(A_h) = \bar{A}_{\nu(h)}$ *for all* $h \in H_-$,

sPL $r(A_i) = \bar{A}_{\pi(i)}$ *for all* $i \in I_-$,

sPTH $r(A_-(z)) = \bar{A}_-(\bar{z})$ *for all* $z \in Z$ *and* $\bar{z} \in \Theta(z)$.

A SWI from $\Gamma \in \mathcal{E}$ to $\bar{\Gamma} \in \mathcal{E}$ is a SWI of the underlying forms which satisfies

sPY *for all $i \in I_-$, there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that*
$\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i(\mathbf{a}) + \beta_i$ *for all* $\mathbf{a} \in \mathbf{A}$
where $\mathbf{r} = (\mathbf{r}_{\bar{h}})_{\bar{h} \in \bar{H}_-} : \mathbf{A} \to \bar{\mathbf{A}}$, $\mathbf{r}_{\nu(h)}(\mathbf{a}) = r(\mathbf{a}_h)$.

SWI games, *SWI invariant* solution concepts and *SWI invariant* behavior-strategy profiles are defined in analogy to their WI counterparts. Obviously, r uniquely determines the bijections ν and π. In addition, **sISA** secures that the mapping $\mathbf{r}$ used in **sPY** is well-defined and bijective. $\mathbf{r}$ is extended to behavior-strategy profiles by CA03 (Equation (3.2)).

3.2. Condition sPTH. RTH determine a possibly non-atomic partition $[Z]$ of Z, $[Z] := \{[z] \,|z \in Z\}$, $z' \in [z]$ iff $A_-(z) = A_-(z')$ where $[z]$ is called the *terminal cell* containing z and $A_-([z])$ its RTH. Denote by $[Z](\mathbf{a}) \subseteq [Z]$ the set of terminal cells reachable by $\mathbf{a}$, and by $\mathbf{A}([z]) \subseteq \mathbf{A}$ its converse, $\mathbf{a} \in \mathbf{A}([z])$ iff $[z] \in [Z](\mathbf{a})$.

The correspondence Θ from **sPTH** is unique in the following sense: By **sPTH**, we have $\bar{A}_-(\bar{z}) = \bar{A}_-(\bar{z}')$ for $\bar{z}, \bar{z}' \in \Theta(z)$ and $\Theta(z) \cap \Theta(z') = \emptyset$ if $z' \notin [z]$. Since Θ is surjective, r uniquely defines a bijection $\theta : [Z] \rightarrow [\bar{Z}]$,

$$r(A_-([z])) = \bar{A}_-(\theta([z])), \quad [z] \in [Z]. \tag{3.1}$$

In fact, **sPTH** and the existence of such a bijection θ are equivalent, and we sometimes refer to (3.1) by **sPTH**. Similar to WI, there is a characterization of **sPTH** for $\mathcal{E}^*$ involving θ. Its proof is referred to the Appendix.

LEMMA 3.2. *(i)* ***sISA*** *and* ***sPTH*** *imply* ***sPTH***$^-$*:* $[\bar{Z}](\mathbf{r}(\mathbf{a})) = \theta([Z](\mathbf{a}))$ *for all* $\mathbf{a} \in \mathbf{A}$. *(ii) In* $\mathcal{EF}^*$, ***sISA*** *and* ***sPTH***$^-$ *imply* ***sPTH***.

3.3. SWI vs. weak isomorphism. The following theorem establishes the relation between SWI and WI. Part (i) says that SWI weakens WI, and part (ii) says that, compared with WI, SWI just disregards the structure of the chance mechanism. While part (i) is immediate from CA03 (Lemma A.3), part (ii) follows from $|[Z](\mathbf{a})| = 1$ and $[z] = \{z\}$ for $\Gamma \in \mathcal{E}^{\text{nc}}$. [110]

THEOREM 3.3. *(i) For any WI* $r : \Gamma \rightarrow \bar{\Gamma}$, *the restriction to* A_- *is a SWI* $r|_{A_-} : \Gamma \rightarrow \bar{\Gamma}$. *(ii) For* $\Gamma, \bar{\Gamma} \in \mathcal{E}^{nc}$, *any SWI* $r : \Gamma \rightarrow \bar{\Gamma}$ *also is a WI.*

The following example shows that SWI non-trivially weakens WI.[2] Casajus (2005) presents general constructions that yield SWI games: the spurious addition of chance nodes and shifting the chance mechanism to the root. Also, alternative but equivalent decompositions of a chance node's lottery do not affect SWI.

EXAMPLE 3.4. Consider $\gamma, \bar{\gamma} \in \mathcal{EF}$ in Figure 3.1 where all information sets are controlled by different players. In both forms, the root is the only chance node, and the chance actions are non-redundant in the following sense. There is an information set that follows a_0 ($\bar{a}_0$) but not a_0' ($\bar{a}_0'$). Consider the bijection $r : A_- \rightarrow \bar{A}_-$, $a \mapsto \bar{a}$ for $a \in \{L,$

[2] I wish to thank an anonymous referee for suggesting to look for such an example.

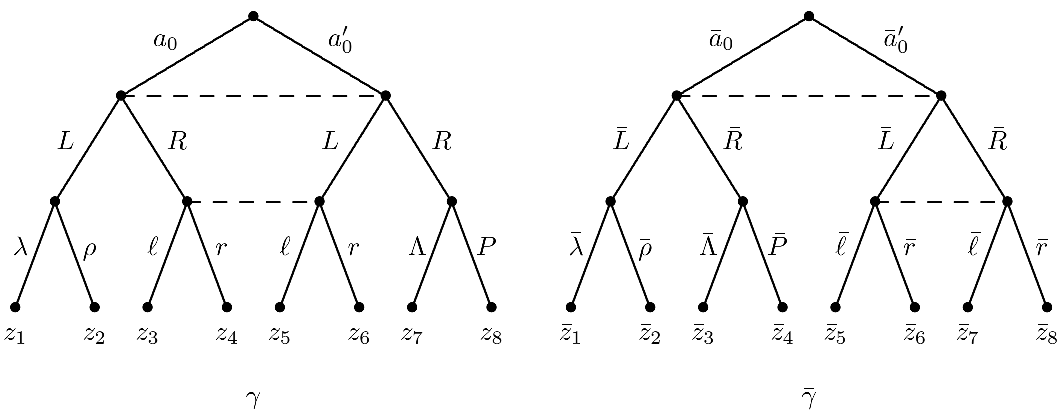

FIGURE 3.1. SWI forms that are not weakly isomorphic

$R, \ell, r, \Lambda, P, \lambda, \rho\}$. Obviously, this mapping satisfies **sISA** and **sPL**. In addition, it easy to check that r satisfies **sPTH** via the bijection $\theta : [Z] \to [\bar{Z}]$, $\theta([z_k]) = [\bar{z}_k]$ for $k = 1, 2, 5, 6$ and $\theta([z_3]) = [\bar{z}_7]$, $\theta([z_4]) = [\bar{z}_8]$, $\theta([z_7]) = [\bar{z}_3]$, $\theta([z_8]) = [\bar{z}_4]$. Hence, r is a SWI from γ to $\bar{\gamma}$. Yet, γ and $\bar{\gamma}$ cannot be WI: In γ, the action λ and the action Λ are contained in exactly one terminal history, and these terminal histories contain different chance actions, a_0 and a'_0, respectively. In contrast in $\bar{\gamma}$, just the actions $\bar{\lambda}$, $\bar{\rho}$, $\bar{\Lambda}$, and $\bar{P}$ are contained in exactly one terminal history where all these terminal histories contain the same chance action, $\bar{a}_0$.

3.4. SWI vs. ANF isomorphism. Obviously, any SWI $r : \Gamma \to \bar{\Gamma}$ induces an isomorphism $(\nu, (r|_{A_h})_{h \in H_-}) : \text{ANF}(\Gamma) \to \text{ANF}(\bar{\Gamma})$ where ν is determined via **sISA**. The converse, however, does not hold in general. Yet by Theorem 3.3, CA03 (Theorem 4.8) also applies to SWI: For $\mathcal{E}^* \cap \mathcal{E}^{\text{nc}}$, SWI and ANF isomorphism are generically equivalent. [111]
Even though SWI largely disregards the chance mechanism, the following example reveals that this does not hold true for the whole set $\mathcal{E}^*$.

EXAMPLE 3.5. Consider $\gamma, \bar{\gamma} \in \mathcal{EF}$ in Figure 3.2 where just the roots are chance nodes (chance probabilities in brackets) and where the non-chance information sets are controlled by different players. γ and $\bar{\gamma}$ are not SWI: While in γ all RTH contain two actions, there is singleton one in $\bar{\gamma}$, $\bar{A}_-(\bar{z}_1) = \{\bar{L}\}$. Yet in the Appendix, we show that for all $\delta \in \mathbb{D}(\gamma)$ there is some $\bar{\delta} \in \mathbb{D}(\bar{\gamma})$ (and vice versa) such that $\gamma(\delta)$ and $\bar{\gamma}(\bar{\delta})$ are ANF isomorphic, contradicting genericity.

For SWI, CA03 (Theorem 4.8) can be extended to the set $\mathcal{E}^{\text{reg}}$ ($\mathcal{EF}^{\text{reg}}$) of regular games (forms). Let $H_-([z])$ denote the set of non-chance information sets corresponding to $A_-([z])$. A game (form) is called *regular* iff for all $[z], [z'] \in [Z]$, $[z] \neq [z']$, $H_-([z]) \cap H_-([z']) \neq \emptyset$ implies $\mathbf{A}([z]) \cap \mathbf{A}([z']) = \emptyset$, i.e., iff the RTH induced by the same strategy profile do not intersect. Of course, regularity is a strong property. Since $|[Z](\mathbf{a})| = 1$ in

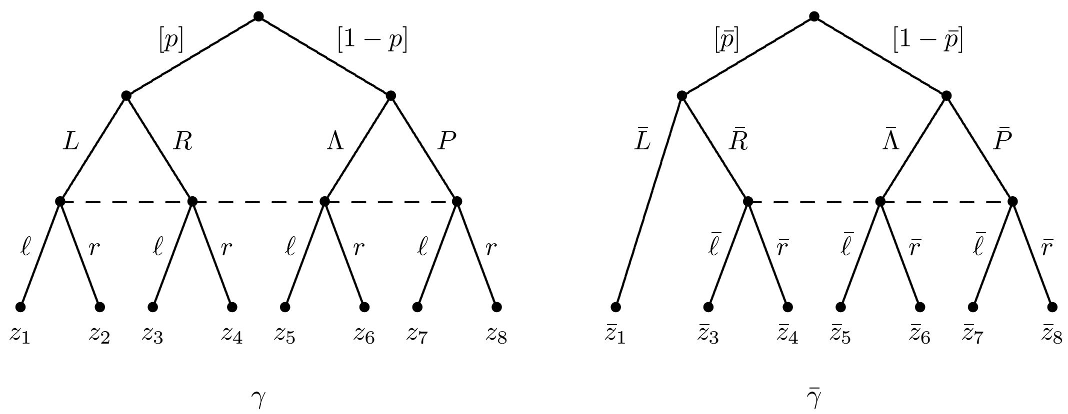

FIGURE 3.2. Non-SWI game forms

$\mathcal{E}^{\mathrm{nc}}$, we have $\mathcal{E}^{\mathrm{nc}} \subseteq \mathcal{E}^{\mathrm{reg}}$. For example, we obtain regular forms by connecting the root of two forms from $\mathcal{EF}^{\mathrm{nc}}$ with a chance node as the new root; the forms in Figure 3.2 are not regular. The proof of the following Theorem is referred to the Appendix.

THEOREM 3.6. *In $\mathcal{E}^{*} \cap \mathcal{E}^{reg}$, generically, any ANF isomorphism $\mathbf{f} = (\nu, (r_h)_{h \in H_-})$ from Γ to $\bar{\Gamma}$ induces a SWI $r : A_- \to \bar{A}_-$, $a \mapsto r_{V(a)}(a)$.*

3.5. Invariance under SWI. Since SWI preserves the (agent) normal form, the arguments for CA03 (Theorems 5.1 and 5.4) apply: SWI invariant perfect equilibria do always exist. Moreover, solution concepts that are based on the fixed (agent) normal form are SWI invariant, e.g. Nash and perfect equilibrium.

This argument does not work for sequential equilibrium because the Kreps & Wilson (1982, Proposition 6) characterization involves a sequence of payoff functions of the extensive game. Nevertheless, sequential equilibrium remains invariant of under SWI. But there are ANF isomorphic extensive games which are not SWI while any ANF isomorphism establishes a bijection of the set of sequential equilibria. By arguments in the proofs to Example 3.5 and of Theorem 3.7, one can show that the game forms in Figure 3.2 give rise to such games. A proof of the following Theorem can be found in the Appendix.

THEOREM 3.7. *Sequential equilibrium is SWI invariant.*

4. Concluding remarks [112]

In this note, we tried to answer the following question: Is it possible (via some concept of extensive game isomorphism) both to keep as less information as enables this concept and ANF isomorphism to be *generically* equivalent and to keep as much information as needed for the determination of *all* sequential equilibria?

Our answer is a partial one: For extensive games without chance mechanism, WI already does the job. SWI goes a little farther: Being equivalent to WI for games with out chance mechanism, it relaxes WI for general games in a way such that sequential equilibrium remains invariant. But even in spite of its disregard of the chance mechanism to a large extent and of the players' detailed preferences over individual terminal nodes, SWI makes only a small step towards generic equivalence which now extends to games that satisfy a strong regularity requirement. Even generically, the presence of a chance mechanism seems to enhance the structure of extensive games far beyond the ANF. Remains the question whether SWI can be further relaxed towards generic equivalence to ANF isomorphism without loosing the invariance of sequential equilibrium.

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Appendix A

Proof of Lemma 3.2. (i) $[z] \in [Z](\mathbf{a}) \Leftrightarrow A_-([z]) \subseteq \{\mathbf{a}_h | h \in H_-\}$ by (2.1), $\Leftrightarrow r(A_-([z])) \subseteq \{r(\mathbf{a}_h) | h \in H_-\}$ by bijectivity of r, $\Leftrightarrow r(A_-([z])) \subseteq \{\mathbf{r}_{\nu(h)}(\mathbf{a}) | h \in H_-\}$ by **sPY**, $\Leftrightarrow \bar{A}_-(\theta([z])) \subseteq \{\mathbf{r}_{\nu(h)}(\mathbf{a}) | h \in H_-\}$ by **(3.1)**, $\Leftrightarrow \bar{A}_-(\theta([z])) \subseteq \{\mathbf{r}_{\bar{h}}(\mathbf{a}) | \bar{h} \in \bar{H}_-\}$ by bijectivity of r, $\Leftrightarrow \theta([z]) \in [\bar{Z}](\mathbf{r}(\mathbf{a}))$ by (2.1).

(ii) Let r be as in the Lemma. By **sPTH$^-$**, r induces a bijection $\theta : [Z] \to [\bar{Z}]$. Consider $\mathbf{a} \in \mathbf{A}([z])$ and $a \in A_-([z])$. As $\Gamma \in \mathcal{E}^*$, there is some $a' \in A_{V(a)}$, $a' \neq a$. Consider $\mathbf{a}' \in \mathbf{A}$, $\mathbf{a}'_{V(a)} = a'$ and $\mathbf{a}'_h = \mathbf{a}_h$ for $h \neq V(a)$. Obviously, $[z] \notin [Z](\mathbf{a}')$. Suppose, $r(a) \notin \bar{A}_-(\theta([z]))$. We then had $\bar{A}_-(\theta([z])) \subseteq \{\mathbf{r}_{\bar{h}}(\mathbf{a}) | \bar{h} \in \bar{H}_-\} \setminus \{r(a)\}$ by **sPY**, **sPTH$^-$**, and (2.1), $\subseteq \{\mathbf{r}_{\bar{h}}(\mathbf{a}') | \bar{h} \in \bar{H}_-\}$, i.e. $\theta([z]) \in [\bar{Z}](\mathbf{r}(\mathbf{a}'))$ by (2.1), contradicting s**PTH$^-$**. Hence, $r(A_-([z])) \subseteq \bar{A}_-(\theta([z]))$. Since the inverse r^{-1} satisfies **sISA** and **sPTH$^-$**, the converse inclusion is immediate.

Proof to Example 3.5. For all assignments $\delta = (p, u)$, $p \in]0, 1[$ and $u_i^k := u_i(z_k) \in \mathbb{R}$, $i \in \{1, 2, 3\}$, $k \in \{1, 2, \ldots, 8\}$ there is an assignment $\bar{\delta} = (\bar{p}, \bar{u})$, $\bar{p} \in]0, 1[$ and $\bar{u}_i^k := \bar{u}_i(\bar{z}_k) \in \mathbb{R}$, $k \in \{1, 3, \ldots, 8\}$ (and vice versa) such that $r : A_- \to \bar{A}_-$, $a \mapsto \bar{a}$ (satisfying **sISA**) induces an isomorphism ANF$(\gamma(\delta)) \to$ ANF$(\bar{\gamma}(\bar{\delta}))$, i.e. satisfies **sPY**. Just set $p = \bar{p}$, $\bar{u}_i^3$

$= u_i^3 + \bar{u}_i^1 - u_i^1$, $\bar{u}_i^4 = u_i^4 + \bar{u}_i^1 - u_i^2$, $\bar{u}_i^5 = u_i^5 + \frac{p}{1-p}\left(u_i^1 - \bar{u}_i^1\right)$, $\bar{u}_i^6 = u_i^6 + \frac{p}{1-p}\left(u_i^2 - \bar{u}_i^1\right)$, $\bar{u}_i^7 = u_i^7 + \frac{p}{1-p}\left(u_i^1 - \bar{u}_i^1\right)$, $\bar{u}_i^8 = u_i^8 + \frac{p}{1-p}\left(u_i^2 - \bar{u}_i^1\right)$ or $u_i^k = \bar{u}_i^k$ for $k \neq 2$, and $u_i^1 = u_i^2$, respectively.

Proof of Theorem 3.6. We denote by $\text{prob}(z) := \prod_{a \in A_0 \cap A(\psi(z))} p_{V(a)}(a)$ the probability [113]
that $z \in Z(\mathbf{a})$ is reached by $\mathbf{a}$ which gives

$$u_i(\mathbf{a}) = \sum_{z \in Z(\mathbf{a})} \text{prob}(z)\, u_i(z) = \sum_{[z] \in [Z](\mathbf{a})} v_i([z]) \qquad i \in I_-, \mathbf{a} \in \mathbf{A} \tag{5.1}$$

where $v_i([z]) := \sum_{z' \in [z]} \text{prob}(z')\, u_i(z')$ is called player i's *valuation* of $[z]$. Since we wish to prove a generic result within $\mathcal{E}^*$, we are allowed to focus on assignments with the following properties: (*) For all $i \in I_-$ and $\chi : [Z] \to \{0, \pm 1, \pm 2\}$, $\sum_{[z] \in [Z]} \chi([z])\, v_i([z]) = 0$ implies $\chi([z]) = 0$ for all $[z] \in [Z]$. (**) The players' preferences are pairwise different, i.e. there is no positive affine transformation between the payoff functions of any two players.

Let $\mathbf{f} = (\nu, (r_h)_{h \in H_-})$ be an isomorphism from ANF (Γ) to ANF $(\bar{\Gamma})$. The bijection $r : A_- \to \bar{A}_-$, $a \mapsto r_{V(a)}(a)$ then satisfies **sISA** and **sPY**. By (**) and **sPY,** r induces the bijection $\pi : I_- \to \bar{I}_-$, $i(h) \mapsto \bar{\imath}(\nu(h))$ which satisfies **sPL**.

Remains show that there is a bijection $\theta : [Z] \to [\bar{Z}]$ that satisfies (3.1), hence **sPTH**. Consider the correspondences $Y : [Z] \rightrightarrows [\bar{Z}]$ and $\bar{Y} : [\bar{Z}] \rightrightarrows [Z]$,

$$Y([z]) := \{[\bar{z}] \in [\bar{Z}] \,|\, \bar{A}_-([\bar{z}]) \subseteq r(A_-([z]))\} \tag{5.2a}$$

$$\bar{Y}([\bar{z}]) := \{[z] \in [Z] \,|\, r(A_-([z])) \subseteq \bar{A}_-([\bar{z}])\} \tag{5.2b}$$

By (5.2), $[\bar{z}] \in Y([z])$ and $[z'] \in \bar{Y}([\bar{z}])$ imply $r(A_-([z'])) \subseteq \bar{A}_-([\bar{z}]) \subseteq r(A_-([z]))$, hence $A_-([z']) \subseteq A_-([z])$. Regularity then implies $[z'] = [z]$, hence $r(A_-([z])) = \bar{A}_-([\bar{z}])$. I.e., if both Y and $\bar{Y}$ are nowhere empty then both are single-valued and inverse to each other. Thus, $\{\theta([z])\} = Y([z])$ determines the desired bijection θ. In view of the bijectivity of r, Y and $\bar{Y}$ are defined symmetrically. Therefore, it suffices to show $Y([z]) \neq \emptyset$ for all $[z] \in [Z]$. For $H_-([z]) = H_-$, we have $\mathbf{A}([z]) = \{\mathbf{a}\}$ and $[\bar{Z}](\mathbf{r}(\mathbf{a})) \subseteq Y([z])$. For $H_-([z]) \subsetneq H_-$, we proceed by a series of claims where the first one merely is a restatement of (2.1) and last one implies $Y([z]) \neq \emptyset$.

Claim 1: $[z] \in [Z](\mathbf{a})$ iff $\mathbf{a}_h \in A_-([z])$ for all $h \in H_-([z])$.

Claim 2: $[Z](\mathbf{a}') \subseteq [Z](\mathbf{a})$ implies $[Z](\mathbf{a}') = [Z](\mathbf{a})$.

It suffices to show that $Z(\mathbf{a}') \subseteq Z(\mathbf{a})$ implies $Z(\mathbf{a}') = Z(\mathbf{a})$. For $z^* \in Z(\mathbf{a})$, by (2.1), there is some $\mathbf{a}_0^* \in \mathbf{A}_0$ such that $z^* = z(\mathbf{a}, \mathbf{a}_0^*)$. We then have $z(\mathbf{a}', \mathbf{a}_0^*) \in Z(\mathbf{a}') \subseteq Z(\mathbf{a})$, i.e. by (2.1), there is some $\mathbf{a}_0 \in \mathbf{A}_0$ such that $z(\mathbf{a}', \mathbf{a}_0^*) = z(\mathbf{a}, \mathbf{a}_0)$. By CA03 (Equation (2.3)), we then have $z(\mathbf{a}', \mathbf{a}_0^*) = z(\mathbf{a}, \mathbf{a}_0^*)$ and therefore $z^* \in Z(\mathbf{a}')$.

Claim 3: If (a) $H_-([z]) \cap H_-([z']) = \emptyset$ and (b) $H_-([z]) \cap H_-([z'']) \neq \emptyset$ then (c) $H_-([z']) \cap H_-([z'']) = \emptyset$.

Suppose on the contrary that $[z], [z'], [z''] \in [Z]$ satisfy (a) and (b) but not (c). Then there are $h \in H_-([z])$ and $h' \in H_-([z'])$ that intersect $\psi(z'')$ as close as possible to the root, respectively. Set $\{x\} := h \cap \psi(z'')$ and $\{x'\} := h' \cap \psi(z'')$. W.l.o.g. we assume $x' \lhd x$. By the choice of h, there are $\mathbf{a}^{\#} \in \mathbf{A}([z])$ and $[z^{\#}] \in [Z](\mathbf{a}^{\#})$ such that $x \in \psi(z^{\#})$. We then have $[z], [z^{\#}] \in [Z](\mathbf{a}^{\#})$ and $h \in H_-([z]) \cap H_-([z^{\#}])$. By $x' \lhd x$, we also have $h' \in H_-([z^{\#}])$, and by (a), $h' \notin H_-([z])$, hence $[z] \neq [z^{\#}]$, contradicting regularity.

Fix some $[z]$ and $\mathbf{a} \in \mathbf{A}([z])$. Since $\Gamma \in \mathcal{E}^*$, there is some $\mathbf{a}^{\bullet} \in \mathbf{A}$ such that $\mathbf{a}^{\bullet}_h \neq \mathbf{a}_h$, [114]
$h \in H_-$. Setting

$$H^*_-([z]) := \bigcup_{[z'] \in [Z](\mathbf{a}^{\bullet}): H_-([z']) \cap H_-([z]) \neq \emptyset} H_-([z']), \tag{5.3}$$

we construct $\mathbf{a}^{\circ}, \mathbf{a}^* \in \mathbf{A}$ as follows:

$$\mathbf{a}^{\circ}_h = \begin{cases} \mathbf{a}_h & , h \in H_-([z]) \\ \mathbf{a}^{\bullet}_h & , h \in H_- \backslash H_-([z]) \end{cases} \qquad \mathbf{a}^*_h = \begin{cases} \mathbf{a}^{\bullet}_h & , h \in H^*_-([z]) \\ \mathbf{a}_h & , h \in H_- \backslash H^*_-([z]) \end{cases} \tag{5.4}$$

Claim 4: $H^*_-([z]) \neq \emptyset$.

Suppose on the contrary, $H^*_-([z]) = \emptyset$, i.e. by (5.3) there is no $[z'] \in [Z](\mathbf{a}^{\bullet})$ such that $H_-([z']) \cap H_-([z]) \neq \emptyset$. Then $[Z](\mathbf{a}^{\bullet}) \subseteq [Z](\mathbf{a}^{\circ})$ by (5.4) and *Claim 1*, hence $[Z](\mathbf{a}^{\bullet}) = [Z](\mathbf{a}^{\circ})$ by *Claim 2*. By (5.4) and *Claim 1*, however, $[z] \in [Z](\mathbf{a}^{\circ})$ but $[z] \notin [Z](\mathbf{a}^{\bullet})$. A contradiction.

Claim 5: For all $i \in I_-$, $u_i(\mathbf{a}) - u_i(\mathbf{a}^{\circ}) - u_i(\mathbf{a}^*) + u_i(\mathbf{a}^{\bullet}) = 0$.

Set $M_1 := \{[z]\}$, $M_2 := [Z](\mathbf{a}) \setminus \{[z]\}$, $M_3 := \{[z'] \in [Z](\mathbf{a}^{\bullet}) | H_-([z']) \subseteq H^*_-([z])\}$, and $M_4 := \{[z'] \in [Z](\mathbf{a}^{\bullet}) | H_-([z']) \cap H^*_-([z]) = \emptyset\}$. In the following, we show (i) $[Z](\mathbf{a}) = M_1 \cup M_2$, (ii) $[Z](\mathbf{a}^{\circ}) = M_1 \cup M_4$, (iii) $[Z](\mathbf{a}^*) = M_3 \cup M_2$, and (iv) $[Z](\mathbf{a}^{\bullet}) = M_3 \cup M_4$. By (5.1), this proves the claim.

By $[z] \in [Z](\mathbf{a})$, (i) is immediate. By (5.3), either $H_-([z']) \subseteq H^*_-([z])$ or $H_-([z']) \cap H^*_-([z]) = \emptyset$ for $[z'] \in [Z](\mathbf{a}^{\bullet})$. This proves (iv). By (5.4) and *Claim 1*, we have $M_1 \subseteq [Z](\mathbf{a}^{\circ})$. If $[z'] \in [Z](\mathbf{a}^{\circ}) \setminus M_1$ then $H_-([z']) \cap H_-([z]) = \emptyset$ by regularity. Then (5.4), (5.3), and *Claim 1* imply $[z'] \in M_4$. This proves (ii). By (5.4), (5.3), and *Claim 1*, we have $M_3 \subseteq [Z](\mathbf{a}^*)$. Together with regularity, we have $H_-([z']) \subseteq H_- \backslash H^*_-([z])$ for $[z'] \in [Z](\mathbf{a}^*) \setminus M_3$, hence $[z'] \in [Z](\mathbf{a}) = M_1 \cup M_2$. *Claim 4* and regularity imply $[z'] \in M_2$, i.e. $[Z](\mathbf{a}^*) \setminus M_3 \subseteq M_2$. If $[z'] \in M_2$ and $[z''] \in M_3$ then $H_-([z']) \cap H_-([z]) = \emptyset$ by regularity, and $H_-([z'']) \cap H_-([z]) \neq \emptyset$ by definition of M_3. *Claim 3* then implies $H_-([z']) \cap H_-([z''])= \emptyset$. Then, again by (5.4), (5.3), and *Claim 1*, we have $[z'] \in [Z](\mathbf{a}^*) \setminus M_3$, hence $M_2 \subseteq [Z](\mathbf{a}^*) \setminus M_3$ which proves (iii).

Claim 6: $[\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^{\circ})) \neq \emptyset$ where $\mathbf{r}$ is induced by r via **sPY**.

By (5.2), (5.4), and *Claim 1*, we have $Y([z]) = [\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^\circ))$. Hence, the claim shows $Y([z]) \neq \emptyset$. Suppose on the contrary, $[\bar{Z}](\mathbf{r}(\mathbf{a})) \cap [\bar{Z}](\mathbf{r}(\mathbf{a}^\circ)) = \emptyset$. Consider any $[\bar{z}] \in [\bar{Z}](\mathbf{r}(\mathbf{a}))$, hence $[\bar{z}] \notin [\bar{Z}](\mathbf{r}(\mathbf{a}^\circ))$.

Suppose there is some $\bar{h}' \in \bar{H}_-([\bar{z}])$ such that $\bar{h}' \in \nu(H^*_-([z]))$. Then by (5.4) and **sPY**, $\mathbf{r}_{\bar{h}'}(\mathbf{a}) \neq \mathbf{r}_{\bar{h}'}(\mathbf{a}^*) = \mathbf{r}_{\bar{h}'}(\mathbf{a}^\bullet)$, hence by *Claim 1*, $[\bar{z}] \notin [\bar{Z}](\mathbf{r}(\mathbf{a}^*)), [\bar{Z}](\mathbf{r}(\mathbf{a}^\bullet))$. Since $\mathbf{r}$ satisfies **sPY**, for all $i \in I_-$ and $\mathbf{a}' \in \mathbf{A}$ there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that $\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}')) = \alpha_i u_i(\mathbf{a}') + \beta_i$. Hence by *Claim 5*,

$$\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) - \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^\circ)) - \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^*)) + \bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a}^\bullet)) = 0. \tag{5.5}$$

Express (5.5) by valuations according to (5.1). Since $[\bar{z}]$ is contained in $[\bar{Z}](\mathbf{r}(\mathbf{a}))$ only, the coefficient of $\bar{v}_{\bar{\imath}(\nu(h))}([\bar{z}])$ is 1 while all other coefficients are between -2 and 2, contradicting (*), i.e. genericity.

Remains the possibility that $\bar{H}_-(\bar{z}) \subseteq \bar{H}_- \backslash \nu(H^*_-([z]))$. Then by (5.4), *Claim 1,* and **sPY**, $[\bar{z}] \in [\bar{Z}](\mathbf{r}(\mathbf{a}^*))$, hence $[\bar{Z}](\mathbf{r}(\mathbf{a})) \subseteq [\bar{Z}](\mathbf{r}(\mathbf{a}^*))$ (since $[\bar{z}]$ was arbitrary) and therefore $[\bar{Z}](\mathbf{r}(\mathbf{a})) = [\bar{Z}](\mathbf{r}(\mathbf{a}^*))$ by *Claim 2.* By *Claims 4* and *5* ((i), (iii)), and regularity, however, $[Z](\mathbf{a}) \neq [Z](\mathbf{a}^*)$. Arguments similar to those for the other case show [115] that this contradicts genericity.

Proof of Theorem 3.7. We denote by μ^* the mapping that assigns to $b' \in B^0$ the system of beliefs $\mu^*(b')$ associated with b' according to Bayes' rule. Let (μ, b) be a sequential equilibrium of $\Gamma \in \mathcal{E}$. By Kreps & Wilson (1982, Proposition 6), there is a sequence (b^k, u^k), $b^k \in B^0$, $u^k \in \mathbb{R}^{I_- \times Z}$ such that $b = \lim_{k\to\infty} b^k$, $\mu = \lim_{k\to\infty} \mu^*(b^k)$, $u = \lim_{k\to\infty} u^k$ and $u^k_i(b_i b^k_{-i}) \geq u^k_i(b'_i b^k_{-i})$ for all $k \in \mathbb{N}$, $i \in I_-$, and $b'_i \in B$.

Further, let r be a SWI from Γ to $\bar{\Gamma} \in \mathcal{E}$ which induces bijections $\pi : I_- \to \bar{I}_-$, $\nu : H_- \to \bar{H}_-$, $\theta : [Z] \to [\bar{Z}]$, $\mathbf{r} : \mathbf{A} \to \bar{\mathbf{A}}$ such that for all $i \in I_-$ there are $\alpha_i, \beta_i \in \mathbb{R}$, $\alpha_i > 0$ such that $\bar{u}_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i(\mathbf{a}) + \beta_i$ for all $\mathbf{a} \in \mathbf{A}$. Since $\bar{\mu}^*$ is continuous, there is some system of beliefs $\bar{\mu}$ of $\bar{\Gamma}$ such that $\lim_{k\to\infty} \bar{\mu}^*(\mathbf{r}(b^k)) = \bar{\mu}$. We show that $(\bar{\mu}, \mathbf{r}(b))$ is a sequential equilibrium.

Fix any payoff function $\upsilon \in \mathbb{R}^Z$ and consider the following system of linear equations where the payoff function $\bar{\upsilon} \in \mathbb{R}^{\bar{Z}}$ is variable:

$$\bar{\upsilon}(\mathbf{r}(\mathbf{a})) = \sum_{\bar{z} \in \bar{Z}(\mathbf{r}(\mathbf{a}))} \text{prob}(\bar{z})\, \bar{\upsilon}(\bar{z}) = \sum_{z \in Z(\mathbf{a})} \text{prob}(z)\, \upsilon(z) = \upsilon(\mathbf{a}) \qquad \mathbf{a} \in \mathbf{A} \tag{5.6}$$

Let $\bar{\Upsilon}$ denote the correspondence $\mathbb{R}^Z \rightrightarrows \mathbb{R}^{\bar{Z}}$ which assigns to υ the set $\bar{\Upsilon}(\upsilon)$ of solutions of (5.6). Using Lemma 3.2 (i), one shows that $\bar{\upsilon}(\upsilon) \in \mathbb{R}^{\bar{Z}}$,

$$\bar{\upsilon}(\upsilon)(\bar{z}) := \frac{\sum_{z \in \theta^{-1}([\bar{z}])} \text{prob}(z)\, \upsilon(z)}{\sum_{\bar{z}' \in [\bar{z}]} \text{prob}(\bar{z}')} \qquad , \bar{z} \in \bar{Z}$$

satisfies (5.6). Hence, $\bar{\Upsilon}(\upsilon)$ is non-empty for all $\upsilon \in \mathbb{R}^{Z}$. Moreover, the set $\bar{\Upsilon}(\upsilon)$ is an affine subspace $\bar{\upsilon}^* + \bar{\Upsilon}_0 \subseteq \mathbb{R}^{\bar{Z}}$ where $\bar{\upsilon}^* \in \bar{\Upsilon}(\upsilon)$ and $\bar{\Upsilon}_0$ denotes the solution set of the homogenous system associated with (5.6). Since the right side of (5.6) is continuous in υ, $\bar{\Upsilon}$ is continuous.

By assumption, we have $\bar{u}_{\pi(i)} \in \bar{\Upsilon}(\alpha_i u_i + \beta_i)$ for all $i \in I_-$. Since $\lim_{k\to\infty} u_i^k = u_i$ and $\bar{\Upsilon}$ is continuous, there is a sequence $(\bar{u}^k_{\pi(i)})_{k\in\mathbb{N}}$, $\bar{u}^k_{\pi(i)} \in \bar{\Upsilon}\left(\alpha_i u_i^k + \beta_i\right)$ such that $\lim_{k\to\infty} \bar{u}^k_{\pi(i)} = \bar{u}_{\pi(i)}$. By (5.6) and (5.1), we then have $\bar{u}^k_{\pi(i)}(\mathbf{r}(\mathbf{a})) = \alpha_i u_i^k(\mathbf{a}) + \beta_i$ for all $\mathbf{a} \in \mathbf{A}$, $i \in I_-$, and $k \in \mathbb{N}$, hence

$$\bar{u}^k_{\pi(i)}(\mathbf{r}(b)) = \alpha_i u_i^k(b) + \beta_i, \qquad b \in B. \tag{5.7}$$

Since $\mathbf{r}$ is continuous, $\lim_{k\to\infty} \mathbf{r}\left(b^k\right) = \mathbf{r}(b)$. Suppose there were some $k \in \mathbb{N}$, $\bar{\imath} \in \bar{I}_-$, $\bar{b}'_{\bar{\imath}} \in \bar{B}_{\bar{\imath}}$ such that

$$\bar{u}^k_{\bar{\imath}}\left(\mathbf{r}\left(b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}\right)\right) < \bar{u}^k_{\bar{\imath}}\left(\bar{b}'_{\bar{\imath}} \mathbf{r}_{-\bar{\imath}}\left(b^k\right)\right)$$

where $\bar{b}'_{\bar{\imath}} \mathbf{r}_{-\bar{\imath}}\left(b^k\right)$ denotes the behavior strategy profile where all players follow $\mathbf{r}\left(b^k\right)$ except for $\bar{\imath}$ who follows $\bar{b}'_{\bar{\imath}}$, analogously for $b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}$. By (5.7) we then had

$$u^k_{\pi^{-1}(\bar{\imath})}\left(b_{\pi^{-1}(\bar{\imath})} b^k_{-\pi^{-1}(\bar{\imath})}\right) < u^k_{\pi^{-1}(\bar{\imath})}\left(\mathbf{r}^{-1}_{\pi^{-1}(\bar{\imath})}\left(\bar{b}'_{\bar{\imath}}\right) b^k_{-\pi^{-1}(\bar{\imath})}\right)$$

with the interpretation of the arguments as above. Since this contradicts the assumptions [116]
on $\left(b^k, u^k\right)$, the sequence $\left(\mathbf{r}\left(b^k\right), \bar{u}^k\right)$ establishes $(\bar{\mu}, \mathbf{r}(b))$ to be a sequential equilibrium. Since the inverse r^{-1} also is a SWI, this proves the claim.

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CHAPTER IV

Strong agent normal form isomorphism

Abstract

Sequential equilibrium and quasi-perfect equilibrium are not invariant under isomorphism of the standard form. In this note, we advocate a concept of isomorphism for extensive games which is generically equivalent to isomorphism of the agent normal form and under which these solution concepts are invariant. Though this concept relies on details of the extensive form, it is essentially weaker than other such concepts as strong, weak, and super weak isomorphism.

Key Words: Equivalence, Invariance, Genericity, Sequential equilibrium, Quasi-perfect equilibrium

JEL classification: C72.

1. Introduction

It is well known that sequential rationality considerations in games already can be made within the normal form: Proper equilibria (Myerson 1978) of the normal form induce sequential equilibria (Kreps & Wilson 1982, henceforth SEQ) and quasi-perfect equilibria (van Damme 1984, henceforth QPE) in every extensive form having this normal form. Weakening the reduced normal form invariance requirement of Kohlberg & Mertens (1986), however, Govindan & Wilson (2004) "accept the relevance of extensive form analysis" and employ QPE in their axiomatic justification of stable equilibria (Kohlberg & Mertens 1986). Moreover, Mertens (1995) argues that QPE seems to be the right combination of admissability and backward induction. Hence, one might be interested in concepts of isomorphism for extensive games under which QPE is invariant.

Consider the games in Figure 1.1 due to van Damme (1984) which both have the same standard form (Harsanyi & Selten 1988) G below.

G

	ℓ	r
L	$1,1$	$1,1$
R	$1,1$	$0,0$

Hence, the identity mapping on the action set establishes an isomorphism of the standard form (Harsanyi & Selten 1988, henceforth SFI). Since this mapping switches the order of information sets, QPE is not SFI invariant. While (R,ℓ) is a QPE in Γ, it is not so in $\bar{\Gamma}$. Despite its drawbacks, SEQ is frequently employed in applications because it is easier to compute than perfect equilibrium (Selten 1975). As QPE, however, SEQ is not SFI invariant (see e.g. Kreps & Wilson 1982, Figures 2 and 13). In contrast, perfect equilibrium can be defined in terms of the agent normal form (Selten 1975, henceforth ANF) and therefore is invariant under isomorphism of the ANF (ANFI). Yet, it shares a major disadvantage with SEQ: It may put positive weight on (conditionally) dominated strategies (Mertens 1995, Example 1). Hence, isomorphism concepts which genuinely rely on the extensive form seem to be needed.

Successively weakening strong isomorphism (Elmes & Reny 1994, Peleg et al. 1999), we introduced weak isomorphism (Casajus 2003, henceforth WI and CA03[1]) and super weak isomorphism (Casajus 2006, henceforth SWI and CA06[2]) under which SEQ is invariant. The latter note observes that ANFI and SWI are generically different (CA06, Example 3.5 and the proof) and then concludes with the question whether there is a concept of isomorphism of extensive games which is generically equivalent to ANFI but under which SEQ remains invariant.

[1]Also Chapter II of this thesis.

[2]Also Chapter III of this thesis.

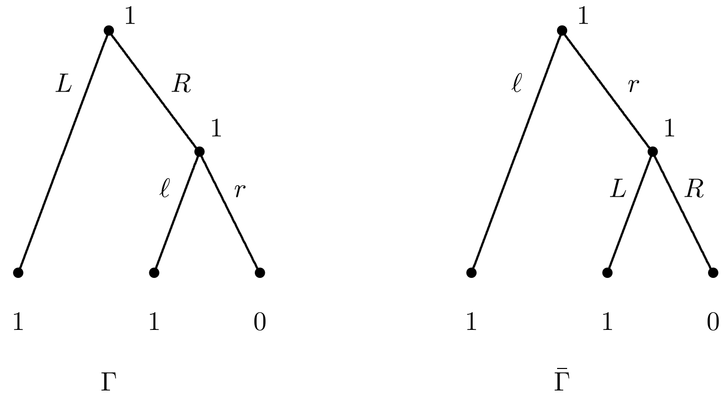

FIGURE 1.1. QPE is not invariant under isomorphism of the standard form

In this note, we propose such a concept, strong ANF isomorphism (strong ANFI), which also preserves QPE. We depart from the observation that in our leading example the identity mapping on the action set no longer is an ANFI if one slightly perturbs the payoffs. Basically, a strong ANFI is an ANFI which remains an ANFI under (slight) perturbations of payoffs.

Since this note is an addendum to CA03 and CA06, for expositional parsimony, we rely on the definitions and notation provided there and just explain the most important notation in the next section. The third one introduces the concept of strong ANFI and explores its main properties. The concluding remarks relate the concepts of isomorphism considered in this paper. The appendix contains one lengthier proof.

2. Notation

Assuming a large enough set which contains the labels of players, pure strategies, and nodes, we consider the set $\mathcal{E}$ of finite extensive games Γ with perfect recall and the underlying set $\mathcal{EF}$ of extensive forms γ, i.e. extensive games without payoff functions and chance probabilities. $\mathcal{E}^{\mathrm{nc}}$ denotes the set of games without chance mechanism and $\mathcal{E}^*$ denotes the set of games where $|A_h| > 1$ for all $h \in H_-$. For $\gamma \in \mathcal{EF}$, $\mathbb{D}(\gamma)$ denotes the set of *assignments* $\delta = (u, p)$ of payoff functions u and chance probabilities p; $\gamma(\delta)$ denotes the extensive game based on γ and specified by the assignment δ. Associated with $\Gamma \in \mathcal{E}$ are the set I_- of genuine players, the set H_- of their information sets h, and the set $\mathbf{A}$ of pure-strategy profiles $\mathbf{a}$; $i(h)$ denotes the player who controls h.

An *ANFI* $\mathbf{r}$ from Γ to $\bar{\Gamma}$ is system of bijections $(\nu, (r_h)_{h \in H_-})$ where $\nu : H_- \to \bar{H}_-$, $r_h : A_h \to \bar{A}_{\nu(h)}$ where A_h denotes the set of actions at h. Abusing notation, $\mathbf{r}$ also denotes the induced bijections $\mathbf{A} \to \bar{\mathbf{A}}$ and $B \to \bar{B}$ (CA03 eqs. (2.1) and (3.2)) which preserve the player's preferences. A *SFI* is an ANFI together with a bijection $\pi : I_- \to \bar{I}_-$ such that

$\nu(H_i) = \bar{H}_{\pi(i)}$ for all $i \in I_-$. $Z(\mathbf{a})$ denotes the set of terminal nodes reached by $\mathbf{a}$ (CA06, eq. (2.1)). We define *genericity* as CA06 (Section 2).

3. Strong agent normal form isomorphism

DEFINITION 3.1. *A* strong ANF isomorphism *from* $\Gamma = \gamma(\delta)$ *to* $\bar{\Gamma} = \bar{\gamma}(\bar{\delta})$, $\gamma, \bar{\gamma} \in \mathcal{EF}$, $\delta \in \mathbb{D}(\gamma)$, $\bar{\delta} \in \mathbb{D}(\bar{\gamma})$ *is an ANFI* $\mathbf{r}$ *from* Γ *to* $\bar{\Gamma}$ *such that there are neighborhoods* U *of* δ *in* $\mathbb{D}(\gamma)$ *and* $\bar{U}$ *of* $\bar{\delta}$ *in* $\mathbb{D}(\bar{\gamma})$ *such that for all* $\delta^\circ \in U$ *and* $\bar{\delta}^\circ \in \bar{U}$ *there are* $\delta^\bullet \in \mathbb{D}(\gamma)$ *and* $\bar{\delta}^\bullet \in \mathbb{D}(\bar{\gamma})$ *such that* $\mathbf{r}$ *is an ANFI from* $\gamma(\delta^\circ)$ *to* $\bar{\gamma}(\bar{\delta}^\bullet)$ *and* $\mathbf{r}^{-1}$ *is an ANFI from* $\bar{\gamma}(\bar{\delta}^\circ)$ *to* $\gamma(\delta^\bullet)$.

Non-technically, a strong ANFI is an ANFI which remains an ANFI under slight perturbations of payoffs and chance probabilities. Our leading example already reveals that ANFI and its strong cousin do not coincide in general. Generically, however, both concepts coincide. In a sense, the definition of strong ANFI already incorporates genericity considerations. In view of the proof of the following theorem (referred to the Appendix), one could sharpen the definition by dropping the restriction to neighborhoods of the assignments without loosing this property. The latter indicates that strong ANFI has all the properties one would expect from an isomorphism: The identity on actions, the composite of strong ANFI, and the inverse of a strong ANFI again is a strong ANFI.

THEOREM 3.2. *Generically, ANFI and strong ANFI coincide*

The following Corollary sheds light on the relation between SWI and strong ANFI: SWI is non-trivially stronger than strong ANFI. Part (i) is immediate from CA06 (Proof of Theorem 3.12, eqs. (5.6–7)). CA06 (Example 3.5 and the proof) establishes a counterexample for part (ii). Since strong isomorphism and WI imply SWI (CA03, Section 3.3; CA06, Theorem 3.3), the theorem can be extended to strong isomorphism and WI.

COROLLARY 3.3. *(i) Any SWI is a strong ANFI. (ii) The converse may fail, even generically.*

By similar arguments (see CA03, proof of Theorem 4.8, (A.1)) and CA06 (Theorem 3.3), we have a limited converse of Corollary 3.3(i). Note that CA03 (Example 3.4) establishes a counterexample for $\mathcal{E} \backslash \mathcal{E}^*$.

COROLLARY 3.4. *For* $\mathcal{E}^{nc} \cap \mathcal{E}^*$, *strong ANFI, SWI, and WI coincide generically.*

Since ANFI is not sensitive to the assignment of information sets to players, it may not imply SFI or (reduced) normal form isomorphism in non-generic cases (see CA03, pp. 281, for an example). Generically, however, this holds true because generically the players' preferences are pairwise different and ANFI preserves these preferences (see CA03, proof of

Theorem 4.8). As strong ANFI is insensitive to slight perturbations of the players payoffs, it is sensitive to the assignment of information sets to players. This implies the following Lemma.

LEMMA 3.5. *Any strong ANFI* $\mathbf{r} : \Gamma \to \bar{\Gamma}$ *induces a bijection* $\pi : I_{-} \to \bar{I}_{-}$ *such that* $\nu(H_i) = \bar{H}_{\pi(i)}$ *for all* $i \in I_{-}$*, hence an isomorphism of the standard forms and the (reduced) normal forms.*

As strong ANFI strengthens ANFI, CA06 (Section 3.5) also applies to strong ANFI. While Nash equilibrium is not invariant under ANFI, not even generically, Lemma 3.5 indicates that Nash equilibrium is strong ANFI invariant. Unsurprisingly, subgame perfect equilibrium is not invariant under strong ANFI. In contrast, the sequence characterization of SEQ (Kreps & Wilson 1982, Proposition 6) and arguments in CA06 (Proof of Theorem 3.7) imply that SEQ is so.

Since we consider games with perfect recall, the information sets of a player are partially ordered. For $i \in I_{-}$, $h, h' \in H_i$, we write $h \lhd h'$ iff for all (equivalently, some) $x' \in h'$ there is some $x_{x'} \in h$ such that $x_{x'} \lhd x'$. Let $H_i^h := \{h' \in H_i | h \lhd h' \vee h = h'\}$ denote the set comprising h itself and all $h' \in H_i$ that come after h. The following Lemma shows that strong ANFI, other than SFI, preserves this order. It is the main ingredient in the subsequent proof of QPE being strong ANFI invariant.

LEMMA 3.6. *If* $\mathbf{r} = (\nu, (r_i)_{i \in I_{-}})$ *is a strong ANFI from* Γ *to* $\bar{\Gamma}$ *then (i) for all* $i \in I_{-}$ *and* $h, h' \in H_i$*, we have* $h \lhd h'$ *iff* $\nu(h) \lhd \nu(h')$ *and (ii)* $\nu\left(H_i^h\right) = \bar{H}_{\pi(i)}^{\nu(h)}$ *where* $\pi : I_{-} \to \bar{I}_{-}$ *is determined by Lemma 3.5.*

PROOF. Let $\mathbf{r}$ be as in the Lemma. Fix $i \in I_{-}$, $h, h' \in H_i$, such that $h \lhd h'$. By perfect recall, there is some $a_{h'} \in A_h$ that comes before all nodes in h, i.e. $\varphi(x) \cap a_{h'} \neq \emptyset$ for all $x \in h'$. Hence if $a_{h'}$ is chosen with a probability of 0 then changes of the local strategy at h' do not affect the payoffs because then h' is not reached. I.e., for all $b \in B^0$ and $a_{h'} \neq a \in A_h$, we have

$$u\left(a', a, b_{H_{-} \setminus \{h,h'\}}\right) = u\left(a'', a, b_{H_{-} \setminus \{h,h'\}}\right).$$

for all $a', a'' \in A_{h'}$, where the subscripts at b indicate restrictions to the respective subset. Suppose we had $\nu(h) \ntriangleleft \nu(h')$. Then, $\nu(h')$ is reached under $\mathbf{r}\left(a', a, b_{H_{-} \setminus \{h,h'\}}\right)$ and $\mathbf{r}\left(a'', a, b_{H_{-} \setminus \{h,h'\}}\right)$. By Section 2 and CA03 (eqs. (2.1) and (3.2)), we then had

$$\bar{u}\left(\mathbf{r}\left(a', a, b_{H_{-} \setminus \{h,h'\}}\right)\right) = \bar{u}\left(\mathbf{r}\left(a'', a, b_{H_{-} \setminus \{h,h'\}}\right)\right),$$

even under (small) perturbations of $\bar{u}$ at some $\nu(h) \ni \bar{x} \lhd \bar{z} \in \bar{Z}$, contradicting $\mathbf{r}$ to be a strong ANFI. Claim (i) then follows from the fact that the inverse of a strong ANFI is a strong ANFI. Claim (ii) is an immediate consequence of claim (i). □

Remark 3.7. From the original definition (van Damme 1984, Definition 1) it is immediate that QPE can be characterized as follows: A behavior-strategy profile b is a *QPE* if there is a sequence $(b^k)_{k\in\mathbb{N}}$, $b^k \in B^0$, $\lim_{k\to\infty} b^k = b$ such that

$$u_i\left(b_{H_i^h} b^k_{H_-\setminus H_i^h}\right) \geq u_i\left(b'_{H_i^h} b^k_{H_-\setminus H_i^h}\right)$$

for all $i \in I_-$, $h \in H_i$, and $b'_i \in B_i$.

Theorem 3.8. *QPE is invariant under strong ANFI.*

Proof. Let $\mathbf{r} = (\nu, (r_i)_{i\in I_-})$ be a strong ANFI from Γ to $\bar{\Gamma}$. By Theorem $\mathbf{r}$ is a SFI, i.e. there is a bijection $\pi : I_- \to \bar{I}_-$ such that $\nu(H_i) = H_{\nu(i)}$ for all $i \in I_-$. Let b be a QPE of Γ. Then there is sequence $\left(b^k\right)_{k\in\mathbb{N}}$ as in Remark 3.7. Since $\mathbf{r} : B \to \bar{B}$ is continuous and both B and $\bar{B}$ are compact, the sequence $\left(\mathbf{r}\left(b^k\right)\right)_{k\in\mathbb{N}}$, $\mathbf{r}\left(b^k\right) \in \bar{B}^0$ converges to $\mathbf{r}(b)$. By Section 2 and CA03 (eqs. (2.1) and (3.2)) and by Lemma 3.6, this sequence also satisfies the characterization of QPE in Remark 3.7. □

This Theorem can be extended to all stronger concepts of isomorphism. By CA03 (Section 3.3), CA06 (Theorem 3.3(i)), and Theorem 3.4, we have the following Corollary.

Corollary 3.9. *QPE is invariant under strong isomorphism, WI, and SWI.*

4. Concluding remarks

Figure 3.1 summarizes the relation between the different concepts of isomorphism of extensive games. For the obvious reasons, we restrict attention to the set $\mathcal{E}^*$. We write **stI** for strong isomorphism, **stANFI** for strong ANF isomorphism, and **NFI** for isomorphism of the normal form. Equivalent concepts are framed. Implications which are not indicated do not hold in general.

In the most general case, the upper left entry, strong ANFI is the weakest concept of isomorphism of extensive games under which SEQ and QPE are invariant. Hence, strong ANFI seems to identify that part of the sequential structure of an extensive game which enables us to apply considerations of sequential rationality in a comprehensible way. Nevertheless, quite often, it will be easier to work with SWI, for example.

Appendix

Proof of Theorem 3.2. For all $\gamma \in \mathcal{EF}$, $u \in \mathbb{U}(\gamma)$, and $i \in I_-$, set $u_i^{\max} := \max_{\mathbf{a}\in\mathbf{A}} u_i(\mathbf{a})$ and $u_i^{\min} := \min_{\mathbf{a}\in\mathbf{A}} u_i(\mathbf{a})$. We then are allowed to restrict attention to $\mathbb{D}^{\neq}(\gamma) := \{(u,p) \in \mathbb{D}(\gamma) \,|\, \forall i \in I_- : u_i^{\max} \neq u_i^{\min}\}$ which is open and dense in $\mathbb{D}(\gamma)$. Consider the payoff normation $\tau : \mathbb{D}^{\neq}(\gamma) \to \mathbb{D}^{\neq}(\gamma)$, $(u,p) \mapsto (\tau u, p)$ given by

$$\tau u_i(z) = \frac{u_i(z) - u_i^{\min}}{u_i^{\max} - u_i^{\min}} \qquad , i \in I_-, z \in Z \tag{4.1}$$

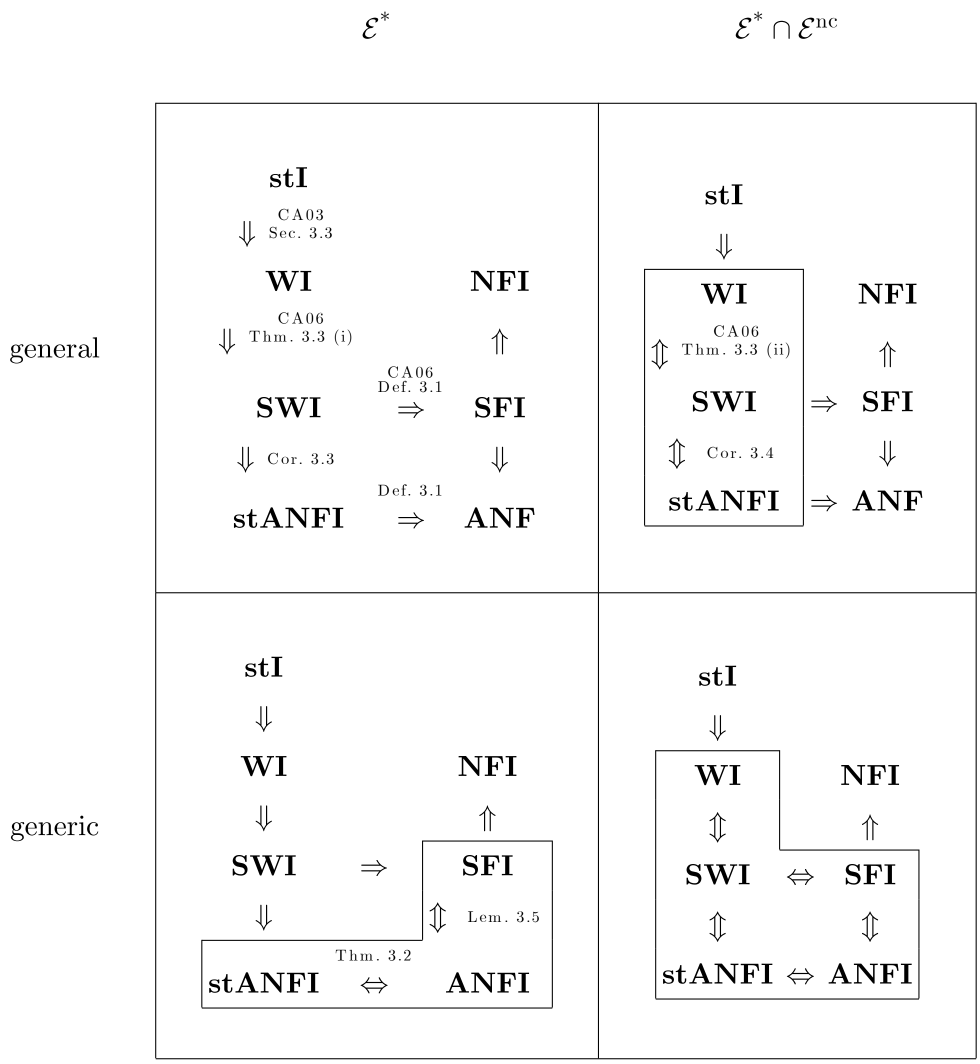

FIGURE 3.1. Relation between concepts of isomorphism for extensive games

which implies

$$\min_{\mathbf{a}\in\mathbf{A}} \tau u_i(\mathbf{a}) = 0 \quad \text{and} \quad \max_{\mathbf{a}\in\mathbf{A}} \tau u_i(\mathbf{a}) = 1 \qquad , i \in I_-. \tag{4.2}$$

Since τ is a system of positive affine transformations, $\mathbf{r}$ is a (strong) ANFI from $\gamma(\tau(\delta))$ to $\bar{\gamma}\left(\tau\left(\bar{\delta}\right)\right)$ iff $\mathbf{r}$ is a (strong) ANFI from $\gamma(\delta)$ to $\bar{\gamma}\left(\bar{\delta}\right)$. Let $\tau\mathbb{D}^{\neq}(\gamma)$ denote the image of τ. Since τ also is continuous, the counterimage of any open and dense subset of $\tau\mathbb{D}^{\neq}(\gamma)$ is open and dense in $\mathbb{D}^{\neq}(\gamma)$ and $\mathbb{D}(\gamma)$. Therefore, we are allowed to restrict attention to the sets $\tau\mathbb{D}^{\neq}(\gamma)$.

For any $\gamma \in \mathcal{EF}$ consider the subset

$$\mathcal{D}(\gamma) := \left\{(u,p) \in \tau\mathbb{D}^{\neq}(\gamma) \,|\forall \zeta \in \mathbb{Z}^Z, \zeta \neq \mathbf{0} : \sum_{z\in Z} \zeta(z)\,\text{prob}(z)\,u_i(z) \neq 0\right\}$$

of $\tau\mathbb{D}^{\neq}(\gamma)$ where $\mathbb{Z}^Z$ denotes the set of mappings from Z into the integers and $\mathbf{0} \in \mathbb{Z}^Z$, $\mathbf{0}(z) = 0$, $z \in Z$. Since $\mathbb{Z}^Z$ is countable, $\mathcal{D}(\gamma)$ is open and dense in $\tau\mathbb{D}^{\neq}(\gamma)$.

Let $\mathbf{r} = (\nu, (r_i)_{i \in I_-})$ be an ANFI from $\gamma(u,p)$ to $\bar{\gamma}(\bar{u},\bar{p})$, $(u,p) \in \mathcal{D}(\gamma)$ and $(\bar{u},\bar{p}) \in \mathcal{D}(\bar{\gamma})$. By Section 2 and CA03 (eqs. (2.1) and (3.2)) and in view of (4.2), we then have

$$\bar{u}_{\bar{\imath}(\nu(h))}(\mathbf{r}(\mathbf{a})) = u_{i(h)}(\mathbf{a}) \qquad , \mathbf{a} \in \mathbf{A}, h \in H_-. \tag{4.3}$$

Enumerate $\mathbf{A}$ and $\bar{\mathbf{A}}$ such that $\mathbf{r}\left(\mathbf{a}^{(k)}\right) = \bar{\mathbf{a}}^{(k)}$. Further, enumerate Z and $\bar{Z}$. Consider the $|\mathbf{A}| \times |Z|$ and $\left|\bar{\mathbf{A}}\right| \times \left|\bar{Z}\right|$ matrices $\Phi = (\Phi_{ij})$ and $\bar{\Phi} = \left(\bar{\Phi}_{ij}\right)$, respectively, given by

$$\Phi_{ij} = \begin{cases} 1, & z^{(j)} \in Z\left(\mathbf{a}^{(i)}\right), \\ 0, & , z^{(j)} \notin Z\left(\mathbf{a}^{(i)}\right), \end{cases} \qquad \text{and} \qquad \bar{\Phi}_{ij} = \begin{cases} 1, & , \bar{z}^{(j)} \in \bar{Z}\left(\bar{\mathbf{a}}^{(i)}\right), \\ 0, & , \bar{z}^{(j)} \notin \bar{Z}\left(\bar{\mathbf{a}}^{(i)}\right). \end{cases} \tag{4.4}$$

It is easy to see that there is a regular $\left|\bar{\mathbf{A}}\right| \times \left|\bar{\mathbf{A}}\right|$ matrix $\bar{C}$ with integer entries such that $\bar{C}\bar{\Phi}$ is in row echelon form where the leading non-zeros may differ from 1. Then in $\bar{C}\bar{\Phi}$, the last $\left|\bar{\mathbf{A}}\right| - \operatorname{rank}\left(\bar{\Phi}\right)$ rows are zero.

Given this, by CA06 (eq. (5.1)), (4.3) can be written as

$$\bar{\Phi}\bar{v}_{\bar{\imath}(\nu(h))} = \Phi v_{i(h)} \qquad , h \in H_-, \tag{4.5}$$

where $v_{i(h)} \in \mathbb{R}^Z$, $\bar{v}_{\bar{\imath}(\nu(h))} \in \mathbb{R}^{\bar{Z}}$, $v_{i(h)}(z) = \operatorname{prob}(z)\, u_{i(h)}(z)$, $z \in Z$, and $\bar{v}_{\bar{\imath}(\nu(h))}(\bar{z}) = \operatorname{prob}(\bar{z})\, \bar{u}_{\bar{\imath}(\nu(h))}(\bar{z})$. Hence, the systems of linear equations in $x \in \mathbb{R}^{\bar{Z}}$

$$\bar{\Phi}x = \Phi v_{i(h)} \qquad , h \in H_- \tag{4.6}$$

have a solution. From the theory of systems of linear equations, we then know that

$$\operatorname{rank}\left(\bar{\Phi}\right) = \operatorname{rank}\begin{pmatrix} \bar{\Phi} & \Phi v_{i(h)} \end{pmatrix}. \tag{4.7}$$

If the jth row of $\bar{C}\bar{\Phi}$ is zero then by (4.7) the jth entry of $\bar{C}\Phi v_{i(h)}$ is zero too. Since $(u,p) \in \mathcal{D}(\gamma)$ and since in $\bar{C}\Phi$ all entries are integers, the jth row of $\bar{C}\Phi$ also is zero which implies that (4.7) holds for arbitrary $v_{i(h)}$. Hence, (4.6) has a solution for all $v_{i(h)} \in \mathbb{R}^Z$, i.e. (4.3) holds for arbitrary $(u,p) \in \mathcal{D}(\gamma)$ and some $(\bar{u},\bar{p}) \in \mathbb{D}(\bar{\gamma})$. Since $\mathbf{r}^{-1}$ also is an ANFI, the opposite direction is immediate. Hence, generically, any ANFI is strong.

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Part 2

Outside options and communication restrictions in TU games

CHAPTER V

Outside options, component efficiency, and stability

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Abstract [49]

In this paper, we introduce a component efficient value for TU games with a coalition structure which reflects the outside options of players within the same structural coalition. It is based on the idea that splitting a coalition should affect players who stay together in the same way. We show that for all TU games there is a coalition structure that is stable with respect to this value.

Key Words: TU game, partition function form game, outside option, splitting, coalition structure, stability

JEL classification: C71

1. Introduction

Consider a gloves game (Shapley & Shubik 1969) with two left-glove holders and four right-glove holders (players) where the worth of a coalition is the number of right-hand-left-hand pairs (matching pairs) it contains. Suppose the players have formed two matching pairs with the two remaining right-glove players unattached. How should the players in a matching-pair coalition split the worth of 1?

One well known way to divide the total worth of a set of players in a game is to assign each player his Shapley (1953) value. The Shapley value, however, does not take into account coalition structures. In order to fill this gap, a number of values for TU games with a given coalition structure (henceforth CS-values and CS-games) have been proposed: In their pioneering work, Aumann & Drèze (1974) introduce a component efficient CS-value (henceforth AD-value) where the payoffs depend on a player's own coalition only. In [50]
contrast, the Owen (1977) value is efficient and sensitive to how the players outside ones own coalition are organized. Table 1.1 lists the payoffs for the leading example.

	AD-value Owen value	Wiese value	Shapley value	χ-value	core
left with right	.5000	.7167	.7333	.8000	1
right with left	.5000	.2833	.1333	.2000	0
single right	0	0	.1333	0	0

Table 1.1. Payoffs for the gloves game

Interestingly, both the AD-value and the Owen value split the worth of 1 equally between the members of a matching-pair coalition, i.e., these values are insensitive to outside options which in the present context means that they do not respond to the relative scarcity of the left gloves. However, outside options might be important:

> Any particular alliance describes only one particular consideration which enters the minds of the participants when they plan their behavior. Even if a particular alliance is ultimately formed, the division of the proceeds between the allies will be decisively influenced by the other alliances which each one might alternatively have entered. [...] Even if [...] one particular alliance is actually formed, the others are present in "virtual" existence: Although they have not materialized, they have contributed essentially to shaping and determining the actual reality. (von Neumann & Morgenstern 1944, p. 36)

> During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each coalition has still to adjust the final share of its proceeds. (This decision may depend on options outside the coalition, even though the chances of defection are slim). (Maschler 1992, pp. 595)

In contrast to the AD- and the Owen value, the unique CS-game core payoffs (Aumann & Drèze 1974) give the whole worth of 1 to the left-glove players; the core neglects the productive role of a right-glove player within a *given* matching-pair coalition. Only recently, Wiese (2007) suggested another component efficient CS-value which steers a course between these extreme positions. This can be seen from the Wiese payoffs listed in Table 1.1. On the one hand, the payoff of a left-glove player is higher than that of the right-glove player in his coalition; the Wiese value accounts for outside options. On the other hand, a right-glove player in a matching-pair coalition obtains a higher payoff than the single right-glove players; the Wiese value recognizes the productive role of right-glove players in the matching-pair coalitions.

Nevertheless, the Wiese value has some drawbacks. Most notably, it lacks a "nice" axiomatization. In essence, there is a non-intuitive ad-hoc specification of the payoffs for unanimity games which is expanded by linearity to the whole class of games (see Section 3). Further, it is not yet clear whether there are stable coalition structures (in the sense of Hart & Kurz 1983) with respect to the Wiese value for all TU-games.

In order to remedy these deficiencies, we introduce a component efficient CS-value—the χ-value. The main idea underlying the χ-value is that splitting a structural coalition affects players who remain together in the same structural coalition in the same way. Besides additivity and component restricted symmetry, we adhere to the Null player axiom for the grand coalition. These axioms uniquely characterize the χ-value which easily can be [51]
computed from the Shapley value. Further, it turns out that stable coalition structures with respect to the χ-value exist for all TU games. The χ-payoffs for our leading example in Table 1.1 indicate that the χ-value balances outside options and the contribution to ones own coalition. Finally, recent experiments within the framework of gloves games indicate that the χ-value allows for better predictions on the outcome of bargaining between a left-glove holder and a right-glove holder on splitting the worth of a matching pair than the Wiese value (Pfau 2007).

The paper is organized as follows: Basic definitions and notation are given in the next section. In the third one, we discuss axioms for CS-games. The χ-value is introduced in the fourth section. In the fifth section, the relation between the χ-value and the Wiese value is explored. The sixth section establishes the general stability results. In the seventh section, the χ-value and χ-stability are applied to a range of games. Some remarks conclude the paper.

2. Basic definitions and notation

A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players N and the coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. $v(K)$ is called the N and the coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. $v(K)$ is called the N and the coalition function $v : 2^N$ $\to \mathbb{R}$, $v(\emptyset) = 0$. $v(K)$ is called the worth of $K \subseteq N$. For $\emptyset \neq T \subseteq N$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a unanimity game. A value is an operator φ that assigns payoff vectors to all games, $\varphi(N, v) \in \mathbb{R}^N$. A coalition structure for (N, v) is a partition $\mathcal{P} \subseteq 2^N$ where $\mathcal{P}(i)$ denotes the cell containing player i. In general, subsets of N are called coalitions; elements of $\mathcal{P}$ are referred to as structural coalitions (components). A partition $\mathcal{P}' \subseteq 2^N$ is finer than $\mathcal{P} \subseteq 2^N$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all $i \in N$. A CS-game is a game together with a coalition structure, $(N, v, \mathcal{P})$. The sum $v + v'$ of two coalition functions on N is given by $(v + v')(K) = v(K) + v'(K)$ for all $K \subseteq N$. A CS-value is an operator φ that assigns payoff vectors to all CS-games, $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$. For $K \subseteq N$, we denote by $\varphi_K(N, v, \mathcal{P})$ the sum $\sum_{i \in K} \varphi_i(N, v, \mathcal{P})$. When it is clear which game is meant, we sometimes drop the argument of the value operator.

An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted by $K_i(\sigma) = \{j : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. A player i is called a Null player iff $v(K \cup \{i\}) = v(K)$ for all $K \subseteq N$. Players i, j are called symmetric if $v(K \cup \{i\}) = v(K \cup \{j\})$ for all $K \subseteq N \setminus \{i, j\}$. Player i dominates player j if $v(K \cup \{i\}) - v(K) \geq v(K \cup \{j\}) - v(K)$ for all $K \subseteq N \setminus \{i, j\}$ and the inequality is strict for some K. The Shapley value, Sh, is defined as the average marginal contribution over all orderings of players, $\text{Sh}_i(N, v) = |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i(\sigma)$. Since this definition does not make use of the players' names, the Shapley value satisfies the following strong symmetry axiom **SS**: A bijection $\pi : N \to N$ is called a symmetry of (N, v) iff $v(K) = v(\pi(K))$ for all $K \subseteq N$. Then, we have $\text{Sh}_i(N, v) = \text{Sh}_{\pi(i)}(N, v)$ for all symmetries π of (N, v).

3. Axioms for CS-values

In this section, we discuss a range of axioms with respect to the desired properties of the CS-value to be introduced.

Axiom 3.1 (Additivity, A). *$\varphi(N, v + v', \mathcal{P}) = \varphi(N, v, \mathcal{P}) + \varphi(N, v', \mathcal{P})$ for all coalition functions v, v'.*

This is a powerful standard axiom. It is among the Shapley axioms as well as among [52]
the axioms for the AD-value, the Owen value, and the Wiese value. So, **A** does not seem to be in conflict with outside options. For a motivation of this axiom, we refer to Roth (1977), for example.

AXIOM 3.2 (Component restricted symmetry, CS). *If $i, j \in N$ are symmetric and $j \in \mathcal{P}(i)$ and then $\varphi_i(N, v, \mathcal{P}) = \varphi_j(N, v, \mathcal{P})$.*

CS-values should be insensitive to the labelling of players. This is expressed with the following symmetry axiom **SCS** which takes into account the coalition structure: A symmetry of $(N, v, \mathcal{P})$ is a symmetry π of (N, v) such that $\pi(\mathcal{P}(i)) = \mathcal{P}(\pi(i))$ for all i. For all i and all symmetries π of $(N, v, \mathcal{P})$, $\varphi_i(N, v, \mathcal{P}) = \varphi_{\pi(i)}(N, v, \mathcal{P})$. Therefore, **CS** as a relaxation of **SCS** should be satisfied. For example, Hart & Kurz (1983) employ an axiom like **SCS** in their axiomatization of the Owen value.

AXIOM 3.3 (Component efficiency, CE). *For all $i \in N$, $\varphi_{\mathcal{P}(i)}(N, v, \mathcal{P}) = v(\mathcal{P}(i))$.*

This axiom indicates that the components are the productive units; the players within a component cooperate in order to produce that component's worth. **CE** is also met by the AD-value and the Wiese value as well as by the approaches of Myerson (1977) and Shenoy (1979), for example. In contrast, the Owen value satisfies efficiency within the grand coalition, $\varphi_N(N, v, \mathcal{P}) = v(N)$, which we call axiom **E**. This corresponds to interpreting the components as bargaining blocs which bargain on the distribution of the grand coalition's worth.

AXIOM 3.4 (Null player, N). *If $i \in N$ is a Null player then $\varphi_i(N, v, \mathcal{P}) = 0$.*

While the Shapley value as well as the AD- and the Owen value satisfy **N**, the Wiese value violates this axiom. Yet, **N** together with **CE** may make a CS-value insensitive to outside options. To see this, consider the unanimity game (N, u_T) where $N = \{1, 2, 3\}$ and $T = \{1, 2\}$ together with the coalition structure $\mathcal{P} = \{\{1\}, \{2, 3\}\}$. Since 3 is a Null player, we have $\varphi_3 = 0$, and by **CE**, $\varphi_2 + \varphi_3 = 0$, and therefore $\varphi_2 = 0$. Yet, player 2 has an outside option to create the worth 1 together with player 1. Therefore, one could argue that payer 2 should obtain a higher payoff than player 3. Hence for our purpose, **N** seems to be too strong.

Several alternatives to the Null player axiom have been proposed. For the class of simple games, Napel & Widgren (2001) define so-called inferior players who form a superset of the set of Null players. All inferior players get the payoff 0 according to their Strict Power Index, a close relative of the Banzhaf index. Nowak & Radzik (1994) present a solidarity value where Null players in unanimity games obtain a positive payoff. Since Null players obtain a non-negative payoff under these values, the same objections as for **N** apply.

AXIOM 3.5 (Grand coalition Null player, GN). *If $i \in N$ is a Null player then $\varphi_i(N, v, \{N\}) = 0$.*

In the grand coalition, there are no outside options. Hence for $\mathcal{P} = \{N\}$, **N** should be satisfied. Note that **GN** together with **CE**, **CS**, and **A** characterizes the Shapley value for $\mathcal{P} = \{N\}$, while **N** together with **CE**, **CS**, and **A** characterizes the AD-value.

Axiom 3.6 (Component restricted dominance, CD). *If* $i \in N$ *dominates* $j \in \mathcal{P}(i)$ *then* $\varphi_i(N, v, \mathcal{P}) > \varphi_j(N, v, \mathcal{P})$.

This axiom seems to capture the idea that outside options as well as contributions to ones own coalition matter in a very weak sense. Therefore, our CS-value should satisfy **CD**. However, **CD** and **CE** together are incompatible with **N**. Reconsider the CS-game $(N, u_T, \mathcal{P})$ as above. Player 3 is dominated by player 2. By **CD,** we then have $\varphi_2 > \varphi_3$, and by **CE**, $\varphi_2 + \varphi_3 = 0$, hence, $0 > \varphi_3$; a Null player may obtain a negative payoff. Note that this is also possible for the Wiese value. At first glance, this seems to be odd. Player 3 could avoid this negative payoff by forming a singleton coalition. Yet, this does not speak against **CD** or the Wiese value but against the coalition structure $\mathcal{P}$ to evolve or—in other words—against $\mathcal{P}$ being stable. We have to distinguish between the payoffs for a (hypothetically) given coalition structure and the payoffs under a stable coalition structure that might or might not exist. Within a stable coalition structure, of course, a Null player should obtain a non-negative payoff (see Corollary 6.3). [53]

Axiom 3.7 (Component independence, CI). *If* $\mathcal{P}(i) = \mathcal{P}'(i)$, $i \in N$ *then* $\varphi_i(N, v, \mathcal{P}) = \varphi_i(N, v, \mathcal{P}')$.

This axiom says that way the players outside ones own component are organized does not affect ones own payoff. At first glance, this does not seem to be a good axiom for a CS-value, in particular for one which is intended to account for outside options. Yet, even though the Wiese value satisfies **CI**, it accounts for outside options. Moreover, this axiom is justified if one maintains the view that the players produce their components' worth which in TU games does not depend on the whole coalition structure. Further, outside options come into play when the coalitions ultimately have been formed, i.e. the players do not consider to change their coalition. Therefore, the players are not necessarily restricted to the actual coalition structure when they bargain *within* their component on the distribution of that component's worth. Compare Maschler (1992, pp. 595) cited in the Introduction. Besides, **CI** is an advantage of a CS-value concerning stability issues which are explored in Section 6.

Axiom 3.8 (Outside option, OO). *For all* $P \in \mathcal{P}$ *and* $\emptyset \neq T \subseteq N$, *we have* $\varphi_{P \backslash T}(N, u_T, \mathcal{P}) = 0$ *if* $|\mathcal{P}(T)| = 1$ *and*

$$\varphi_{P \backslash T}(N, u_T, \mathcal{P}) = -\frac{|P \cap T|}{|T|} \frac{|P \backslash T|}{|P \cup T|}$$

if $|\mathcal{P}(T)| > 1$, *where* $\mathcal{P}(T) := \{\mathcal{P}(i) \,|\, \mathcal{P}(i) \cap T \neq \emptyset\}$.

This axiom replaces the Null player axiom in the axiomatization of the Wiese (2007) value. Together with **A**, (in fact, Wiese employs linearity which can be relaxed into **A**), **CE**, and **CS**, **OO** characterizes the Wiese value. However, this axiom has not too much intuitive appeal. In essence, **OO** (together with **CE** and **CS**) determines the payoffs for unanimity games in a way such that the definition via marginal contributions (see Section 5) is met.

AXIOM 3.9 (Splitting, SP). *If $\mathcal{P}'$ is finer than $\mathcal{P}$ then for all $i \in N$ and $j \in \mathcal{P}'(i)$, we have*

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i\left(N, v, \mathcal{P}'\right) = \varphi_j(N, v, \mathcal{P}) - \varphi_j\left(N, v, \mathcal{P}'\right).$$

We feel that this axiom is much more appealing than **OO**. It can be paraphrased as follows: Splitting a structural coalition affects all players who remain in the same structural coalition in the same way. As the value is already meant to reflect the outside options of the players, one could argue that the gains/losses of splitting/separating should be distributed equally within a resulting structural coalition. As it turns out, **SP** fills the [54]
gap concerning uniqueness issues which arises when **N** is relaxed into **GN**.

In a different setting, Myerson (1977) employs a similar axiom. Instead of partitions, he considers undirected graphs on the player set. The related fairness axiom requires that connecting two players—other things being equal—changes these players' payoffs by the same amount. The resulting value, however, is very different from ours. For completely connected components, the Myerson value and the AD-value coincide. Moreover, Slikker & van den Nouweland (2001, p. 93) suggest an axiom for CS-values that somewhat resembles **SP** but which together with **CE** characterizes the AD-value.

4. The χ-value

In this section, we show that some of the axioms advocated in the previous section already characterize a CS-value which satisfies the remaining such axioms. Further, we determine the payoffs for unanimity games and demonstrate that this CS-value accounts for outside options.

THEOREM 4.1. *There is a unique CS-value that satisfies* ***CE, CS, A, GN,*** *and* ***SP****.*

PROOF. Let φ be a value that satisfies **CE**, **CS**, **A**, **GN**, and **SP**. Since the first four axioms are the Shapley ones for $\mathcal{P} = \{N\}$, we have $\varphi_i(N, v, \{N\}) = \mathrm{Sh}_i(N, v)$. By **SP**,4.2 we have

$$\varphi_i(N, v, \mathcal{P}) - \mathrm{Sh}_i(N, v) = \varphi_j(N, v, \mathcal{P}) - \mathrm{Sh}_j(N, v) \tag{4.1}$$

for $j \in \mathcal{P}(i)$. Summing up (4.1) over $j \in \mathcal{P}(i)$ gives

$$|\mathcal{P}(i)|\left(\varphi_i(N, v, \mathcal{P}) - \mathrm{Sh}_i(N, v)\right) = v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N, v)$$

by **CE**. Hence,

$$\varphi_i(N,v,\mathcal{P}) = \mathrm{Sh}_i(N,v) + \frac{v(\mathcal{P}(i)) - \mathrm{Sh}_{\mathcal{P}(i)}(N,v)}{|\mathcal{P}(i)|}. \tag{4.2}$$

The CS-value defined by (4.2) inherits **CS** and **A** from the Shapley value. For $\mathcal{P} = \{N\}$, this CS-value actually is the Shapley value, hence satisfies **GN**. Finally, **CE** and **SP** follow from simple calculations. □

We call this CS-value "the χ-value". It employs the Shapley value as a yardstick to distribute the payoff within a structural coalition. The difference of a player's payoff from the average payoff of his structural coalition equals the difference between his Shapley payoff and the average Shapley payoff of his structural coalition. In other words, the players within a structural coalition depart from their Shapley payoffs and then compare the worth of the coalition with the sum of the Shapley payoffs; the difference, positive or negative, is distributed equally. Hence, whenever the Shapley payoffs are component efficient, the χ-value coincides with the Shapley value. This can be paraphrased as that the Shapley value reflects outside options "up to component efficiency".

From (4.2), it is immediate that the χ-value satisfies **CI**. Since $\mathrm{Sh}_i(N,v) > \mathrm{Sh}_j(N,v)$ whenever i dominates j, (4.2) also implies $\chi_i(N,v,\mathcal{P}) > \chi_j(N,v,\mathcal{P})$ if $j \in \mathcal{P}(i)$. Hence, the χ-value satisfies **CD**.

The AD-value and the Owen value can be characterized by auxiliary games. So can the χ-value. Given a CS-game $(N,v,\mathcal{P})$, we construct an auxiliary game (P,v_P) for every structural coalition $P \in \mathcal{P}$. Basically, this game is the inessential game which is generated by assigning the Shapley payoff to the singleton coalitions $\{i\}$. The only deviation is the [55]
grand coalition P of (P,v_P) which is assigned its worth $v(P)$, i.e. $v_P(K) = \mathrm{Sh}_K(N,v)$ if $K \subsetneq P$ and $v_P(P) = v(P)$. With probability $|P|^{-1}$, player $i \in P$ is the last player for some order in $\Sigma(P)$. In this case, i's marginal contribution is $v(P) - \mathrm{Sh}_{P\backslash\{i\}}(N,v)$. As the game is inessential elsewhere, with probability $1 - |P|^{-1}$, i's marginal contribution is the Shapley payoff $\mathrm{Sh}_i(N,v)$. By (4.2), we then have $\mathrm{Sh}_i(\mathcal{P}(i), v_{\mathcal{P}(i)}) = \chi_i(N,v,\mathcal{P})$.

The unanimity games $(N,u_T,\mathcal{P})$ form a basis of the linear space of games based on fixed N and $\mathcal{P}$. Therefore, the χ-payoffs for these games are of particular interest. One easily checks that

$$\chi_i(N,u_T,\mathcal{P}) = \begin{cases} \frac{1}{|T|}, & i \in T, |\mathcal{P}(T)| = 1, \\ 0, & i \notin T, |\mathcal{P}(T)| = 1, \\ \frac{|\mathcal{P}(i)\backslash T|}{|\mathcal{P}(i)||T|}, & i \in T, |\mathcal{P}(T)| > 1, \\ -\frac{|\mathcal{P}(i)\cap T|}{|\mathcal{P}(i)||T|}, & i \notin T, |\mathcal{P}(T)| > 1. \end{cases} \tag{4.3}$$

The last two lines of (4.3) indicate that the χ-value accounts for outside options. This can be seen from the following justification of these payoffs.

If a structural coalition contains players of the same type only, **CS** and **CE** distribute the coalition's payoff equally among the players as all players are symmetric to their likes, and if a coalition contains all T-players and some non-T-players, then the players get their Shapley payoffs because then the Shapley payoffs are component efficient.

The interesting cases are those where outside options come into play, i.e. where a structural coalition $P \in \mathcal{P}$ contains both types of players, but not all T-players. Then, we have $v(P) = 0$. One could argue that a T-player in P has the outside option to create the worth of 1 together with the T-players outside P and thus foregoes the payoff $\frac{1}{|T|}$ which should be refunded by *all* players of his structural coalition because, for some reason, all of them were interested in forming just this coalition. Hence, a T-player obtains $\frac{1}{|T|}$ but has to pay an amount of $\frac{|P \cap T|}{|P||T|}$, i.e. he obtains a net payoff $\frac{|P \backslash T|}{|P||T|}$. Every non-$T$-player, i.e. Null player in P pays $\frac{1}{|P||T|}$ to every T-player in P, i.e. a Null player has to pay $\frac{|P \cap T|}{|P||T|}$. Moreover, one could think of that the players had some preferences—beyond the payoffs—for being in a particular coalition. The transfers within a structural coalition then reflect a (hypothetical) trade-off between being in the preferred coalition and the payoff obtained. Both types of players face some cost. The T-players in P obtain the payoff $\frac{|P \backslash T|}{|P||T|}$ which is less than the payoff $\frac{1}{|T|}$ which they obtained in a coalition with all other T-players. The Null players in P pay $\frac{|P \cap T|}{|P||T|}$ instead of nothing in the case they formed singleton coalitions, for example.

Further examples are given in Section 7.

5. Relation to the Wiese value

In this section, we compare the χ-value and the Wiese (2007) value. It turns out that both concepts are close relatives. Though the Wiese value W lacks a convincing justification in terms of intuitive axioms (see Section 3), the following definition via marginal contributions has some appeal: For all $i \in N$,

$$W_i(N, v, \mathcal{P}) = \frac{1}{|\Sigma(N)|} \sum_{\sigma \in \Sigma(N)} \begin{cases} v(\mathcal{P}(i)) - \sum_{j \in \mathcal{P}(i) \backslash \{i\}} MC_j(\sigma), & \sigma \in \Sigma_i(N, \mathcal{P}), \\ MC_i(\sigma), & \sigma \notin \Sigma_i(N, \mathcal{P}), \end{cases}$$

where $\Sigma_i(N, \mathcal{P}) \subseteq \Sigma(N)$ denotes the set of orders which satisfy $|K_i(\sigma) \cap \mathcal{P}(i)| = |\mathcal{P}(i)|$, [56]
i.e. player i is the last player of his component in σ. This has a nice interpretation: For a given order, the last player of a component can be viewed as the "owner" of the component, i.e. its residual claimant. While the other players of this coalition obtain their marginal contribution, the last one obtains the worth of the coalition but has to pay the marginal contributions of the other players. Of course, one could ask oneself what is particular about being the last player of one's own structural coalition within some order? Alternatively,

one could think of the first or any other position. From (5.1) and (5.2) it clear that the χ-value is the average over all possible positions.

Applying the Shapley formula, we obtain the following equations which make explicit the close relation between the Wiese value and the χ-value:

$$W_i(N,v,\mathcal{P}) = \\ \mathrm{Sh}_i(N,v) + \frac{v(\mathcal{P}(i))}{|\mathcal{P}(i)|} - \frac{1}{|\mathcal{P}(i)|}\frac{1}{|\Sigma_i(N,\mathcal{P})|}\sum_{\sigma\in\Sigma_i(N,\mathcal{P})}\sum_{j\in\mathcal{P}(i)} MC_j(\sigma) \quad (5.1)$$

$$\chi_i(N,v,\mathcal{P}) = \\ \mathrm{Sh}_i(N,v) + \frac{v(\mathcal{P}(i))}{|\mathcal{P}(i)|} - \frac{1}{|\mathcal{P}(i)|}\frac{1}{|\Sigma(N)|}\sum_{\sigma\in\Sigma(N)}\sum_{j\in\mathcal{P}(i)} MC_j(\sigma) \quad (5.2)$$

The last terms in (5.1) and (5.2) make the difference between these concepts. Ignoring the factor $-|\mathcal{P}(i)|^{-1}$, it gives the average sum of the marginal contributions of the structural coalition $\mathcal{P}(i)$. While the χ-value takes this average over all orders on N, the Wiese value focuses on those orders where i is the last player of $\mathcal{P}(i)$. This implies that the Wiese value and the χ-value coincide for symmetric games.

6. Stability

Since the χ-value is component independent, all of the Hart & Kurz (1983) stability concepts coincide and can be characterized as follows (Wiese 2007): A coalition structure $\mathcal{P}$ for (N,v) is stable with respect to the χ-value (χ-stable) iff for all $\emptyset \neq K \subseteq N$ there is some $i \in K$ such that $\chi_i(N,v,\mathcal{P}) \geq \chi_i(N,v,\{K, N\backslash K\})$.

Theorem 6.1. *For all TU games, there are χ-stable coalition structures.*

In contrast, Hart & Kurz (1984) provide examples of TU games that do not allow for stable coalition structures with respect to the Owen value. For the Wiese value it is not yet clear whether there are games without stable coalition structures.

Proof. We mimic the Wiese (2007) proof for symmetric games. Construct a partition $\mathcal{P} = \{K_1, K_2, \ldots, K_k\}$ as follows: Set $P_1 = \emptyset$ and continue by induction: $P_{n+1} = P_n \cup K_n$ for $n \geq 1$ and

$$K_n \in \underset{K\subseteq N\backslash P_n}{\operatorname{argmax}}\, \Delta(K), \qquad \Delta(K) := \frac{v(K) - \mathrm{Sh}_K(N,v)}{|K|} \quad (6.1)$$

for $n > 1$ until $P_{k+1} = N$. Suppose, $\mathcal{P}$ were not χ-stable. Then, there were some coalition $C \notin \mathcal{P}$ such that $\chi_i(N,v,\{C,N\backslash C\}) > \chi_i(N,v,\mathcal{P})$ for all $i \in C$. The only reason for C not being in $\mathcal{P}$ is that $\mathcal{P}$ contains a structural coalition K_j such that $C \cap K_j \neq \emptyset$ and $\Delta(C) \leq \Delta(K_j)$. Hence by (4.2), we had $\chi_i(N,v,\{C,N\backslash C\}) \leq \chi_i(N,v,\mathcal{P})$ for $i \in C \cap K_j$, a contradiction. □

From this proof it is clear that [57]

Corollary 6.2. *All χ-stable coalition structures can be constructed in this way.*

In (4.3), we have seen that a Null player may obtain a negative χ-payoff. This is impossible within a χ-stable coalition structure.

Corollary 6.3. *Within χ-stable coalition structures, Null players obtain the χ-payoff* 0.

Proof. Let i be a Null player and $\mathcal{P}$ be a χ-stable coalition structure. Since $\Delta(\{i\}) = 0$, we have $\Delta(\mathcal{P}(i)) = \chi_i(N, v, \mathcal{P}) \geq 0$ by (4.2) and stability. If $\chi_i(N, v, \mathcal{P}) > 0$ then $\Delta(\mathcal{P}(i) \setminus \{i\}) > \Delta(\mathcal{P}(i))$—contradicting $\mathcal{P}$ being χ-stable. □

The following theorem provides a first stability result. In Section 7, we apply χ-stability to a range of games.

Corollary 6.4. *The grand coalition is χ-stable iff the Shapley value lies in the core.*

Unsurprisingly, of course, this implies that the grand coalition is χ-stable for convex games for which the Shapley value lies in the core.

Proof. By Corollary 6.2, the grand coalition is χ-stable iff

$$0 = \frac{v(N) - \mathrm{Sh}_N(N, v)}{|N|} = \Delta(N) \geq \Delta(K) = \frac{v(K) - \mathrm{Sh}_K(N, v)}{|K|}$$

i.e. iff $\mathrm{Sh}_K(N, v) \geq v(K)$ for all $K \subseteq N$, i.e. iff the Shapley payoff lies in the core. □

7. Examples

In this section, we apply the χ-value and χ-stability to a range of games.

7.1. Simple monotonic non-contradictory games. A game (N, v) is called simple if $v(2^N) \subseteq \{0, 1\}$ and monotonic if $K \subseteq K'$ implies $v(K) \leq v(K')$ for all $\emptyset \neq K, K' \subseteq N$. Such a game is characterized by the set of winning coalitions $\mathbb{W} := \{K \subseteq N | v(K) = 1\}$. A winning coalition K is called minimal if $v(K') < v(K)$ for all $K' \subsetneq K$. We denote by $\mathbb{W}_{\min}$ the set of these coalitions. A simple monotonic game is called non-contradictory if $K \in \mathbb{W}$ implies $N \backslash K \notin \mathbb{W}$. For these games, we have $\Delta(K) = -\frac{\mathrm{Sh}_K}{|K|}$ if $K \notin \mathbb{W}$ and $\Delta(K) = \frac{1 - \mathrm{Sh}_K}{|K|}$ if $K \in \mathbb{W}$.

It is clear that the Shapley payoff $\mathrm{Sh}_i(N, v)$ is non-negative and that it is 0 iff i is a Null player, i.e. if i is not member of any winning coalition. Hence, $\Delta(K)$ is negative if K is not winning but contains non-Null players. Suppose there is a unique minimal winning coalition T. If $T \subseteq K$ then $\Delta(K) = 0$. Therefore, the χ-stable coalition structures are those where $|\mathcal{P}(T)| = 1$, i.e. all T-players are united. If there is more than one minimal

winning coalition then we have $\mathrm{Sh}_K < 1$, hence $\Delta(K) > 0$ for all minimal winning coalitions K. Since $\Delta(K)$ decreases in $|K|$ for $\mathrm{Sh}_K < 1$ and $K \in \mathbb{W}$, a χ-stable coalition structure contains a minimal winning coalition that maximizes $\frac{1-\mathrm{Sh}_K}{|K|}$. Since the game is non-contradictory, the other players form structural coalitions containing players with the [58]
same Shapley payoff. The latter follows from $\Delta(K) = -\frac{\mathrm{Sh}_K}{|K|}$ if $K \notin \mathbb{W}$.

Theorem 7.1. *In simple monotonic non-contradictory games, we have the following χ-stable coalition structures:*

1. *If $\mathbb{W}_{\min} = \{T\}, T \subseteq N$ then $\mathcal{P}$ is χ-stable iff $|\mathcal{P}(T)| = 1$.*
2. *If $|\mathbb{W}_{\min}| > 1$ then $\mathcal{P}$ is χ-stable iff there is some $T \in \mathbb{W}_{\min} \cap \mathcal{P}$ such that $\frac{1-\mathrm{Sh}_T}{|T|} \geq \frac{1-\mathrm{Sh}_K}{|K|}$ for all $\emptyset \neq K \in \mathbb{W}_{\min}$, and for all $i, j \in N \backslash T$, $j \in \mathcal{P}(i)$ implies $\mathrm{Sh}_i = \mathrm{Sh}_j$.*

On the one hand, χ-stability favors small winning coalitions since we have $|K|$ as denominator in $\Delta(K)$. Besides the payoff as the main effect, one could argue that it is easier to keep smaller coalitions together, for example because of lower negotiation costs. On the other hand, χ-stability favors coalitions with a low sum of Shapley payoffs. Since the Shapley payoffs—in a sense—measure the outside options of the players, coalitions where the players have less outside options tend to be more stable. This seems to be full in line with our intuitions. Altogether, χ-stability balances individual payoffs and stability.

We apply this result to some special classes of simple monotonic non-contradictory games.

Unanimity games (N, u_T) have the unique minimal winning coalition T. Hence by Theorem 7.1, the coalition structures $\mathcal{P}$ satisfying $|\mathcal{P}(T)| = 1$ are the χ-stable ones. The χ-payoffs have already been given by (4.3).

Consider now the apex games A_n, $n \geq 2$ with the set of players $\{0, 1, \ldots, n\}$ where we call 0 the apex player and the other players minor ones. All coalitions which contain the apex player and at least one minor player as well the coalition which contains all minor players produce the worth of 1 while all other coalitions produce the worth of 0. We then have $\mathrm{Sh}_0 = \frac{n-1}{n+1}$ and $\mathrm{Sh}_i = \frac{2}{n(n+1)}$, $i \neq 0$. This gives the χ-payoffs for the apex player 0 and the minor players i as follows

$$\chi_0(A_n, \mathcal{P}) = \begin{cases} 0, & \mathcal{P}(0) = \{0\} \\ \frac{n-2}{n} + \frac{2}{n|\mathcal{P}(I)|}, & \mathcal{P}(0) \neq \{0\} \end{cases} \tag{7.1a}$$

$$\chi_i(A_n, \mathcal{P}) = \begin{cases} \frac{2}{n|\mathcal{P}(i)|}, & 0 \in \mathcal{P}(i) \\ \frac{1}{n} & \mathcal{P}(i) = N \backslash \{0\} \\ 0,, & \mathcal{P}(i) \subsetneq N \backslash \{0\} \end{cases} \tag{7.1b}$$

The interesting cases are those where the apex player and some minor players are in the same structural coalition. Both χ-payoffs decrease with the number of minor players and finally become the Shapley payoffs when the grand coalition is formed. Since the Shapley value assigns a positive payoff to all players, all players in this structural coalition gain in comparison to their Shapley payoffs. If just one minor player joined the apex player, his payoff then is $\frac{1}{n}$ and equals his χ-payoff in the coalition structure where all minor players form a structural coalition.

The apex games A_n are simple monotonic non-contradictory games with more than one minimal winning coalition. In particular, the minimal winning coalitions are the coalitions $\{0, i\}$ containing the apex player and some minor player i and the coalition $N\backslash\{0\}$ containing all minor players. By (7.1), we have $\Delta(\{0,i\}) = \frac{n-1}{n(n+1)} = \Delta(N\backslash\{0\})$. Hence by Theorem 7.1, A_n has the following χ-stable coalition structures: (a) The apex player forms a coalition with one minor player and the other players are organized arbitrarily. [59]
(b) All minor players form a coalition excluding the apex player. Thus, all minimal winning coalitions are χ-stable. This result is in line with Bennet (1983), whereas Hart & Kurz (1984) and Aumann & Myerson (1988) obtain (b) as the outcome, while Chatterjee, Dutta, Ray & Sengupta (1993) favor (a).

7.2. The gloves game. Shapley & Shubik (1969) consider a simple market game—the gloves game $[\lambda, \rho]$. There are $\lambda > 0$ left-glove holders (ℓ) in L and $\rho > 0$ right-glove holders (r) in R. The coalition function is given by $v(K) = \min(|R \cap K|, |L \cap K|)$ for $K \subseteq R \dot{\cup} L$ $=: N$, i.e. the worth of a coalition is the number of its matching pairs of gloves. For symmetry reasons, we focus on the case $\rho \geq \lambda$. The Shapley payoffs then are given by

$$\mathrm{Sh}_r(\lambda, \rho) = \frac{1}{2} - \frac{\rho - \lambda}{2\rho} \sum_{k=0}^{\lambda} \frac{\binom{\lambda}{k}}{\binom{\rho+k}{k}}, \qquad \mathrm{Sh}_\ell(\lambda, \rho) = \frac{1}{2} + \frac{\rho - \lambda}{2\lambda} \sum_{k=1}^{\lambda} \frac{\binom{\lambda}{k}}{\binom{\rho+k}{k}}, \tag{7.2}$$

i.e. $\mathrm{Sh}_\ell(\lambda, \rho) > \mathrm{Sh}_r(\lambda, \rho)$ (> 0) iff $\lambda < \rho$. Hence, the Shapley value reflects the relative scarcity of the resources.

By (4.2), the χ-payoffs easily can be calculated from (7.2). Further, it is easy to see that the χ-value inherits the sensitivity of the Shapley value with respect to the relative scarcity of the resources: For $r \in \mathcal{P}(\ell)$, we have $\chi_\ell(\lambda, \rho, \mathcal{P}) - \chi_r(\lambda, \rho, \mathcal{P}) = \mathrm{Sh}_\ell(\lambda, \rho) - \mathrm{Sh}_r(\lambda, \rho)$. Hence, the χ-value captures outside options in the gloves game. Further, forming balanced coalitions is rewarded by the χ-value. A structural coalition $\mathcal{P}(i)$ is called balanced if it contains the same number of left- and right-glove holders. Obviously, we have $\mathrm{Sh}_\ell(\lambda, \rho) + \mathrm{Sh}_r(\lambda, \rho) \leq 1$, hence by (4.2), $\chi_i(\lambda, \rho, \mathcal{P}) \geq \mathrm{Sh}_i(\lambda, \rho)$ for balanced $\mathcal{P}(i)$ where equality holds iff $\lambda = \rho$.

For the gloves game, Shapley & Shubik (1969) show that under replication ($[\alpha\lambda, \alpha\rho]$, $\alpha \to \infty$) the Shapley value converges to the core. If $\rho > \lambda$, the Shapley payoff of a left-glove holder converges to 1 and that one of right-glove holder to 0. For $\rho = \lambda$, both payoffs are $\frac{1}{2}$. A coalition structure $\mathcal{P}$ is called balanced iff all structural coalitions containing left-glove holders are balanced. For balanced coalition structures, the core is component efficient.

Hence in the limit, the Shapley payoffs become component efficient. Therefore by (4.2), the χ-payoffs converge to the Shapley payoffs, hence to the core. Note that this is not the case for unbalanced structural coalitions containing left-glove holders. Whereas the worth "counts" the matching pairs within a structural coalition, at the limit, the Shapley value "counts" the number of left-glove holders.

THEOREM 7.2. *For balanced coalition structures and* $[\alpha\lambda, \alpha\rho]$, $\alpha \to \infty$, *the* χ*-payoffs converge to the core.*

If K contains left- or right-glove owners only then $\Delta(K) = -\mathrm{Sh}_i < 0$, $i \in K$. If K contains both types of players but is unbalanced then it is possible to increase $\Delta(K)$ by removing the glove holders in excess. This is immediate from (6.1), as $v(K)$ does not change but Sh_K and $|K|$ decrease. If K is balanced then $\Delta(K) = \frac{1-(\mathrm{Sh}_\ell + \mathrm{Sh}_r)}{2} \geq 0$. Hence, $\Delta(K)$ is maximal when K is balanced. By Corollary 6.2, we then have

THEOREM 7.3. *In* $[\lambda, \rho]$, *the balanced coalition structures are the* χ*-stable ones.*

8. Conclusion [60]

In this paper, we introduced and advocated a component efficient CS-value—the χ-value—that accounts for outside options as an alternative to the Wiese (2007) value. The main advantages of the χ-value are its intuitive axiomatization where the splitting axiom **SP** is the crucial ingredient and the universal existence of χ-stable coalition structures. Besides, the χ-value can easily be derived from the Shapley value. Nevertheless, both concepts turn out to be close relatives that coincide on the class of symmetric games. Therefore, one could view the χ-value and its axiomatization as means to support the Wiese value which in view of its appealing definition via marginal contributions seems to be desirable.

Further, splitting type axioms similar to our splitting axiom **SP** may serve as means to justify/axiomatize component efficient CS-value concepts that are derived from other (efficient) value concepts via formulae like (4.2). Let us outline an example: Thrall & Lucas (1963) introduce partition function form games (PFFG), (N, p). In PFFG, the worth $p(P, \mathcal{P})$ of a coalition P is given by the partition function p which takes into account not only the coalition itself but also a coalition structure $\mathcal{P}$ it is embedded in, i.e. $P \in \mathcal{P}$. Several values φ for PFFG have been proposed (e.g. Myerson 1977b, Bolger 1989, Potter 2000, Pham Do & Norde 2002) all of which satisfy the efficiency axiom **pE**, $\varphi_N(N, p) = p(N, \{N\})$. Similar to CS-games, define CS-PFFG as PFFG that come with a fixed coalition structure, $(N, p, \mathcal{P})$. Then, one may be interested in component efficient values for such CS-PFFG, i.e. values that satisfy the axiom **pCE**: $\varphi_{\mathcal{P}(i)}(N, p, \mathcal{P}) = p(\mathcal{P}(i), \mathcal{P})$

for all $i \in N$, which are supported by intuitive axioms. Such values may be of interest in the analysis of oligopoly games where not necessarily a single cartel arises.

For example, Pham Do & Norde (2002) adapt the notions of a Null player and of symmetric players as well as the Shapley axioms **A**, **N**, **S**, and **E** to PFFG such that the resulting axioms (indicated by the affix **p**) characterize their value Ψ. Keep **pA** and replace **pS** and **pN** by the new axioms **pCS** and **pGN** in analogy to **CS** and **GN**. Finally, add the axiom **pSP** analogous to **SP**. Arguments similar to those applied in the proof of Theorem 4.1 then show that the axioms **pA**, **pCS**, **pGN**, **pCE**, and **pSP** characterize the CS-PFFG value Ψ^* given by

$$\Psi_i^* (N, p, \mathcal{P}) = \Psi_i (N, p) + \frac{p (\mathcal{P} (i), \mathcal{P}) - \Psi_{\mathcal{P}(i)} (N, p)}{|\mathcal{P} (i)|}$$

for all $i \in N$.

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CHAPTER VI

Outside options in TU games with a cooperation structure

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Abstract

In this paper, we introduce and characterize a component efficient value for TU games with a cooperation structure which in contrast to the Myerson (1977) value accounts for outside options. It is based on the idea that the distribution of the worth within the connected components should be consistent with some "outside-option" graphs which keep the internal link structure of a component, but which consider all links between a component's players and the players outside.

Key Words: TU game, Outside option, Splitting, Consistency, Cooperation structure

JEL classification: C71

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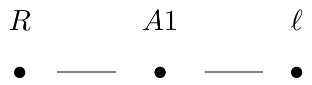
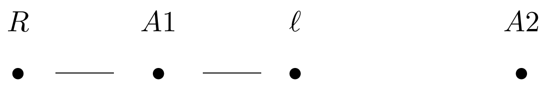
1. Introduction

One right-glove holder, R, and one left-glove holder, ℓ, actually sell their pair of gloves which is worth 1 via some agent $A1$. How should R, ℓ, and $A1$ split the proceeds? Would this split change if there were a second agent $A2$? In order to answer this kind of questions, Myerson (1977), Borm et al. (1992), and Hamiache (1999) consider values for TU games with a cooperation structure, i.e. an undirected graph on the player set (henceforth, CO-games and CO-values). In the following, we focus on the Myerson value as the most eminent one.

Our leading example then corresponds to a TU game with 3 (or 4) players, R, ℓ, $A1$, (and $A2$), where the worth of a coalition is 1 if it contains a matching pair, i.e. the players R and ℓ, and is 0 if it does not so. The fact that R and ℓ sell their pair via $A1$ then can be modelled by the following graphs:



(1.1)



(1.2)

In both cases, the Myerson value μ assigns the same payoffs to R, ℓ, and $A1$, $\mu_R = \mu_\ell = \mu_{A1} = \frac{1}{3}$. Though A is a Null player, he obtains a positive payoff what fits nicely with our intuitions on his role in this transaction—he actually facilitates the sale. Yet, a bit unintuitively, the share of $A1$ is not affected by the presence of the potential competitor $A2$. Thus, the Myerson value does not account for the outside option of R and ℓ to sell their pair of gloves via $A2$. Outside options, however, may be important:

> Even if a particular alliance is ultimately formed, the division of the proceeds between the allies will be decisively influenced by the other alliances which each one might alternatively have entered. (von Neumann & Morgenstern 1944, p. 36)
>
> During the course of negotiations there comes a moment when a certain coalition structure is "crystallized". The players will no longer listen to "outsiders", yet each coalition has still to adjust the final share of its proceeds. (This decision may depend on options outside the coalition, even though the chances of defection are slim). (Maschler 1992, pp. 595)

The Myerson value as well as the values considered by Borm et al. (1992) and Hamiache (1999) share this neglect of outside options with the Aumann & Drèze (1974) value (henceforth AD-value) for TU-games with a coalition structure, i.e. a partition of the player set (henceforth CS-games). In order to remedy this peculiarity of the AD-value, Casajus (2009)[1] and Wiese (2007) introduce the χ-value and the outside-option value. Hence, it seems to be worthwhile to look for a CO-value which generalizes these concepts.

In this paper, we introduce and axiomatize the "graph-χ-value", $\chi^\sharp$, which extends the χ-value to CO-games and thus accounts for outside options. To achieve this, we restrict the Myerson fairness axiom to situations without outside options or where outside option are not affected. An outside-option consistency axiom determines how players within the same component assess their outside options and restores the uniqueness lost by relaxing the fairness axiom. It turns out that the $\chi^\sharp$-value coincides with χ-value for completely connected components. For our leading example, we obtain the following payoffs: If $A2$ is not present then the payoffs are as for the Myerson value. But in presence of $A2$, the payoff of $A1$ decreases. In particular, we then have $\chi^\sharp_R = \chi^\sharp_\ell = \frac{4}{9}$ and $\chi^\sharp_{A1} = \frac{1}{9}$ which shows that the $\chi^\sharp$-value rewards outside options without neglecting the role of player $A1$.

The plan of this paper is as follows: The next section provides basic definitions and notation. In the third section, we discuss several axioms for CO-games with respect to outside options. The $\chi^\sharp$-value is introduced and axiomatized in the fourth section. In the fifth section, we explore some properties of this CO-value. In particular, we clarify its relation to the χ-value and demonstrate the difference to the Myerson value concerning stability issues. Some remarks conclude the paper.

2. Basic definitions and notation

In order to avoid set theoretic complications, we assume that there is a large enough set $\mathcal{U}$ that contains the names of the players. A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players $N \subset \mathcal{U}$ and a coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. $v(K)$ is called the worth of $K \subseteq N$; subsets of N are called coalitions. In general, we consider the set of all TU games, possibly equipped with some additional structure. For $\emptyset \neq T \subseteq N$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a unanimity game. The restriction of v to $N' \subseteq N$ is denoted $v|_{N'}$. Player i is a Null player in (N, v) if $v(S) = v(S \setminus \{i\})$ for all $S \subseteq N$. (N, v) is called superadditive if $K, S \subseteq N$ and $K \cap S = \emptyset$ imply $v(K \cup S) \geq v(K) + v(S)$. A value is an operator φ that assigns payoff vectors $\varphi(N, v) \in \mathbb{R}^N$ to all games (N, v), $N \subset \mathcal{U}$.

An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players

[1] Also Chapter V of this thesis.

not after i in σ is denoted by $K_i(\sigma) = \{j \in N | \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i(\sigma, v) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is given by

$$\text{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i(\sigma, v) \quad , i \in N. \tag{2.1}$$

A coalition structure for (N, v) is a partition $\mathcal{P} \subseteq 2^N$ where $\mathcal{P}(i)$ denotes the component containing player i. A partition $\mathcal{P}' \subseteq 2^N$ is finer than $\mathcal{P} \subseteq 2^N$ if $\mathcal{P}'(i) \subseteq \mathcal{P}(i)$ for all $i \in N$. A CS-game is a game together with a coalition structure, $(N, v, \mathcal{P})$. A CS-value is an operator φ that assigns payoff vectors $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to all CS-games $(N, v, \mathcal{P})$, $N \subset \mathcal{U}$. For $K \subseteq N$, we denote by $\varphi_K(N, v, \cdot)$ the sum $\sum_{i \in K} \varphi_i(N, v, \cdot)$.

The AD-value simply is the restriction of the Shapley value to the components, i.e. $\text{AD}_i(N, v, \mathcal{P}) = \text{Sh}_i\left(\mathcal{P}(i), v|_{\mathcal{P}(i)}\right)$; the χ-value (Casajus 2009) is defined by

$$\chi_i(N, v, \mathcal{P}) := \text{Sh}_i(N, v) + \frac{v(\mathcal{P}(i)) - \text{Sh}_{\mathcal{P}(i)}(N, v)}{|\mathcal{P}(i)|} \quad , i \in N. \tag{2.2}$$

A cooperation structure for (N, v) is an undirected graph (N, L) on N where L is a subset of the set $L^N := \{\{i, j\} | i, j \in N, i \neq j\}$ of unordered pairs from N. Abusing notation, we frequently refer to the link set L as the graph. For $\{i, j\}$ we also write ij; $L + ij$ denotes the graph $L \cup \{ij\}$, analogously for "$-$". Given any graph L on some set N, N splits into (maximal connected) components the set of which is denoted by $C(N, L)$; $C_i(N, L) \in C(N, L)$ denotes the component containing i. $L|_{N'}$ denotes the restriction of L to $N' \subseteq N$, $L|_{N'} := \{ij \in L | i, j \in N'\}$. Any coalition structure $\mathcal{P}$ on N induces a cooperation structure $L^{\mathcal{P}} := \bigcup_{i \in N} L^{\mathcal{P}(i)}$ on N. For $K, K' \subseteq N$, $K \cup K' = \emptyset$, we denote by $[K, K'] \subseteq L^N$ the set of all links that connect players in K with players in K'. A CO-game is a game together with a cooperation structure, (N, v, L). A CO-value is an operator φ that assigns payoff vectors $\varphi(N, v, L) \in \mathbb{R}^N$ to all CO-games (N, v, L), $N \subset \mathcal{U}$. The Myerson (1977) value μ is defined by

$$\mu(N, v, L) := \text{Sh}\left(N, v^L\right) \qquad , v^L(K) := \sum_{S \in C(K, L|_K)} v(S) \qquad , K \subseteq N. \tag{2.3}$$

3. Axioms for CO-values

In this section, we consider several axioms for CO-values with respect to outside options.

Axiom 3.1 (Component efficiency, CE). *For all $C \in C(N, L)$,*

$$\varphi_C(N, v, L) = v(C).$$

Axiom 3.2 (Fairness, F). *For all $ij \in L$, we have*

$$\varphi_i(N, v, L) - \varphi_j(N, v, L) = \varphi_i(N, v, L - ij) - \varphi_j(N, v, L - ij).$$

CE and **F** are the original axioms that characterize the Myerson value. **CE** indicates that the connected components $C \in C(N,L)$ are the productive units. The very nice fairness axiom **F** has strong consequences far beyond pure fairness considerations. In particular, van den Nouweland (1993, pp. 28) shows that μ satisfies the following axiom.

Axiom 3.3 (Component decomposability, CD). *For all* $i \in C \in C(N,L)$,

$$\varphi_i(N,v,L) = \varphi_i(C, v|_C, L|_C).$$

Hence, the payoffs within a component $C \in C(N,L)$ are not affected by the players outside, neither from their actual cooperation structure $L|_{N\backslash C}$ nor from the potential contributions of players in C to coalitions containing players from $N\backslash C$. Therefore, the Myerson value cannot account for outside options. It shares this property with the AD-value for CS-games. In fact, μ and AD coincide for completely connected components, i.e. $\mathrm{AD}(N,v,\mathcal{P}) = \mu\left(N,v,L^{\mathcal{P}}\right)$ (Myerson 1977).

Therefore, one could argue that **F** is to strong an axiom and one could think of restricting **F** to those situations were outside options are not involved: (i) Removing a link ij does not split a component, i.e. outside options do not change. (ii) ij is removed from a connected graph, i.e. from a cooperation structure which lacks outside options. This idea is captured by the following two axioms where **WF1** refers to case (i) and **WF2** to case (ii). Note that **WF2** involves games with connected graphs only. Furthermore, **WF2** may relate games with different player sets while **F** involves a fixed player set.

Axiom 3.4 (Weak fairness 1, WF1). *If* $j \in C_i(N, L-ij)$ *then*

$$\varphi_i(N,v,L) - \varphi_i(N,v,L-ij) = \varphi_j(N,v,L) - \varphi_j(N,v,L-ij).$$

Axiom 3.5 (Weak fairness 2, WF2). *If* L *is connected on* N *and* $ij \in L$ *then*

$$\varphi_i(N,v,L) - \varphi_i\left(C_i(N,L-ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right)$$
$$= \varphi_j(N,v,L) - \varphi_j\left(C_j(N,L-ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right).$$

The following Lemma gives a characterization of μ which separates two aspects of **F**, fairness, **WF2**, and neglect of outside options, **CD**.

Lemma 3.6. *μ is characterized by **CE**, **CD**, and **WF2**.*

Proof. **CD** and **F** imply **WF2**. Hence, μ satisfies **CE**, **CD**, and **WF2**. Since $(C, L|_C)$ is connected for $C \in C(N,L)$, **CD** and **WF2** together with arguments similar to those in the Myerson (1977) proof show that μ is the unique such value. □

Outside options come into play when the links in L ultimately have been formed, i.e. the players do not consider breaking links or creating new ones. Therefore, the players are not necessarily restricted to the actual cooperation structure outside their component when they bargain *within* their component on the distribution of that component's worth. Compare Maschler (1992, pp. 595) cited in the Introduction. Therefore, one could argue that the distribution of the worth within $C \in C(N, L)$ should not be affected by how the players outside C are organized. This is expressed with the following axiom. As **CD**, **CI** neglects the link structure outside C, but in contrast to **CD**, it may recognize the productive potential and the linking potential outside C.

AXIOM 3.7 (Component independence, CI). *If* $L|_{C_i(N,L)} = L'|_{C_i(N,L')}$ *then*

$$\varphi_i(N, v, L) = \varphi_i(N, v, L').$$

The χ-value for CS-games (Casajus 2009) is characterized by five axioms: additivity, component restricted symmetry, component efficiency, the restriction of the Null player axiom to the grand coalition, and the following splitting axiom which determines how outside options are evaluated.

AXIOM 3.8 (Splitting, SP). *If* $\mathcal{P}'$ *is finer than* $\mathcal{P}$ *then for all* $i, j \in P \in \mathcal{P}'$, *we have*

$$\varphi_i(N, v, \mathcal{P}) - \varphi_i(N, v, \mathcal{P}') = \varphi_j(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}').$$

Since all partitions are finer than $\{N\}$, **SP** implies (in fact, **SP** can be replaced by the following property)

$$\varphi_i(N, v, \mathcal{P}) - \varphi_j(N, v, \mathcal{P}) = \varphi_i(N, v, \{N\}) - \varphi_j(N, v, \{N\}) \tag{3.1}$$

for $i, j \in P \in \mathcal{P}$.

While the Myerson value satisfies **CD**, i.e. restricts attention to the graphs $(C, L|_C)$, $C \in C(N, L)$, we make use of the player's outside-option graphs $L(i, N)$ which generalize the transition from $\mathcal{P}$ to $\{N\}$ in (3.1). What is important about $\{N\}$ is that the players in some component $P \in \mathcal{P}$ are connected to those outside, i.e. in $N \backslash P$. Since in CS-games the components do not bear any inner structure, one necessarily ends up at $\{N\}$. Due to their richer structure, in CO-games, there is a range of reasonable alternatives to derive a connected graph $L(i, N)$ from L for $i \in C$. Of course, one would like to keep the inner structure of C, i.e. $L(i, N)|_C = L|_C$. Also, every player in C should be connected with every player in $N \backslash C$ in order to account for outside options in a symmetric way. Again for symmetry reasons, at first glance, one would guess that one is left with just two alternatives if **CI** had to be satisfied:

$$L(i, N) := \quad L|_C \cup [C, N \backslash C] \tag{3.2}$$

$$L^+(i, N) := \quad L|_C \cup [C, N \backslash C] \cup L^{N \backslash C} \tag{3.3}$$

Both graphs agree with L on C and contain all links between the players in i's component C and those in $N \backslash C$. $L(i, N)$ which we call the *lower outside-option graph* (LOOG) contains no further links. Since the players in $N\backslash C$ are completely disconnected internally, $L(i,N)|_{N\backslash C} = \emptyset$, the LOOG reflects the productive as well as the linking potential of the players in C with respect to the players in $N\backslash C$. In contrast, $L^+(i,N)$ which we call the *upper outside-option graph* (UOOG) completely connects the players in $N\backslash C$. Therefore, the UOOG neglects the linking potential. When there is no danger of confusion, we write $L(i)$ or $L^+(i)$. Note that both outside-option graphs are connected and coincide with L whenever L is connected, i.e. if there are no outside options. Further, both graphs coincide for players of the same component, respectively.

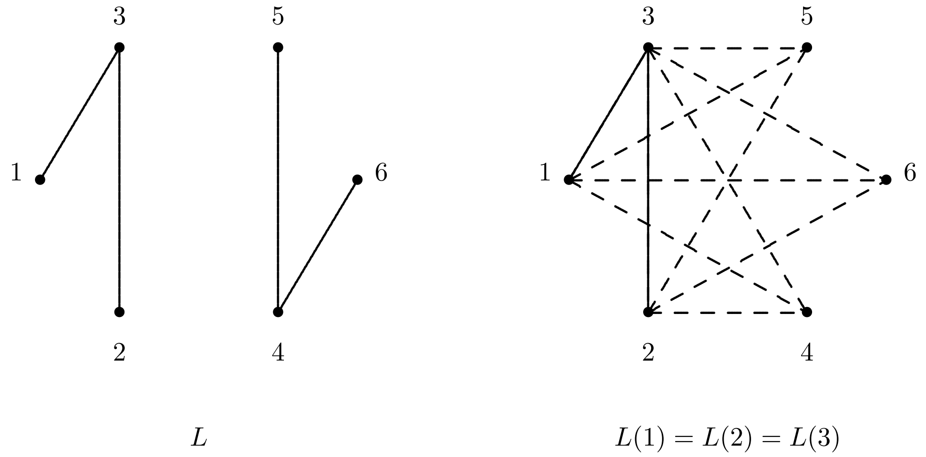


FIGURE 3.1. A lower outside option graph

Figure 3.1 provides an example of a LOOG. On the left side, we have the link set L of some graph. On the right one, the outside-option graph $L(1)$ for player 1 is given where the original links kept are drawn as solid lines while the additional links are drawn as dashed lines. The component of player 1 in L comprises the players in $\{1,2,3\}$. Player 1's outside-option graph hence does not contain the link 12 which is missing in L. Further, all original links among players outside 1's component, i.e. in the set $\{4,5,6\}$, have been removed. Finally, the players in $\{1,2,3\}$ are completely connected with those outside (dashed lines).

Since the LOOG seems to capture outside options in a broadest sense, we employ this graph in our axiomatization of the $\chi^\sharp$-value. As we will see later on (Theorem 4.6), however, it does not matter whether we employ the LOOG or the UOOG since the linking potentials of players of the same component recognized by the LOOG cancel out. Now, the idea of (3.1) can be expressed for CO-games as follows.

AXIOM 3.9 (Outside-option consistency, OO). *If* $i,j \in C \in C(N,L)$ *then*

$$\varphi_i(N,v,L) - \varphi_j(N,v,L) = \varphi_i(N,v,L(i)) - \varphi_j(N,v,L(j)).$$

In presence of **CE**, **CD** is equivalent to

$$\varphi_i(N,v,L) - \varphi_j(N,v,L) = \varphi_i(C,v|_C,L|_C) - \varphi_j(C,v|_C,L|_C)$$

holding for all $i,j \in C \in C(N,L)$ which clarifies the relation between **OO** and **CD**. It is clear that **OO** holds trivially if L is connected.

4. A χ-value for cooperation structure games

In this section, we show that some of the axioms (**CE**, **WF2**, **OO**) advocated in the previous section already characterize a CO-value which satisfies the remaining such axioms.

LEMMA 4.1. ***OO** and **WF2** imply **WF1**.*

In our axiomatization, **OO** replaces **CD**. Similar to Lemma 3.6, **WF1** then has not to be required explicitly.

PROOF. For $j \in C_i(N, L - ij)$, (3.2) implies that $L(i) - ij = (L - ij)(i)$ is connected. We then have

$$\begin{aligned}
\varphi_i(N,v,L) - \varphi_j(N,v,L) &\overset{\textbf{OO}}{=} \varphi_i(N,v,L(i)) - \varphi_j(N,v,L(i)) \\
&\overset{\textbf{WF2}}{=} \varphi_i(N,v,L(i)-ij) - \varphi_j(N,v,L(i)-ij) \\
&= \varphi_i(N,v,(L-ij)(i)) - \varphi_j(N,v,(L-ij)(i)) \\
&\overset{\textbf{OO}}{=} \varphi_i(N,v,L-ij) - \varphi_j(N,v,L-ij)
\end{aligned}$$

which proves the claim. □

LEMMA 4.2. *If φ satisfies **CE**, **WF1**, and **WF2** then it coincides with μ on all connected graphs.*

This result is in line with our intention to model outside options. Connected graphs lack outside options. Therefore, one could argue that all arguments in favor of μ apply in these situations.

PROOF. μ is characterized by **CE** and **F** where the latter strengthens **WF1** and together with **CD** implies **WF2**. We mimic the Myerson (1977) proof of uniqueness. Suppose φ and $\bar{\varphi}$ both satisfy **CE**, **WF1**, and **WF2**. Suppose N is a minimal player set such that φ and $\bar{\varphi}$ differ on a connected graph. By **CE**, N contains at least two players. Further, suppose L is a minimal connected graph on N such that they do so. If $j \in C_i(N, L-ij)$ then **WF1** and the minimality of L imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. And if $j \notin C_i(N,L-ij)$ then **WF2** and again the minimality of N imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. Since L is connected, we have $\varphi_i(N,v,L) - \bar{\varphi}_i(N,v,L) = \Delta$ for some Δ and all $i \in N$. **CE** then implies $\Delta = 0$. A contradiction. □

THEOREM 4.3. *There is a unique CO-value that satisfies* ***CE, WF2,*** *and* ***OO.***

PROOF. Suppose φ satisfies **CE**, **WF2**, and **OO**. By **OO**, we have $L(i) = L(j)$ and

$$\varphi_i(N,v,L) - \varphi_i(N,v,L(i)) = \varphi_j(N,v,L) - \varphi_j(N,v,L(i))$$

for $i,j \in C \in C(N,L)$. Summing up over C combined with **CE** gives

$$|C|\left(\varphi_i(N,v,L) - \varphi_i(N,v,L(i))\right) = v(C) - \varphi_C(N,v,L(j)).$$

Since $L(i)$ is connected, Lemmas 4.1 and 4.2 imply

$$\varphi_i(N,v,L) = \mu_i(N,v,L(i)) + \frac{v(C) - \mu_C(N,v,L(i))}{|C|}. \tag{4.1}$$

Hence, φ were unique.

By construction, the value given by (4.1) satisfies **CE**. If $C_i(N,L) = N$ then $L(i) = L(j) = L$ by (3.2), and therefore

$$\begin{aligned}
&\varphi_i(N,v,L) - \varphi_j(N,v,L) \\
&\quad = \mu_i(N,v,L) - \mu_j(N,v,L) \\
&\quad = \mu_i(N,v,L-ij) - \mu_j(N,v,L-ij) \\
&\quad = \mu_i\left(C_i(N,L-ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad\qquad -\mu_j\left(C_j(N,L-ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right) \\
&\quad = \varphi_i\left(C_i(N,L-ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\
&\qquad\qquad -\varphi_j\left(C_j(N,L-ij), v|_{C_j(N,L-ij)} L|_{C_j(N,L-ij)}\right)
\end{aligned}$$

by (4.1), μ satisfying **F**, μ satisfying **CD**, and again (4.1) together with $L|_{C_k(N,L-ij)}$ being connected on $C_k(N,L-ij)$, for $k = i,j$. Hence, φ satisfies **WF2**. If $j \in C_i(N,L)$ then $L(i) = L(j)$ by (3.2) and therefore

$$\begin{aligned}
\varphi_i(N,v,L) - \varphi_j(N,v,L) &= \mu_i(N,v,L(i)) - \mu_j(N,v,L(j)) \\
&= \varphi_i(N,v,L(i)) - \varphi_j(N,v,L(j))
\end{aligned}$$

by (4.1) and $L(i)$ being connected on N and Lemma 4.2 which shows that φ satisfies **OO**. □

The CO-value defined by (4.1) is called "the graph-χ-value" and we denote it by "$\chi^\sharp$" where the musical "sharp" symbol ♯ is intended to indicate a graph. It employs the Myerson value of the outside-option graph $L(i)$ as a yardstick to distribute the payoff *within* the component $C_i(N,L)$. The players within a component depart from their outside-option payoffs $\mu_i(N,v,L(i))$ and then compare the worth of the coalition with the sum of the outside-option payoffs; the difference, positive or negative, is distributed equally. By (4.1) and (3.2) or Lemma 4.2, $\chi^\sharp$ coincides with the Myerson value if a game lacks outside options, i.e. if the graph is connected. From (3.2) and (4.1) it is clear that $\chi^\sharp$ satisfies **CI**.

It is easy to see that the characterization of $\chi^\sharp$ is non-redundant. μ satisfies **CE** and **WF2**, but not **OO**. Further, χ applied to $C(N,L)$ satisfies **CE** and **OO**, but not **WF2**. Finally, the CO-value φ given by $\varphi_i(N,v,L) = \chi_i^\sharp(N,v,L) + 1$ obviously satisfies **WF2** and **OO**, but not **E**.

In Section 3, we also suggested the UOOG. It is easy to check that Lemma 4.1 and 4.3 remain true if one replaces **OO** by **OO**$^+$ where $L(i)$ is replaced by $L^+(i)$. Denote the resulting CO-value by $\chi^\#$ which is given by replacing the $L(j)$ in (4.1) by $L^+(j)$. Interestingly, $\chi^\sharp$ and $\chi^\#$ coincide. We show this by proving that $\chi^\sharp$ and $\chi^\#$ satisfy the following strong version of **OO**.

AXIOM 4.4 (Strong outside option consistency, SOO). *If $i,j \in C \in C(N,L)$, $L|_C = L'|_C$, and $[C, N\backslash C] \subseteq L'$ then*

$$\varphi_i(N,v,L) - \varphi_j(N,v,L) = \varphi_i(N,v,L') - \varphi_j(N,v,L').$$

The outside-option graphs L' in **SOO** all coincide with the original graph on the component C, i.e. they express the same inside options as the original graph. Further, they all reflect comprehensive and symmetric productive outside options via the links in $[C, N\backslash C]$. The difference between two such outside-option graphs lies on the link set $L^{N\backslash C}$, the links in $N\backslash C$. The more links from $L^{N\backslash C}$ the graph L' contains the less linking outside option are modelled. If $L^{N\backslash C} \subseteq L'$, i.e. $L' = L^+(i)$, then any subset of $N\backslash C$ is connected internally. Hence, L' does not reflect linking outside options. Vice versa, if $L^{N\backslash C} \cap L' = \emptyset$, i.e. $L' = L(i)$, then the players in $N\backslash C$ are connected only via players in C, i.e. L' also reflects comprehensive linking outside options. Yet, as the following Lemma and Theorem reveal, this difference cancels out since **SOO** applies the same outside-option graph to both players involved.

LEMMA 4.5. *$\chi^\sharp$ and $\chi^\#$ satisfy **SOO**.*

PROOF. Consider $j \in C_i(N,L)$ and let L' be as in **SOO**. Since L' is connected, $\chi^\sharp = \mu$ on connected graphs, and by (4.1), the claim on $\chi^\sharp$ is equivalent to

$$\mu_i(N,v,L(i)) - \mu_j(N,v,L(i)) = \mu_i(N,v,L') - \mu_j(N,v,L') \tag{4.2}$$

By (3.2) and (3.3), replacing $L(i)$ by $L^+(i)$, etc., the proof also runs through for $\chi^\#$.

Consider $\sigma, \rho \in \Sigma(N)$ such that $\sigma(i) > \sigma(j)$, $\sigma(i) = \rho(j)$, $\sigma(j) = \rho(i)$, and $\sigma(k) = \rho(k)$ for all $k \in N\backslash\{i,j\}$. In order to show (4.2), by (2.1) and (2.3) it suffices to prove

$$\begin{aligned} MC_i\left(\sigma, v^{L(i)}\right) + MC_i\left(\rho, v^{L(i)}\right) - \left(MC_j\left(\sigma, v^{L(i)}\right) + MC_j\left(\rho, v^{L(i)}\right)\right) \quad (4.3)\\ = MC_i(\sigma, v^{L'}) + MC_i(\rho, v^{L'}) - (MC_j(\sigma, v^{L'}) + MC_j(\rho, v^{L'})). \end{aligned}$$

If $K \subseteq C_i(N,L)$ then $L(i)|_K = L|_K = L'|_K$ and if $j \in K \nsubseteq C_i(N,L)$ then $C(K, L(i)|_K) = \{K\} = C(K, L'|_K)$ by (3.2) and our assumption on L'. Hence by (2.3), $v^{L(i)}(K) = v^{L'}(K)$ for all $K \subseteq N$ such that $i \in K$ or $j \in K$. By our choice of σ and ρ, this already implies

$$MC_i\left(\sigma, v^{L(i)}\right) = MC_i(\sigma, v^{L'}) \quad \text{and} \quad MC_j\left(\rho, v^{L(i)}\right) = MC_j(\rho, v^{L'}). \tag{4.4}$$

Further, we have $K_i(\rho) \setminus \{i\} = K_j(\sigma) \setminus \{j\}$ and therefore

$$\begin{aligned} MC_i\left(\rho, v^{L(i)}\right) - MC_j\left(\sigma, v^{L(i)}\right) &= v^{L(i)}(K_i(\rho)) - v^{L(i)}(K_j(\sigma)) \\ &= v^{L'}(K_i(\rho)) - v^{L'}(K_j(\sigma)) = MC_i(\rho, v^{L'}) - MC_j(\sigma, v^{L'}) \end{aligned} \tag{4.5}$$

where the second equation follows from the arguments above. Then, (4.4) and (4.5) together imply (4.3). □

By Theorem 4.3 and since **SOO** implies both **OO** and **OO**$^+$, there is a unique CO-value that satisfies **CE**, **WF2**, and **SOO**. Since $\chi^\sharp$ and $\chi^\#$ are characterized by **CE**, **WF2**, and **OO** or **OO**$^+$, respectively, by Lemma 4.5, this CO-value coincides with $\chi^\sharp = \chi^\#$.

THEOREM 4.6. $\chi^\sharp = \chi^\#$ *is the unique CO-value that satisfies* ***CE***, ***WF2***, *and* ***SOO***.

5. Properties

First, we explore the relation between the $\chi^\sharp$-value and χ-value which already indicates that $\chi^\sharp$ accounts for outside options. An example demonstrates that this property extends to cases which are not already covered by the χ-value. Further, we investigate properties of $\chi^\sharp$ for superadditive games. Finally, we compare network formation under μ and under $\chi^\sharp$ with an example.

5.1. Relation to the χ-value. If a connected component $C \in C(N,L)$ is completely connected internally, i.e. $L|_C = L^C$, then $L^+(i) = L^N$ for $i \in C$ by (3.3). Since $\mu(N, v, L^N) = \mathrm{Sh}(N,v)$ (Myerson 1977), Theorem 4.6, (4.1), and (2.2) imply the following Theorem where part (ii) is immediate from part (i). Since L^N is connected, we also have $\chi^\sharp(N, v, L^N) = \mathrm{Sh}(N,v)$. Hence, $\chi^\sharp$ generalizes the Shapley value and the χ-value, the latter justifying its name "graph-χ-value".

THEOREM 5.1. *(i) If* $i \in C \in C(N,L)$, $L|_C = L^C$, *and* $\mathcal{P}(i) = C$ *then* $\chi_i^\sharp(N,v,L) = \chi_i(N,v,\mathcal{P})$.

(ii) $\chi^\sharp(N, v, L^{\mathcal{P}}) = \chi(N, v, \mathcal{P})$.

For any CO-game (N,v,L) and $k \in \mathbb{N}$ define the CO-game (N_k, v_k, L_k) where $N_k = N \cup N_{0k}$, $N_{0k} := \{0j | j = 1, \ldots, k\}$, $L_k = L$, and $v_k(K) = v(K \cap N)$ for all $K \subseteq N_k$. I.e., (N_k, v_k, L_k) is derived from (N,v,L) by adding k Null players which are not linked to any other player.

THEOREM 5.2. $\lim_{k\to\infty}\chi_i^\sharp(N_k,v_k,L_k)=\chi_i(N,v,C(N,L))$ *for all* $i\in N$.

The intuition behind this property should be clear. The more Null players outside the original player set are present, the less the internal link structure of the connected components influences the distribution of worth within them. At the limit, all what matters is whether players are connected or not, i.e. the partition $C(N,L)$.

PROOF. In view of Theorem 4.6, (4.1) and (2.2), it suffices to show that

$$\lim_{k\to\infty}\mu_i\left(N_k,v_k,L_k^+(i)\right)=\mathrm{Sh}_i(N,v)$$

for all $i\in N$. By (3.2), $L_k^+(i)|_{N'}$ is connected if $N'\cap N_{0k}\neq\emptyset$. Further, any $\sigma_k\in\Sigma(N_k)$ induces some $\sigma_k|_N=\Sigma(N)$ such that $\sigma_k|_N(i)\geq\sigma_k|_N(j)$ iff $\sigma_k(i)\geq\sigma_k(j)$ for all $i,j\in N$. Hence for $i\in N$, we have

$$v^{L_k^+(i)}(K_i(\sigma_k))=v(K_i(\sigma_k|_N))\quad\text{and}\quad v^{L_k^+(i)}(K_i(\sigma_k)\setminus\{i\})=v(K_i(\sigma_k|_N)\setminus\{i\})\tag{5.1}$$

if $K_i(\sigma_k)\cap N_{0k}\neq\emptyset$. Further, the probability that $\sigma_k\in\Sigma(N_k)$ induces $\sigma\in\Sigma(N)$ is $|\Sigma(N)|^{-1}$ for all $k\in\mathbb{N}$. Let $\mathrm{prob}(K_i(\sigma_k)\cap N_{0k}\neq\emptyset|\sigma_k|_N=\sigma)$ denote the probability that some added Null player comes before player i in some order $\sigma_k\in\Sigma(N_k)$ conditional on inducing the order $\sigma\in\Sigma(N)$, $\sigma_k|_N=\sigma$. It is clear that

$$\lim_{k\to\infty}\mathrm{prob}(K_i(\sigma_k)\cap N_{0k}\neq\emptyset|\sigma_k|_N=\sigma)=1\tag{5.2}$$

for all $\sigma\in\Sigma(N)$ and $i\in N$. Together with (2.3) and (2.1), (5.1) and (5.2) then prove the claim. □

5.2. Outside options—an example. The formula (4.1) together with (3.2) shows that a player's payoff depends on the link structure of his component as well as his productive potential with players outside his component. This indicates that the $\chi^\sharp$-value has the potential to account for outside options. Reconsidering our leading example shows that it actually does so. It should be clear that this property extends to other situations involving outside options.

EXAMPLE 5.3. In the first situation of our leading example, the graph (1.1) is connected. By Lemma 4.2, the $\chi^\sharp$-payoffs are the same as for μ. In the second situation, we have $N=\{R,\ell,A1,A2\}$ with the productive players in $T=\{R,\ell\}$ and the graph L from (1.2). By (3.2), we then have $L(R)=L(\ell)=L(A1)=L^N\backslash R\ell$, i.e. the complete graph minus the link $R\ell$. It is easy to check that this gives the μ-payoffs $\mu_{A1}(N,u_T,L(A1))=\frac{1}{12}$, $\mu_R(N,u_T,L(R))=\mu_\ell(N,u_T,L(\ell))=\frac{5}{12}$, and then by (4.1) the $\chi^\sharp$-payoffs

$$\begin{aligned}\chi_{A1}^\sharp(N,u_T,L)&=\frac{1}{12}+\frac{1-\left(\frac{5}{12}+\frac{5}{12}+\frac{1}{12}\right)}{3}=\frac{1}{9}\\ \chi_R^\sharp(N,u_T,L)=\chi_\ell^\sharp(N,u_T,L)&=\frac{5}{12}+\frac{1-\left(\frac{5}{12}+\frac{5}{12}+\frac{1}{12}\right)}{3}=\frac{4}{9}\end{aligned}$$

as indicated in the Introduction. **CE**, of course, implies $\chi^\sharp_{A2}(N, u_T, L) = 0$. Hence, indeed, the $\chi^\sharp$-value recognizes the potential competition between the linking agents in this situation.

5.3. Link monotonicity and improvement. In this section, we explore to which extend two properties of the Myerson value concerning superadditive games, link monotonicity and the improvement property, are satisfied. These properties are of particular interest in the study of stability issues for such games which we will touch in Section 5.4. It turns out that $\chi^\sharp$ satisfies component restricted versions of these axioms only. This seems to make it much more difficult to derive general stability properties similar to those of Dutta, van den Nouweland & Tijs (1998).

For superadditive games, μ satisfies the following axiom (Myerson 1977).

Axiom 5.4 (Link monotonicity, LM). *For all* $i, j \in N$,

$$\varphi_i(N, v, L + ij) \geq \varphi_i(N, v, L).$$

The following example reveals that this may not be the case for $\chi^\sharp$ when $j \notin C_i(N, L)$.

Example 5.5. Consider the game (N, u_T), $N = \{1, 2, 3\}$, $T = \{1, 2\}$, and the graph $L = \emptyset$. It is easy to check that we then have $\chi^\sharp_3(N, u_T, L) = 0$ but $\chi^\sharp_3(N, u_T, L + 23) = -\frac{1}{4}$.

However, $\chi^\sharp$ satisfies the following component restricted version of **LM** for superadditive games.

Theorem 5.6 (Component restricted link monotonicity, CLM). *If (N, v) is superadditive then $\chi^\sharp$ satisfies the following axiom: For all $i, j \in C \in C(N, L)$,*

$$\varphi_i(N, v, L + ij) \geq \varphi_i(N, v, L).$$

The proof is prepared by a Lemma on the Myerson value and a Corollary. Obviously, the axiom in Lemma 5.8 strengthens the following axiom which thus and also by Slikker (2000) is satisfied by μ.

Axiom 5.7 (Improvement property, IP). *If (N, v) is superadditive then $\varphi_k(N, v, L+ij) > \varphi_k(N, v, L)$ implies $\varphi_i(N, v, L + ij) > \varphi_i(N, v, L)$ or $\varphi_j(N, v, L + ij) > \varphi_j(N, v, L)$ for all $i, j \in N$ and $k \in N \setminus \{i, j\}$.*

Lemma 5.8 (Strong improvement, SI). *μ satisfies the following axiom: If (N, v) is superadditive then*

$$\varphi_i(N, v, L + ij) - \varphi_i(N, v, L) \geq \varphi_k(N, v, L + ij) - \varphi_k(N, v, L)$$

for all $i, j, k \in N$.

Proof. Let (N,v) be superadditive. Consider some i,j,k and the orders σ and ρ on N, $\sigma(i)=\rho(k)>\sigma(k)=\sigma(i)$, and $\sigma(\ell)=\rho(\ell)$ for $\ell\in N\backslash\{i,k\}$. By (2.3) and the superadditivity of (N,v), we have

$$MC_i\left(\rho,v^{L+ij}\right)-MC_i\left(\rho,v^L\right)\geq 0=MC_k\left(\sigma,v^{L+ij}\right)-MC_k\left(\sigma,v^L\right). \tag{5.3}$$

Further, we have

$$MC_i\left(\sigma,v^{L+ij}\right)-MC_i\left(\sigma,v^L\right)=v^{L+ij}\left(K_i\left(\sigma\right)\right)-v^L\left(K_i\left(\sigma\right)\right)$$

since $i\notin S$ implies $v^{L+ij}(S)=v^L(S)$. Hence by $K_i(\sigma)=K_k(\rho)$, we have

$$\begin{aligned}MC_k\left(\rho,v^{L+ij}\right)-MC_k\left(\rho,v^L\right) &= MC_i\left(\sigma,v^{L+ij}\right)-MC_i\left(\sigma,v^L\right)\\ &\quad +v^L\left(K_i\left(\sigma\right)\backslash k\right)-v^{L+ij}\left(K_i\left(\sigma\right)\backslash k\right).\end{aligned}$$

Since $C\left(K_i(\sigma)\backslash k, L|_{K_i(\sigma)\backslash k}\right)$ is finer than $C\left(K_i(\sigma)\backslash k, L+ij|_{K_i(\sigma)\backslash k}\right)$, the superadditivity of (N,v) and (2.3) imply $v^L\left(K_i(\sigma)\backslash k\right)\leq v^{L+ij}\left(K_i(\sigma)\backslash k\right)$, i.e.

$$MC_i\left(\sigma,v^{L+ij}\right)-MC_i\left(\sigma,v^L\right)\geq MC_k\left(\rho,v^{L+ij}\right)-MC_k\left(\rho,v^L\right). \tag{5.4}$$

In view of (2.3) and (2.1), (5.3) and (5.4) together then prove the claim. □

The following example shows that—as one would expect—$\chi^\sharp$ neither satisfies **SI** nor **IP**.

Example 5.9. Consider the CO-game (N,u_T,L), $N=\{1,2,\ldots,8\}$, $T=\{1,6,7\}$, and $L=\{12,23,34,45,56,78\}$. Simple but tedious calculations (or a bit of thinking) show

$$\begin{aligned}\chi_6^\sharp(N,v,L+67)-\chi_6^\sharp(N,v,L) &= \frac{1}{7}-\frac{2}{9}=-\frac{5}{63}<0\\ \chi_7^\sharp(N,v,L+67)-\chi_7^\sharp(N,v,L) &= \frac{1}{7}-\frac{1}{6}=-\frac{1}{42}<0\\ \chi_8^\sharp(N,v,L+67)-\chi_8^\sharp(N,v,L) &= 0-\left(-\frac{1}{6}\right)=\frac{1}{6}>0\end{aligned}$$

Since u_T is superadditive, **SI** is violated.

Again, $\chi^\sharp$ satisfies a component restricted version of **SI**. By (3.2), $j,k\in C_i(N,L)$ implies $L+ij(i)=L(i)+ij=L(k)+ij$. Lemma 5.8 together with (4.1) then implies the following Corollary.

Corollary 5.10 (Component restricted strong improvement, CSI). *For superadditive games, $\chi^\sharp$ satisfies the following axiom: For all $j,k\in C_i(N,L)$, we have*

$$\varphi_i(N,v,L+ij)-\varphi_i(N,v,L)\geq\varphi_k(N,v,L+ij)-\varphi_k(N,v,L).$$

Proof. (Theorem 5.6) Let (v, N) be superadditive and $j \in C_i(N, L)$. Suppose $\chi_i^\sharp(N, v, L + ij) < \chi_i^\sharp(N, v, L)$. Since $C_i(N, L) = C_i(N, L + ij)$ and by Corollary 5.10, we then had

$$0 \overset{\mathbf{CE}}{=} \sum_{k \in C_i(N,L)} \left(\chi_k^\sharp(N, v, L + ij) - \chi_k^\sharp(N, v, L)\right) < 0.$$

A contradiction. □

5.4. Stable networks. Dutta et al. (1998) study network formation in superadditive TU games by the following network formation game (φ-NFG) which was formally introduced by Myerson (1991, p. 448). Given a TU game (N, v) and a CO-value φ, we consider the strategic form game Γ^φ. The player set is N and player $i \in N$ has the strategy set $S_i = \{s_i | s_i \subseteq N \setminus \{i\}\}$. Any strategy profile $s = (s_i)_{i \in N} \in S := \prod_{i \in N} S_i$ induces a graph $L(s) := \{ij \in L^N | i \in s_j \wedge j \in s_i\}$. The players' payoffs are given by φ, i.e. $u_i^\varphi(s) := \varphi_i(N, v, L(s))$. Dutta et al. (1998) consider a class of CO-values including μ and then apply some solution concepts to theses games: the Nash equilibrium, the undominated Nash equilibrium (UNE), and the coalition-proof Nash equilibrium (CPNE). In order to illustrate the difference between the Myerson value and the $\chi^\sharp$-value, we focus on the Nash equilibrium and the CPNE.

Bernheim, Peleg & Whinston (1987) define the CPNE inductively: For all $T \subseteq N$ and $s_{N \setminus T} \in S_{N \setminus T} := \prod_{i \in N \setminus T} S_i$, the game $\Gamma^\varphi(s_{N \setminus T})$ consists of the players set T, the strategy sets S_i, $i \in T$, and the payoff functions $u_i^\varphi[s_{N \setminus T}]$, $i \in T$ where $u_i^\varphi[s_{N \setminus T}](s_T) = u_i^\varphi(s_T, s_{N \setminus T})$ for all $i \in T$ and $s_T \in \prod_T S_i$. For $|N| = 1$, a strategy profile $s^* \in S$ is a CPNE if $u_i^\varphi(s_i^*)$ maximizes u_i^φ over S. For $|N| > 1$, a strategy profile $s^* \in S$ is called *self-enforcing* if for all $T \subsetneq N$, s_T^* is a CPNE of $\Gamma^\varphi(s_{N \setminus T}^*)$. A strategy profile s^* is a CPNE if it is self-enforcing and if there is no self-enforcing strategy profile $s \in S$ such that $u_i^\varphi(s) > u_i^\varphi(s^*)$ for all $i \in N$.

Dutta et al. (1998, Proposition 1) show that any network can be supported by a Nash equilibrium of the μ-NFG. This may not be the case in the $\chi^\sharp$-NFG. In Example 5.5, the Null player 3 can avoid the negative payoff under the graph $\{23\}$, by playing $s_3 = \emptyset$ which results in $C_3(N, L(s)) = \{3\}$ and $\chi_3^\sharp(N, u_{\{1,2\}}, L(s)) = 0$. Hence in the $\chi^\sharp$-NFG, the Nash equilibrium already allows for some useful predictions about which networks will prevail. This also indicates that it is not too odd for a Null player to obtain a negative payoff. As with the χ-value (Casajus 2009) and for the same reason, this does not speak against our concept. A negative $\chi^\sharp$-payoff for a Null player under some cooperation structure simply means that this cooperation structure never will evolve, i.e. it is not stable in any reasonable sense.

Stronger solution concepts, UNE and CPNE, yield more clear cut general results. In particular, Dutta et al. (1998, Theorems 1 and 2) show that the complete network may

L	$\mu(N,v,L)$	$\chi^\sharp(N,v,L)$
$\emptyset$	$(0,0,0)$	$(0,0,0)$
$\{12\}$	$(18,18,0)$	$(24,12,0)$
$\{13\}$	$(12,0,12)$	$(21,0,3)$
$\{23\}$	$(0,0,0)$	$(0,3,-3)$
$\{12,13\}$	$(22,10,4)$	$(22,10,4)$
$\{12,23\}$	$(18,18,0)$	$(18,18,0)$
$\{13,23\}$	$(16,4,16)$	$(16,4,16)$
L^N	$(22,10,4)$	$(22,10,4)$

TABLE 5.1. Payoffs for the example

arise from a UNE or a CPNE of the μ-NFG. Moreover, any UNE or CPNE of the μ-NFG leads to the same payoffs as the complete network. The latter may not be the case in the $\chi^\sharp$-NFG.

The following example illuminates the difference between the Myerson value and the $\chi^\sharp$-value concerning network formation. Consider the TU game with player set $N = \{1,2,3\}$ and the coalition function v given by

$$v(S) = \begin{cases} 0 & , S = \{2,3\} \vee |S| < 2, \\ 24 & , S = \{1,3\} \\ 36 & , S = \{1,2\}, N. \end{cases} \tag{5.5}$$

It is easy to check that this game is superadditive but not convex. Straightforward calculations give the payoffs listed in Table 5.1. For connected networks (the bottom four rows) the μ-payoffs and the $\chi^\sharp$-payoffs coincide since there are no outside options. The second to fourth row show that $\chi^\sharp$ accounts for outside option while μ does not so. The Myerson value splits the payoffs of any two-player coalition equally among its members. Yet, if just player 1 and 2 formed a link, for example, the $\chi^\sharp$-value rewards player 1's outside option to create the worth of 24 together with player 3—player 1 obtains a much higher payoff than player 2, $\chi_1^\sharp = 24 > 12 = \chi_2^\sharp$. Similar for the other one-link networks. This makes a big difference concerning stability issues.

By Dutta et al. (1998, Theorem 2), the complete network L^N can be supported by the CPNE of the μ-NFG where all players wish to form all links. In our example, one easily checks that this is the unique such network. In the $\chi^\sharp$-NFG, besides the complete network, however, the network $\{12\}$ is stable in this sense. Note that the resulting partition $\{\{1,2\},\{3\}\}$ is the unique χ-stable (Casajus 2009) coalition structure.

In the following, we frequently refer to the $\chi^\sharp$-payoffs in Table 5.1 without mentioning this explicitly. The network $\{12\}$ can be supported by the CPNE $s^* = (\{2\},\{1\},\emptyset)$. Obviously, there are no profitable one-player deviations. Thus, (s_1^*, s_2^*), (s_1^*, s_3^*), and (s_2^*, s_3^*) are self-enforcing in $\Gamma^{\chi^\sharp}(s_3^*)$, $\Gamma^{\chi^\sharp}(s_2^*)$, and $\Gamma^{\chi^\sharp}(s_1^*)$, respectively. In $\Gamma^{\chi^\sharp}(s_1^*)$, there are two other self-enforcing strategy profiles involving the strategies $s_2 = \{1,3\}$ and $s_3 = \{2\}$ or $s_3 = \{2,3\}$. The resulting networks are $\{12\}$ and $\{12,23\}$, respectively, which both result in a zero payoff for player 3. Hence, (s_2^*, s_3^*) is a CPNE in $\Gamma^{\chi^\sharp}(s_1^*)$. Moreover, in $\Gamma^{\chi^\sharp}(s_2^*)$, just the links 12 and 13 can be formed. Since player 1 strictly prefers the network $\{12\}$ and since he can enforce it, (s_1^*, s_3^*) is self-enforcing and any other self-enforcing strategy profile also gives the network $\{12\}$. Hence, (s_1^*, s_3^*) a CPNE in $\Gamma^{\chi^\sharp}(s_2^*)$. In $\Gamma^{\chi^\sharp}(s_3^*)$, the players 1 and 2 just can form the link 12 or not but both prefer to do so. Therefore, (s_1^*, s_2^*) is self-enforcing and all self-enforcing strategy profiles lead to the network $\{12\}$. Hence, (s_1^*, s_2^*) is a CPNE in $\Gamma^{\chi^\sharp}(s_3^*)$. Since player 1 strictly prefers the graph $\{12\}$ over all other graphs, s^* is a CPNE of the $\chi^\sharp$-NFG.

One important thing about s^* is that player 3 does not wish to form a link with player 2. At first glance, this seems to be odd. But since player 1 does not wish to form a link with player 3, there is—in principle—the possibility that player 2 just wishes to form a link with player 3. In this case, player 3 prefers to be isolated and to obtain a zero payoff since under the network $\{23\}$ his payoff were negative. Moreover, player 3 does not gain by forming the link 23. Hence, if there were (even very small) costs for establishing links as studied by Slikker & van den Nouweland (2000), then player 3 would prefer not to form this link.

However, the fact that players 1 and 2 both gain by deviating from the complete network does not prevent it from being supported by a CPNE. Let $\bar{s}$ denote the unique strategy profile that creates L^N. Obviously, there are no profitable one-player deviations. Thus, $(\bar{s}_1, \bar{s}_2)$, $(\bar{s}_1, \bar{s}_3)$, and $(\bar{s}_2, \bar{s}_3)$ are self-enforcing in $\Gamma^{\chi^\sharp}(\bar{s}_3)$, $\Gamma^{\chi^\sharp}(\bar{s}_2)$, and $\Gamma^{\chi^\sharp}(\bar{s}_1)$, respectively. Moreover, in $\Gamma^{\chi^\sharp}(\bar{s}_3)$, there is no other such strategy profile. In particular, player 2 can profitably deviate from $(s_1, s_2) = (\{2\},\{1\})$ by choosing $\bar{s}_2$. Therefore, $(\bar{s}_1, \bar{s}_2)$ is a CPNE in $\Gamma^{\chi^\sharp}(\bar{s}_3)$. In $\Gamma^{\chi^\sharp}(\bar{s}_2)$, there is one other self-enforceable strategy combination, $(s_1, s_3) = (\{2\},\{2\})$, but which is dominated by $(\bar{s}_1, \bar{s}_3)$. Hence, $(\bar{s}_1, \bar{s}_3)$ is CPNE in $\Gamma^{\chi^\sharp}(\bar{s}_2)$. In $\Gamma^{\chi^\sharp}(\bar{s}_1)$, again, there is one other self-enforceable strategy combination, $(s_2, s_3) = (\{1\},\{1\})$, but which gives the same payoffs as $(\bar{s}_1, \bar{s}_3)$. Hence, $(\bar{s}_2, \bar{s}_3)$ is CPNE in $\Gamma^{\chi^\sharp}(\bar{s}_1)$. Since there is no other network where all players gain, $\bar{s}$ is a CPNE.

6. Conclusion

In this paper, we introduced and advocated a CO-value, $\chi^\sharp$, which combines the ideas underlying the Myerson value and the χ-value. In contrast to the Myerson value, this value accounts for the outside options of the players. This way, $\chi^\sharp$ may recognize e.g. the potential competition between linking agents. In Section 5.4, we have demonstrated that network formation under the $\chi^\sharp$-value and under the Myerson value, respectively, may lead to different networks. Moreover, this difference may be related to χ-stability. Hence, further research on stability under the $\chi^\sharp$-value, both in general and in specific applications, and on their relation to χ-stability and to stability under the Myerson value seems to be worthwhile.

The Myerson value was extended by van den Nouweland, Borm & Tijs (1992) to the class of TU games with a conference structure (hypergraph on the player set) (henceforth CF-games and CF-value) which we will call the Myerson CF-value. Since the characterization of the Myerson CF-value is analogous to that of the Myerson value, slightly adapting the arguments of this paper and of van den Nouweland et al. (1992), it should be hardly more than a five-finger exercise to extend our CO-value into a CF-value with analogous properties.

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CHAPTER VII

An efficient value for TU games with a cooperation structure

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Abstract

In this note, we introduce and characterize an efficient value for TU games with a cooperation structure which generalizes the Owen (1977) value for games with a coalition structure but which does not deviate too much from the Myerson (1977) value.

Key Words: Efficiency, Consistency, Graph, Owen value, Myerson value

JEL classification: C71

1. Introduction

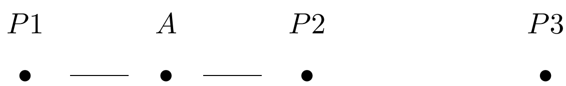
Consider the TU game with the player set $N = \{P1, P2, P3, A\}$ and the coalition function given by

$$v(K) = \begin{cases} 1 & , |K \cap \{P1, P2, P3\}| > 1, \\ 0 & , |K \cap \{P1, P2, P3\}| \leq 1, \end{cases} \qquad , K \subseteq N.$$

A is a Null player and the presence of any two of the productive players $P1, P2$, and $P3$ already suffices to produce the worth of 1. Suppose all these players cooperate in order to create the grand coalition's worth of $v(N) = 1$. If the players do not form any coalitions when bargaining on the distribution of $v(N)$, then, for symmetry reasons, one would expect an equal split between the three productive players. Would/should this split change if $P1$ and $P2$ formed a bargaining bloc? What if these players could form this bloc only via the Null player A?

As an answer to questions like the first one, Owen (1977) introduces and axiomatizes an efficient value for games with a coalition structure (partition of the player set). Hart & Kurz (1983, 1984) provide alternative axiomatizations and explore stability issues with respect to the Owen value. In our leading example, the Owen value assigns the payoff $\frac{1}{2}$ to $P1$ and to $P2$ while $P3$ and A get nothing. Since the players $P1$ and $P2$ already produce the grand coalition's worth and since they bargain as one person as well as for symmetry reasons, this fits nicely with our intuitions.

Yet, the Owen value may not give an adequate answer to the second type of questions. If $P1$ and $P2$ need A in order to form a bargaining bloc then one could argue that—despite being a Null player—A should obtain a positive payoff. However, adding A to the bloc formed by $P1$ and $P2$ does not affect the Owen payoffs. One reason for this is that coalition structures are too coarse structures. From the coalition $\{P1, P2, A\}$ alone one cannot infer whether A is necessary to connect the productive players $P1$ and $P2$ or not. The necessity of A can be modelled by the undirected graph



(1.1)

where $P1$ and $P2$ are connected but only via a chain of links involving A. Of course, this transcends the world of coalition structures and leads into the realm of cooperation structures (undirected graphs).

Generalizing the Shapley (1953) value for TU games and the Aumann & Drèze (1974) value for TU games with a coalition structure, Myerson (1977) introduced a value for TU games with a cooperation structure (henceforth CO-games and CO-value). As an alternative, Meessen (1988) suggests the position value for CO-games which was popularized by Borm et al. (1992). Yet another CO-value has been introduced by Hamiache (1999) which was discussed by Bilbao, Jiménez & López (2006). All these CO-values have in common that they are component efficient. In contrast to efficiency, this corresponds to the interpretation of connected components as productive units. In the following, we focus on the Myerson value as the most eminent one of these CO-values.

Since in our leading example the connected component $\{P1, P2, A\}$ already produces $v(N)$, the Myerson payoffs for the graph in (1.1) actually are efficient, but this is rather accidental. Just increase $v(N)$ by a small amount. Moreover, for the empty graph, the Myerson payoffs vanish due to component efficiency. Hence one would like to have an efficient CO-value which recognizes, for example, the coordinating role of player A in the situation above.

This is what this paper aims at. We introduce and axiomatize a CO-value that generalizes the Owen value to the class of CO-games and which, in a sense, does not deviate too much from the Myerson value. More specific, our CO-value coincides with the Owen value for completely connected components and coincides with the Myerson value for connected graphs. For the graph (1.1) in our leading example, that CO-value assigns the payoffs $\varphi_{P1} = \varphi_{P2} = \frac{5}{12}$, $\varphi_A = \frac{1}{6}$, and $\varphi_{P3} = 0$ which meet our intuitions concerning player A.

The axiomatization involves four axioms. Besides efficiency, we require our CO-value to assign the same payoffs for the complete graph as for the empty graph. Further, merging connected components into single players should not affect the component's payoffs. Finally, we modify the Myerson fairness axiom such that the number of components involved is not affected by removing a link. Yet, the player set involved may shrink.

The plan of this paper is as follows: Basic definitions and notation are given in second section. In the third section, we discuss some axioms related to CO-values. Our CO-value is introduced and axiomatized in the fourth section. The fifth section explores the relation of our CO-value to the Myerson value and to the Owen value as well as consistency properties, and touches stability issues. A few remarks conclude the paper.

2. Basic definitions and notation

In order to avoid set theoretic complications, we assume that there is a large enough set $\mathcal{U}$ that contains the names of the players. A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players $N \subset \mathcal{U}$ and a coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. In general, we consider the set of all TU games, possibly equipped with some additional structure. $v(K)$ is called the worth of $K \subseteq N$; subsets of N are called coalitions. For $\emptyset \neq T \subseteq N$, the game (N, u_T), $u_T(K) = 1$ if $T \subseteq K$ and $u_T(K) = 0$ otherwise, is called a unanimity game. The sum $v + v'$ of two coalition functions on N is given by $(v + v')(K) = v(K) + v'(K)$ for all $K \subseteq N$; $v|_{N'}$ denotes the restriction of v to $N' \subseteq N$. A game is called superadditive iff $v(K \cup K') \geq v(K) + v(K')$ for all $K, K' \subseteq N$, $K \cap K' = \emptyset$.

A value is an operator φ that assigns payoff vectors $\varphi(N, v) \in \mathbb{R}^N$ to all games (N, v), $N \subset \mathcal{U}$. An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted by $K_i(\sigma) = \{j \in N : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is defined by

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma) \quad , i \in N. \tag{2.1}$$

For $K \subseteq N$, we denote by $\varphi_K(N, v, \cdot)$ the sum $\sum_{i \in K} \varphi_i(N, v, \cdot)$.

A coalition structure for (N, v) is a partition $\mathcal{P} \subseteq 2^N$ where $\mathcal{P}(i)$ denotes the cell containing player i. We denote by $\langle S \rangle$, $S \subseteq N$ the atomistic partition on S, $\langle S \rangle := \{\{i\} | i \in S\}$. By $\mathcal{P}|_{N'}$ we denote the restriction of the partition $\mathcal{P}$ on N to $N' \subseteq N$, $\mathcal{P}|_{N'} := \{\mathcal{P}(i) \cap N' | i \in N'\}$. A CS-game is a game together with a coalition structure, $(N, v, \mathcal{P})$. A CS-value is an operator φ that assigns payoff vectors $\varphi(N, v, \mathcal{P}) \in \mathbb{R}^N$ to all CS-games $(N, v, \mathcal{P})$, $N \subset \mathcal{U}$. For any coalition structure $\mathcal{P}$ on N, we define a subset

$$\Sigma(N, \mathcal{P}) := \{\sigma \in \Sigma(N) | \forall i, j \in \mathcal{P}(i) : |\sigma(i) - \sigma(j)| < |\mathcal{P}(i)|\} \tag{2.2}$$

of $\Sigma(N)$. The Owen (1977) value is given by

$$\mathrm{Ow}_i(N, v, \mathcal{P}) := |\Sigma(N, \mathcal{P})|^{-1} \sum_{\sigma \in \Sigma(N, \mathcal{P})} MC_i^v(\sigma) \quad , i \in N. \tag{2.3}$$

Any $\sigma \in \Sigma(N, \mathcal{P})$ uniquely determines some $\sigma|_{\mathcal{P}} \in \Sigma(\mathcal{P})$ and $\sigma|_P \in \Sigma(P)$, $P \in \mathcal{P}$ such that $\sigma|_{\mathcal{P}}(\mathcal{P}(i)) < \sigma|_{\mathcal{P}}(\mathcal{P}(j))$ iff $\sigma(i) < \sigma(j)$ for all $i, j \in N$ or $\sigma|_P(i) < \sigma|_P(j)$ iff $\sigma(i) < \sigma(j)$ for all $i, j \in P$, respectively. For $\sigma_i \in \Sigma(\mathcal{P}(i))$ and $\rho \in \Sigma(\mathcal{P})$, we set

$$\Sigma(N, \mathcal{P}, \sigma_i, \rho) := \left\{\sigma \in \Sigma(N, \mathcal{P}) | \sigma|_{\mathcal{P}} = \rho \wedge \sigma|_{\mathcal{P}(i)} = \sigma_i\right\}. \tag{2.4}$$

A cooperation structure for (N, v) is an undirected graph (N, L), $L \subseteq L^N := \{\{i, j\} | i, j \in N, i \neq j\}$. A typical element of L is written as ij. Given any graph (N, L), N splits into (maximal connected) components the set of which is denoted by $C(N, L)$; $C_i(N, L) \in C(N, L)$ denotes the component containing $i \in N$. $L|_{N'} = \{\{i, j\} \in L | i, j \in N'\}$ denotes the restriction of L to $N' \subseteq N$. For any partition $\mathcal{P} \subseteq 2^N$, $L^{\mathcal{P}}$ denotes the graph $\bigcup_{P \in \mathcal{P}} L^P$ which splits in the completely connected components in $C\left(N, L^{\mathcal{P}}\right) = \mathcal{P}$.

A CO-game is a game together with a cooperation structure. A CO-value is an operator φ that assigns payoff vectors $\varphi(N, v, L) \in \mathbb{R}^N$ to all CO-games (N, v, L), $N \subset \mathcal{U}$. The

Myerson (1977) value μ is defined by

$$\mu(N,v,L) := \mathrm{Sh}\left(N,v^L\right) \qquad, v^L(K) := \sum_{S\in C(K,L|_K)} v(S) \qquad, K\subseteq N. \tag{2.5}$$

3. Axioms for CO-values

In this section, we consider several axioms for CO-values with respect to bargaining within the grand coalition.

Axiom 3.1 (Additivity, A). $\varphi(N,v+v',L) = \varphi(N,v,L) + \varphi(N,v',L)$.

From a mathematical viewpoint, additivity is nice axiom which is satisfied by quite a lot of values for TU games with or without additional structures and which is part of many axiomatizations. Nevertheless, additivity does not reflect any fairness considerations and therefore one may wish to avoid explicit reference to this property.

Axiom 3.2 (Efficiency, E). $\varphi_N(N,v,L) = v(N)$.

We feel that the efficiency axiom presupposes the grand coalition to be the productive unit which creates its worth $v(N)$. This corresponds to the interpretation of the connected components of L as bargaining blocs which are formed via bilateral agreements or communication channels.

Axiom 3.3 (Component efficiency, CE). *For all $C\in C(N,L)$, we have*

$$\varphi_C(N,v,L) = v(C).$$

Component efficiency evokes another interpretation of the graph L. In order to cooperate in the production of worth, players have to be connected via a chain of links. Hence, the connected components C of L are the productive units which produce their worth $v(C)$.

Axiom 3.4 (Fairness, F). *For all $ij\in L$, we have*

$$\varphi_i(N,v,L) - \varphi_i(N,v,L-ij) = \varphi_j(N,v,L) - \varphi_j(N,v,L-ij).$$

CE and **F** are the original axioms that characterize the Myerson value. The very nice fairness axiom **F** has strong consequences far beyond pure fairness considerations. In particular, van den Nouweland (1993, pp. 28) shows that μ satisfies the following axiom which says that (distribution of) payoffs within a component is not affected by the players outside. In general, of course, **CD** and **E** are incompatible.

Axiom 3.5 (Component decomposability, CD). *For all $i\in N$,*

$$\varphi_i(N,v,L) = \varphi_i\left(C_i(N,L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right).$$

AXIOM 3.6 (Equivalence, Q). $\varphi\left(N, v, L^N\right) = \varphi\left(N, v, \emptyset\right).$

This axiom says that—from the bargaining viewpoint—it does not make a difference whether the players do not form any (bargaining) components ($L = \emptyset$) or they form just one such component where all players are completely connected ($L = L^N$). Note that the Owen value has a similar property: The Owen payoffs for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N \rangle$ coincide. We feel that **Q** as a natural generalization of that property should be satisfied by an efficient CO-value.

AXIOM 3.7 (Modified fairness, MF). *For all* $ij \in L$,

$$\varphi_i\left(N, v, L\right) - \varphi_i\left(N_i\left(L, ij\right), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right)$$
$$= \varphi_j\left(N, v, L\right) - \varphi_j\left(N_j\left(L, ij\right), v|_{N_j(L,ij)} L|_{N_j(L,ij)}\right)$$

where

$$N_i\left(L, ij\right) := N \backslash \left(C_i\left(N, L\right) \backslash C_i\left(N, L - ij\right)\right). \tag{3.1}$$

MF is intended to replace the fairness axiom **F**. It is trivially satisfied if $C_i\left(N, L\right)$ does not split by removing the link ij since then $N_i\left(L, ij\right) = N$. Otherwise, $C_i\left(N, L\right)$ splits into two disjoint components. In this case, $N_i\left(L, ij\right) = N \backslash C_j\left(N, L - ij\right)$, i.e. the players in j's component are removed from N. Hence, and this seems to be one important thing about **MF**, all graphs involved have the same number of connected components while the number of players may differ. We feel that this modification of **F** fits nicely with the interpretation of the graph L as a device to model structured bargaining blocs. Compare this with **F**. There, the player set involved is fixed at N but removing a link may increase the number of components. Note also the role of **MF** in the proof of the consistency property of the CO-value to be introduced in Theorem 5.5. Further, compare the player set in (3.1) with those in (5.2) and (5.4).

AXIOM 3.8 (Component merging, CM). *For all* $C \in C\left(N, L\right)$, *we have*

$$\varphi_C\left(N, v, L\right) = \varphi_C\left(C\left(N, L\right), v \circ \cup, \emptyset\right)$$

where $v \circ \cup (K) = v\left(\bigcup_{S \in K} S\right)$ *for all* $K \subseteq C\left(N, L\right).$

CM says the distribution of worth among the components depends only on the game between coalitions, $\left(C\left(N, L\right), v \circ \cup\right)$, which are completely disconnected. This could be paraphrased as that merging all connected components into single players does not affect the component's payoffs. I.e. the inner structure of the components does not matter in this respect. What matters is just the fact that they are connected. Note that **CM** is very similar to Owen's (1977) axiom A3.

Of course, instead of **CM** one could think of a more graph-related axiom which requires the component's payoffs not to be affected by inflating links, i.e. by merging directly connected players i and j, i.e. $ij \in L$, removing the resulting loop at ij, and identifying parallel links. Yet, this would imply **CM** by successively merging links. The other way round, inflating links is equivalent to **CM** if one assumes invariance under the renaming of players.

4. A generalization of the Owen value

In this section, we show that some of the axioms advocated in the previous section, in particular **E**, **Q**, **MF**, and **CM**, already characterize a CO-value which satisfies the remaining such axioms. Further, the non-redundancy of our axiomatization is established.

4.1. Uniqueness. We first consider connected graphs, i.e. all players are contained in one bargaining bloc. In this case, one could argue that the distribution of the grand coalition's worth should be governed by the inner structure of that single bloc and the fairness considerations embodied in the Myerson value. Yet, this is already implied by **E** and **MF**.

LEMMA 4.1. *If a CO-value φ satisfies **E** and **MF** then it coincides with μ on all connected graphs.*

PROOF. We first note that for connected graphs **MF** involves connected graphs only. μ satisfies **CE** which for connected graphs becomes **E**. We also have

$$\begin{aligned}
&\mu_i(N,v,L) - \mu_j(N,v,L) \\
&\overset{\mathbf{CD}}{=} \mu_i\left(C_i(N,L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L), v|_{C_j(N,L)}, L|_{C_j(N,L)}\right) \\
&= \mu_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \mu_j\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right)
\end{aligned}$$

for $ij \in L$ where the second equation follows from $C_i(N,L) = C_i(N_i(L,ij), L|_{N_i(L,ij)})$ and the analogon for j. Hence, μ satisfies **MF**.

We mimic the Myerson (1977) proof of uniqueness. Suppose φ and $\bar{\varphi}$ both satisfy **E** and **MF**. Suppose N is a minimal player set such that φ and $\bar{\varphi}$ differ on a connected graph. Further, suppose L is a minimal connected graph on N such that they do so. By **CE**, L contains at least one edge. If $j \in C_i(N, L - ij)$ then **MF** and the minimality of L imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. And if $j \notin C_i(N, L - ij)$ then again **MF** and the minimality of N imply $\varphi_i(N,v,L) - \varphi_j(N,v,L) = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)$. Since L is connected, we have $\varphi_i(N,v,L) - \bar{\varphi}_i(N,v,L) = \Delta$ for some Δ and all $i \in N$. **E** then implies $\Delta = 0$. Contradiction. □

Applying this Lemma and again the Myerson technique, we are now able to approach the general case.

THEOREM 4.2. *There is at most one CO-value that satisfies **E**, **Q**, **MF**, and **CM**.*

In view of their role in the proof below, one could of course merge **Q** and **CM** into a single axiom. However, we feel that the two axioms refer to essentially different considerations. While **Q** basically is as a very weak expression of invariance with respect to the renaming of players, **CM** requires the payoff of the components to be independent of their inner structure.

PROOF. Let φ be a CO-value that satisfies **E**, **Q**, **MF**, and **CM**. By **CM** and **Q**, we have

$$\varphi_C(N,v,L) = \varphi_C\left(C(N,L), v \circ \cup, L^{C(N,L)}\right)$$

for all $C \in C(N,L)$. Since $\left(C(N,L), L^{C(N,L)}\right)$ is connected, Lemma 4.1 implies

$$\varphi_C(N,v,L) = \mu_C\left(C(N,L), v \circ \cup, L^{C(N,L)}\right). \tag{4.1}$$

Again, we mimic the Myerson (1977) proof of uniqueness. Suppose there were two CO-values, φ and $\bar{\varphi}$, that satisfy **E**, **Q**, **MF**, and **CM**. Let N be a minimal player set such that $\varphi \neq \bar{\varphi}$ and let L be a minimal graph on N such that they do so. By **E**, N then contains more than one player, and by **Q** and Lemma 4.1, L contains at least one link. If $C_i(N,L) = \{i\}$, then $\varphi_i(N,v,L) = \bar{\varphi}_i(N,v,L)$ by (4.1). For $ij \in L|_{C_i(N,L)}$, we have

$$\begin{aligned}
&\varphi_i(N,v,L) - \varphi_j(N,v,L) \\
&\quad = \varphi_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \varphi_j\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \\
&\quad = \bar{\varphi}_i\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \bar{\varphi}_j\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \\
&\quad = \bar{\varphi}_i(N,v,L) - \bar{\varphi}_j(N,v,L)
\end{aligned}$$

by **MF**, the minimality of N and L, and again **MF**. Thus, we have $\varphi_j(N,v,L) - \bar{\varphi}_j(N,v,L) = \varphi_k(N,v,L) - \bar{\varphi}_k(N,v,L)$ for all $j,k \in C_i(N,L)$. In view of (4.1), this implies $\varphi_j(N,v,L) = \bar{\varphi}_j(N,v,L)$ for all $j \in C_i(N,L)$. A contradiction. □

4.2. Existence. We show that there exists a CO-value that combines the Owen value (distribution between components) and the Myerson value (distribution within components) which satisfies our set of axioms.

THEOREM 4.3. *There is a CO-value that satisfies **E**, **Q**, **MF**, and **CM**.*

PROOF. Consider the CO-value φ given by

$$\varphi_i(N,v,L) := |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \mu_i\left(C_i(N,L), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L)}\right) \tag{4.2}$$

where v_C^σ is given by

$$v_C^\sigma(S) := v\left(S \cup \bigcup_{\substack{C' \in C(N,L): \\ \sigma(C') < \sigma(C)}} C'\right) - v\left(\bigcup_{\substack{C' \in C(N,L): \\ \sigma(C') < \sigma(C)}} C'\right), S \subseteq C \tag{4.3}$$

for all $C \in C(N,L)$. We then have

$$\begin{aligned}
\sum_{i \in N} \varphi_i(N,v,L) &= |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \sum_{C \in C(N,L)} \mu_C(C, v_C^\sigma, L|_C) \\
&= |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \sum_{C \in C(N,L)} v_C^\sigma(C) \\
&= |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} v(N) \\
&= v(N)
\end{aligned}$$

by (4.2) and changing the order of summation, by the fact that $(C, L|_C)$ is connected for $C \in C(L,N)$ and that μ is component efficient, and by (4.3). Hence, φ satisfies **E**.

For $L = \emptyset$, we have $C_i(N,L) = \{i\}$, hence $C(N,L) \cong N$ and $\Sigma(C(N,L)) \cong \Sigma(N)$, and therefore

$$\begin{aligned}
\varphi_i(N,v,\emptyset) &= |\Sigma(C(N,L))|^{-1} \sum_{\sigma \in \Sigma(C(N,L))} \mu_i(\{i\}, v_i^\sigma, \emptyset) \\
&= |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} v_i^\sigma(i) \\
&= |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma) \\
&= \mathrm{Sh}_i(N,v)
\end{aligned}$$

by (4.2), again by the fact that $(\{i\}, \emptyset)$ is connected and that μ is component efficient, by (4.3) and the definition of $MC_i^v(\sigma)$, and by (2.1). Further, by (4.2) and (4.3), we have $\varphi(N,v,L^N) = \mu(N,v,L^N)$ since (N,L^N) is connected. Since $\mu(N,v,L^N) = \mathrm{Sh}(N,v)$ (Myerson 1977), φ also satisfies **Q**.

Next, we show that φ satisfies **CM**. In view of (4.2), it suffices to show that

$$\mu_C(C, v_C^\sigma, L|_C) = \mu_{\{C\}}\left(\{C\}, (v \circ \cup)_{\{C\}}^\sigma, \emptyset\right)$$

for all $C \in C(N,L)$ and $\sigma \in \Sigma(C(N,L))$. Since μ is component efficient and $(C, L|_C)$ as well as $(\{C\}, \emptyset)$ are connected for $C \in C(N,L)$ in general, this is equivalent to $v_C^\sigma(C) = (v \circ \cup)_{\{C\}}^\sigma(\{C\})$ which holds by (4.3).

Finally, we show that φ satisfies **MF**. In view of (4.2) and (4.3), it suffices to show that

$$\begin{aligned}\mu_i\left(C_i(N,L), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_i\left(C_i(N,L-ij), v^{\sigma}_{C_i(N,L)}, L|_{C_i(N,L-ij)}\right)\\ = \mu_j\left(C_j(N,L), v^{\sigma}_{C_j(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L-ij), v^{\sigma}_{C_j(N,L)}, L|_{C_i(N,L-ij)}\right)\end{aligned}$$

holds for all $ij \in L$. Yet, this follows from μ satisfying **F** and **CD**. □

Below, we show that the value defined by (4.2) and (4.3) is a generalization of the Owen value. This may justify the notation $\mathrm{Ow}^{\sharp}$ where the musical "sharp" symbol $\sharp$ is intended to indicate a graph.

4.3. Non-redundancy. Next, we show that our axiomatization is non-redundant. Since by Theorem 4.2 and 4.3 $\mathrm{Ow}^{\sharp}$ is characterized by **E**, **Q**, **MF**, and **CM**, it suffices to show that there are CO-values that are different from $\mathrm{Ow}^{\sharp}$ but satisfy any three of theses axioms. The CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by $\varphi_i(N,v,L) = 0$ for all $i \in N$ satisfies **MF**, **CM**, and **Q**. From our leading example it is clear that $\mathrm{Ow}^{\sharp}$ and Ow applied to the coalition structure $C(N,L)$ do not coincide. Yet, the latter satisfies **E**, **CM**, and **Q**. Also, the CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by $\varphi_i(N,v,L) = |N|^{-1} v(N)$ for all $i \in N$ satisfies **E**, **MF**, and **Q**. Finally, consider the CO-value $\varphi \neq \mathrm{Ow}^{\sharp}$ given by

$$\varphi_i(N,v,L) := \mu_i(N,v,L) + \frac{v(N) - \sum_{C \in C(N,L)} v(C)}{|C(N,L)|\,|C_i(N,L)|}. \tag{4.4}$$

Since μ satisfies **CE**, we have

$$\varphi_{C_i(N,L)}(N,v,L) = v(C_i(N,L)) + \frac{v(N) - \sum_{C \in C(N,L)} v(C)}{|C(N,L)|}, \tag{4.5}$$

i.e. $\varphi_{C_i(N,L)}(N,v,L)$ depends only on the worth of the components in $C(N,L)$ and $|C(N,L)|$ which are not affected by considering components as players. Hence, φ satisfies **CM**. Summing up (4.5) over $C(N,L)$ then shows $\varphi_N(N,v,L) = v(N)$, i.e. φ satisfies **E**. Further, we have

$$\begin{aligned}\varphi_i(N,v,L) - {}&\varphi_j(N,v,L)\\ &\overset{(4.4)}{=} \mu_i(N,v,L) - \mu_j(N,v,L)\\ &\overset{\mu,\mathbf{CD}}{=} \mu_i\left(C_i(N,L), v|_{C_i(N,L)}, L|_{C_i(N,L)}\right) - \mu_j\left(C_j(N,L), v|_{C_j(N,L)}, L|_{C_j(N,L)}\right)\\ &\overset{\mu,\mathbf{F}}{=} \mu_i\left(C_i(N,L-ij), v|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right)\\ &\qquad - \mu_j\left(C_j(N,L-ij), v|_{C_j(N,L-ij)}, L|_{C_j(N,L-ij)}\right)\end{aligned}$$

$$\overset{\mu,\mathbf{CD}}{=} \mu_i\left(N_i\left(L,ij\right),v|_{N_i(L,ij)},L|_{N_i(L,ij)}\right)-\mu_j\left(N_j\left(L,ij\right),v|_{N_j(L,ij)},L|_{N_j(L,ij)}\right)$$
$$\overset{(4.4)}{=} \varphi_i\left(N_i\left(L,ij\right),v|_{N_i(L,ij)},L|_{N_i(L,ij)}\right)-\varphi_j\left(N_j\left(L,ij\right),v|_{N_j(L,ij)}L|_{N_j(L,ij)}\right)$$

which finally shows that φ also satisfies **MF**.

4.4. An example. Concluding this section, we reconsider our leading example with the graph in (1.1). There are two orders, σ and ρ, on $C(N,L)=\{C,\{P3\}\}$, $C=\{P1,P2,A\}$ where $\sigma(C)=1$ and $\rho(C)=2$. By (4.3), this gives the payoff functions $v_C^\sigma(S)=v(S)$, $S\subseteq C$ and

$$v_C^\rho(S)=v(S\cup\{P3\})-v(\{P3\})=\begin{cases}1 & ,|S\cap\{P1,P2\}|>0\\ 0 & ,|S\cap\{P1,P2\}|=0\end{cases}\qquad S\subseteq C.$$

Straightforward calculations in accordance with (2.5) yield the Myerson payoffs

$$\mu_{(P1,P2,A)}\left(C,v_C^\sigma,L|_C\right)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\quad\text{and}\quad\mu_{(P1,P2,A)}\left(C,v_C^\rho,L|_C\right)=\left(\frac{1}{2},\frac{1}{2},0\right).$$

By (4.2), this gives the payoffs

$$\mathrm{Ow}^\sharp_{(P1,P2,P3,A)}(N,v,L)=\left(\frac{5}{12},\frac{5}{12},0,\frac{1}{6}\right)$$

as in the Introduction where the payoff for $P3$ is immediate from **E**.

5. Properties

5.1. Relation to the Myerson value and to the Owen value. From (4.2) and (4.3) it is easy to see that $\mathrm{Ow}^\sharp$ and μ coincide on connected graphs and that $\mathrm{Ow}^\sharp$ inherits additivity from the Myerson value in general. The axiomatizations for the Owen value of Owen (1977) itself as well as those of Hart & Kurz (1983) involve the additivity axiom. Khmelnitskaya & Yanovskaya (2007) characterize the Owen value without additivity by employing the Young (1985) marginality axiom. Vázquez-Brage, García-Jurado & Carreras (1996) suggest a generalization of both the Owen and the Myerson value the axiomatization of which also does not involve the additivity axiom. However, their value refers to TU games with both a coalition structure and a cooperation structure. For the complete graph, this value coincides with the Owen value, but if the coalition structure equals the set of the graph's components then the Myerson value results. Hence, that value and our value are essentially different. Yet, in view of the following Theorem, our value indeed extends the Owen value to CO-games and therefore provides another justification of the Owen value without the additivity axiom.

Theorem 5.1. $\mathrm{Ow}^\sharp\left(N,v,L^{\mathcal{P}}\right)=\mathrm{Ow}\left(N,v,\mathcal{P}\right).$

Proof. Since $C_i\left(N, L^{\mathcal{P}}\right) = \mathcal{P}(i)$, we have

$$\begin{aligned}
&\mathrm{Ow}_i^\sharp\left(N, v, L^{\mathcal{P}}\right)\\
&\overset{(4.2)}{=} |\Sigma(\mathcal{P})|^{-1} \sum_{\rho\in\Sigma(\mathcal{P})} \mu_i\left(\mathcal{P}(i), v^\rho_{\mathcal{P}(i)}, L^{\mathcal{P}(i)}\right)\\
&\overset{(5.2)}{=} |\Sigma(\mathcal{P})|^{-1} \sum_{\rho\in\Sigma(\mathcal{P})} \mathrm{Sh}_i\left(\mathcal{P}(i), v^\rho_{\mathcal{P}(i)}, L^{\mathcal{P}(i)}\right)\\
&\overset{(2.1)}{=} |\Sigma(\mathcal{P})|^{-1} \sum_{\rho\in\Sigma(\mathcal{P})} |\Sigma(\mathcal{P}(i))|^{-1} \sum_{\sigma_i\in\Sigma(\mathcal{P}(i))} MC_i^{v^\rho_{\mathcal{P}(i)}}(\sigma_i)\\
&\overset{(4.3),(2.4)}{=} |\Sigma(\mathcal{P})|^{-1} \sum_{\rho\in\Sigma(\mathcal{P})} |\Sigma(\mathcal{P}(i))|^{-1} \sum_{\sigma_i\in\Sigma(\mathcal{P}(i))} |\Sigma(N,\mathcal{P},\sigma_i,\rho)|^{-1} \sum_{\sigma\in\Sigma(N,\mathcal{P},\sigma_i,\rho)} MC_i^v(\sigma)\\
&\overset{(2.4)}{=} |\Sigma(N,\mathcal{P})|^{-1} \sum_{\sigma\in\Sigma(N,\mathcal{P})} MC_i^v(\sigma)\\
&\overset{(2.3)}{=} \mathrm{Ow}_i(N, v, \mathcal{P})
\end{aligned}$$

for all $i \in N$. □

Since the Owen value and the Shapley value coincide for $\mathcal{P} = \{N\}$ and $\mathcal{P} = \langle N\rangle$ the following property is immediate.

Corollary 5.2. $\mathrm{Ow}^\sharp\left(N, v, L^N\right) = \mathrm{Ow}^\sharp(N, v, \emptyset) = \mathrm{Sh}(N, v)$.

Finally, **CM** and **Q** then imply that the distribution of the grand coalition's worth between components is governed by the same principles for $\mathrm{Ow}^\sharp$ and Ow.

Corollary 5.3. *For all* $C \in C(N, L)$, $\mathrm{Ow}^\sharp_C(N, v, L) = \mathrm{Ow}_C(N, v, C(N, L))$.

5.2. Consistency. Owen (1977) shows that for Ow the distribution of worth between coalitions and within coalitions is governed by the same principles. In particular, he shows that his value satisfies the following consistency property:

Theorem 5.4 (Owen 1977). *For all* $i \in N$, *we have*

$$\mathrm{Ow}_i(N, v, \mathcal{P}) = \mathrm{Ow}_i\left(\mathcal{P}(i), v^{N,\mathcal{P}}_{\mathcal{P}(i)}, \{\mathcal{P}(i)\}\right) = \mathrm{Ow}_i\left(\mathcal{P}(i), v^{N,\mathcal{P}}_{\mathcal{P}(i)}, \emptyset\right) \tag{5.1}$$

where the coalition function $v_P^{N,\mathcal{P}}$ *on* $P \in \mathcal{P}$ *is defined by*

$$v_P^{N,\mathcal{P}}(S) := \mathrm{Ow}_S\left(N\backslash(P\backslash S), v|_{N\backslash(P\backslash S)}, \mathcal{P}|_{N\backslash(P\backslash S)}\right) \quad , S \subseteq P. \tag{5.2}$$

$\mathrm{Ow}^\sharp$ satisfies a similar consistency property. In view of $\{\mathcal{P}(i)\} = \mathcal{P}|_{\mathcal{P}(i)}$, the following Theorem is the obvious analogon to Theorem 5.4. Since the components of $\mathcal{P}$ have no inner structure, however, there is no such analogon to the second equation in (5.1).

THEOREM 5.5. *We have* $\mathrm{Ow}^{\sharp} = \mathrm{Ow}^{\#}$ *where the CO-value* $\mathrm{Ow}^{\#}$ *is defined by*

$$\mathrm{Ow}_i^{\#}(N,v,L) = \mathrm{Ow}_i^{\sharp}\left(C_i(N,L), v_{C_i(N,L)}^{N,L}, L|_{C_i(N,L)}\right) \qquad , i \in N \tag{5.3}$$

where the coalition functions $v_C^{N,L}$ *on* $C \in C(N,L)$ *are defined by*

$$v_C^{N,L}(S) := \mathrm{Ow}_S^{\sharp}\left(N\backslash(C\backslash S), v|_{N\backslash(C\backslash S)}, L|_{N\backslash(C\backslash S)}\right) \qquad , S \subseteq C. \tag{5.4}$$

PROOF. By Theorems 4.2 and 4.3, it suffices to show that $\mathrm{Ow}^{\#}$ satisfies **E**, **Q**, **MF**, and **CM**. Since $\mathrm{Ow}^{\sharp}$ satisfies **E** and by (5.3) and (5.4), we have $\mathrm{Ow}_C^{\#}(N,v,L) = \mathrm{Ow}_C^{\sharp}(N,v,L)$ for all $C \in C(N,L)$. Therefore, $\mathrm{Ow}^{\#}$ inherits **E** and **CM** from $\mathrm{Ow}^{\sharp}$.

In order to show **Q**, we prove $\mathrm{Ow}^{\#}(N,v,\emptyset) = \mathrm{Sh}(N,v) = \mathrm{Ow}^{\#}\left(N,v,L^N\right)$. The first equation follows from

$$\begin{aligned}\mathrm{Ow}_i^{\#}(N,v,\emptyset) &\overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(\{i\}, v_{\{i\}}^{N,L}, \emptyset\right) \overset{(5.4)}{=} v_{\{i\}}^{N,L}(\{i\}) \\ &\overset{(5.4)}{=} \mathrm{Ow}_i^{\sharp}(N,v,\emptyset) \overset{\text{Thm. 5.1}}{=} \mathrm{Ow}_i(N,v,\langle N\rangle) \overset{\text{Cor. 5.2}}{=} \mathrm{Sh}_i(N,v).\end{aligned}$$

By (5.3), Theorem 5.1, and (5.4), we have $v_N^{N,\{N\}}(S) = v_N^{N,L^N}(S)$ for all $S \subseteq N$. Since

$$\mathrm{Ow}_i^{\#}\left(N,v,L^N\right) \overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(N, v_N^{N,L^N}, L^N\right) \overset{\text{Thm. 5.1}}{=} \mathrm{Ow}_i\left(N, v_N^{N,L^N}, \{N\}\right),$$

Theorem 5.4 implies $\mathrm{Ow}_i^{\#}\left(N,v,L^N\right) = \mathrm{Ow}_i(N,v,\{N\})$. Hence, $\mathrm{Ow}^{\#}\left(N,v,L^N\right) = \mathrm{Sh}(N,v)$ by Theorem 5.1 and Corollary 5.2.

Let now $ij \in L$ and $C := C_i(N,L)$. We then have

$$\begin{aligned}&\mathrm{Ow}_i^{\#}(N,v,L) - \mathrm{Ow}_j^{\#}(N,v,L) \\ &\overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(C, v_C^{N,L}, L|_C\right) - \mathrm{Ow}_j^{\sharp}\left(C, v_C^{N,L}, L|_C\right) \\ &\overset{\mathbf{MF}}{=} \mathrm{Ow}_i^{\sharp}\left(C_i(N,L-ij), v_C^{N,L}|_{C_i(N,L-ij)}, L|_{C_i(N,L-ij)}\right) \\ &\qquad - \mathrm{Ow}_j^{\sharp}\left(C_j(N,L-ij), v_C^{N,L}|_{C_j(N,L-ij)}, L|_{C_j(N,L-ij)}\right)\end{aligned}$$

and

$$\begin{aligned}&\mathrm{Ow}_i^{\#}\left(N_i(L,ij), v|_{N_i(L,ij)}, L|_{N_i(L,ij)}\right) - \mathrm{Ow}_j^{\#}\left(N_j(L,ij), v|_{N_j(L,ij)}, L|_{N_j(L,ij)}\right) \\ &\overset{(5.3)}{=} \mathrm{Ow}_i^{\sharp}\left(C_i(N,L-ij), v_{C_i(N,L-ij)}^{N_i(L,ij),L|_{N_i(L,ij)}}, L|_{C_i(N,L-ij)}\right) \\ &\qquad - \mathrm{Ow}_j^{\sharp}\left(C_j(N,L-ij), v_{C_j(N,L-ij)}^{N_j(L,ij),L|_{N_i(L,ij)}}, L|_{C_j(N,L-ij)}\right).\end{aligned}$$

Since $C_i(N,L-ij) \subseteq C$ and by (5.4), we also have

$$v_C^{N,L}|_{C_i(N,L-ij)}(S) = \mathrm{Ow}_S^{\sharp}\left(N\backslash(C\backslash S), v|_{N\backslash(C\backslash S)}, L|_{N\backslash(C\backslash S)}\right) = v_{C_i(N,L-ij)}^{N_i(L,ij),L|_{N_i(L,ij)}}(S)$$

for all $S \subseteq C_i(N,L-ij)$, analogously for j. Hence, $\mathrm{Ow}^{\#}$ satisfies **MF**. □

S	$L\|_{N\backslash(C\backslash S)} \cup L\|_{C\backslash S}$	$L\|_{N\backslash(C\backslash S)} \cup L^{C\backslash S}$	$L\|_{N\backslash(C\backslash S)}$	${}^{(k)}v_N^{N,L}(S)$
$\{1\}$	$\{23\}$	$\{23\}$	$\emptyset$	1
$\{2\}$	$\emptyset$	$\{13\}$	$\emptyset$	$\frac{1}{2}$
$\{3\}$	$\{12\}$	$\{12\}$	$\emptyset$	$\frac{1}{2}$
$\{1,2\}$	$\{12\}$	$\{12\}$	$\{12\}$	$\frac{3}{2}$
$\{1,3\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\frac{3}{2}$
$\{2,3\}$	$\{23\}$	$\{23\}$	$\{23\}$	1
N	L	L	L	2

TABLE 5.1. Graphs and worths for the counterexample

In addition, Hart & Kurz (1983) show that for Ow the distribution of worth between coalitions is consistent with the distribution within coalitions in the following sense.

THEOREM 5.6 (Hart and Kurz 1983). *Theorem 5.4 remains true if we replace the coalition function $v_P^{N,\mathcal{P}}$, $P \in \mathcal{P}$ by either of the following ones: For all $S \subseteq P$*

$$ {}^{(1)}v_P^{N,\mathcal{P}}(S) := \mathrm{Ow}_S(N, v, (\mathcal{P}\backslash\{P\}) \cup \{S, P\backslash S\}) \tag{5.5}$$

$$ {}^{(2)}v_P^{N,\mathcal{P}}(S) := \mathrm{Ow}_S(N, v, (\mathcal{P}\backslash\{P\}) \cup \{S\} \cup \langle P\backslash S\rangle) \tag{5.6}$$

The following conjecture tries to transfer the results of Theorem 5.6 to $\mathrm{Ow}^\sharp$. In (5.5), the component $P \in \mathcal{P}$ is split into the components $S, P\backslash S \subseteq P$. In a sense, all "links" between the players in S and those in $P\backslash S$ have been removed. This is the idea of (5.7): Now, the links between players in $S \subseteq C \in C(N,L)$ and $C\backslash S$ have been removed indeed. The idea of (5.8) is the same except that the players in $C\backslash S$ are completely connected which, of course, did not make a difference for coalition functions. In (5.6), the players in S are also separated from those in $P\backslash S$ but the players in $P\backslash S$ are isolated, i.e. they form singleton coalitions. (5.9) mimics this by removing all links outside $N\backslash(C\backslash S)$.

CONJECTURE 5.7. *Theorem 5.5 remains true is we replace the coalition function $v_C^{N,L}$, $C \in C(N,L)$ by either of the following ones: For all $S \subseteq C$*

$$ {}^{(1)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp\left(N, v, L|_{N\backslash(C\backslash S)} \cup L|_{C\backslash S}\right) \tag{5.7}$$

$$ {}^{(2)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp\left(N, v, L|_{N\backslash(C\backslash S)} \cup L^{C\backslash S}\right) \tag{5.8}$$

$$ {}^{(3)}v_C^{N,L}(S) := \mathrm{Ow}_S^\sharp\left(N, v, L|_{N\backslash(C\backslash S)}\right) \tag{5.9}$$

As the following example reveals, however, this conjecture is wrong.

Example 5.8. Set $N = \{1,2,3\}$, $L = \{12,23\}$ and $v = u_{\{1,2\}} + u_{\{1,3\}}$. Since L is connected, one easily obtains $\mathrm{Ow}^{\sharp}(N,v,L) = \mu(N,v,L) = \left(\frac{5}{6},\frac{5}{6},\frac{1}{3}\right)$. Table 5.1 lists the graphs and worths involved in Conjecture 5.7 where the payoff functions coincide. Again, one easily obtains $\mathrm{Ow}^{\#}(N,v,L) = \mathrm{Ow}^{\sharp}_N\left(N, {}^{(k)}v_N^{N,L}, L\right) = \mu\left(N, {}^{(k)}v_N^{N,L}, L\right) = \left(1,\frac{1}{2},\frac{1}{2}\right)$. I.e., $\mathrm{Ow}^{\#} \neq \mathrm{Ow}^{\sharp}$.

5.3. Stability issues. Employing the Owen value, Hart & Kurz (1983) study coalition formation in CS-games by strong equilibria of simultaneous coalition formation games. Yet, Hart & Kurz (1984) provide examples of TU games that do not allow for stable coalition structures. Dutta et al. (1998) study link formation in CO-games by simultaneous link formation games which involve the Myerson value. For superadditive games, they show that the complete network can be supported by undominated Nash equilibria and coalition proof Nash equilibria and that any such equilibrium yields the same payoffs. Partly, this result rests on the following axiom which μ satisfies for superadditive games (Myerson 1977).

Axiom 5.9 (Link monotonicity, LM). *For all $i,j \in N$,*

$$\varphi_i(N,v,L+ij) \geq \varphi_i(N,v,L).$$

As the following example reveals, $\mathrm{Ow}^{\sharp}$ fails this axiom.

Example 5.10. Consider the game (N,u_N), $N = \{1,2,3\}$ which is superadditive and the graph $L = \emptyset$. It is easy to check that we then have $\mathrm{Ow}^{\sharp}_1\left(N,u_{\{1,2\}},L\right) = \frac{1}{3}$ but $\mathrm{Ow}^{\sharp}_1(N,u_N,L+12) = \frac{1}{4}$. Note that $2 \notin C_1(N,L)$.

Hence, since $\mathrm{Ow}^{\sharp}$ combines the Owen value and the Myerson value, it seems to us that one cannot reasonably expect general stability results for $\mathrm{Ow}^{\sharp}$. Nevertheless, in view of (4.2) and (4.3), it is immediate that $\mathrm{Ow}^{\sharp}$ satisfies the following component restricted version of **LM** for superadditive games.

Theorem 5.11 (Component restricted link monotonicity, CLM). *If (N,v) is superadditive then $\mathrm{Ow}^{\sharp}$ satisfies the following axiom: For all $i \in N$ and $j \in C_i(N,L)$,*

$$\varphi_i(N,v,L+ij) \geq \varphi_i(N,v,L).$$

6. Conclusion

In this paper, we introduced and advocated an efficient CO-value, $\mathrm{Ow}^{\sharp}$, which combines the ideas underlying the Owen and the component efficient Myerson value. In contrast to the Owen value, this value is capable to exploit the inner structure of the bargaining blocs modelled by the connected components of a graph. This way, $\mathrm{Ow}^{\sharp}$ may recognize e.g. the role of a coordinating player who keeps a bloc together. As mentioned above, this may be an additional source for instability in network formation. Nevertheless, it seems to be worthwhile to study implications of $\text{Ow}^\sharp$ in this regard, both in general and in specific applications.

The Myerson value was extended by van den Nouweland et al. (1992) to the class of TU games with a conference structure (hypergraph on the player set) (henceforth CF-games and CF-value) which we will call the Myerson CF-value. Remember, a hypergraph is a pair (N, H) consisting of a set N and a subset H of the power set 2^N the elements h of which are called hyperlinks or conference structures. Let $C(N, H)$ denote the set of connected components of (N, H) and $C_i(N, H)$ the component that hosts player i. Since the characterization of the Myerson CF-value is analogous to that of the Myerson value, one may wonder whether the results of this paper could be extended to CF-games.

Indeed, slightly adapting the arguments from this paper and van den Nouweland et al. (1992), it is hardly more than a five-finger exercise to extend our CO-value into a CF-value with analogous properties: In the definition, i.e. in (4.2) and (4.3), the graph has to be replaced by a hypergraph, and in (4.2), the Myerson value has be to replaced by the Myerson CF-value. The characterization then involves extensions of **CE**, **Q**, **CF**, and **CM**. Those of **CE** and **CM** are natural. The obvious extension of **Q** would require $\varphi\left(N, v, 2^N\right) = \varphi(N, v, \emptyset)$, but in view of the definition of the Myerson CF-value, the complete hypergraph 2^N could be replaced by the complete graph L^N as a subset of 2^N. Besides **CE**, the Myerson CF-value is characterized by the following modification of **F**: For all $i, j \in h \in H$, we have

$$\varphi_i(N, v, H) - \varphi_i(N, v, H \setminus \{h\}) = \varphi_j(N, v, H) - \varphi_j(N, v, H \setminus \{h\}).$$

This translates into the following extension of **MF**: For all $i, j \in h \in H$,

$$\begin{aligned}\varphi_i(N, v, H) &- \varphi_i\left(N_i(H, h), v|_{N_i(H,h)}, H|_{N_i(H,h)}\right) \\ &= \varphi_j(N, v, H) - \varphi_j\left(N_j(H, h), v|_{N_j(H,h)}, H|_{N_j(H,h)}\right)\end{aligned}$$

where

$$N_i(H, h) := N \setminus (C_i(N, H) \setminus C_i(N, H \setminus \{h\})).$$

It is easy to see that for hypergraphs containing just two-player hyperlinks the modified axioms become the original ones.

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CHAPTER VIII

On a relation between the Myerson value and the position value

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Abstract

In this note, we characterize the position value for TU games with a cooperation structure in terms of the Myerson value of some natural modification of the original game—the link agent form.

Key Words: TU game, cooperation structure, graph, link agent form

JEL classification: C71

1. Introduction

Generalizing the Shapley (1953) value for TU games and the Aumann & Drèze (1974) value for TU games with a coalition structure (partition of the player set), Myerson (1977) introduced a now well-known value for TU games with a cooperation structure (graph on the player set) (henceforth CO-games and CO-value). As an alternative, Meessen (1988) suggests the position value for CO-games which was popularized by Borm et al. (1992). Yet another CO-value has been introduced by Hamiache (1999) which was discussed by Bilbao et al. (2006).

Besides the elegant Myerson (1977) axioms, there are more or less general alternative axiomatizations of this value (Myerson 1980, Borm et al. 1992, Slikker & van den Nouweland 2001). The position value was axiomatized by Borm et al. (1992) for a restricted class of CO-games. Only recently, Slikker (2005) gave a general characterization.

In this note, we suggest a new way to characterize the position value. In particular, we express the position value in terms of the Myerson value. In contrast to the Myerson value which emphasizes the role of the players, the position value focuses on the links. Therefore, one may be tempted to split the players into separate agents, one for each link, and then to connect a player's agents completely. Based on this idea, we introduce the link agent form (LAF) of a CO-game. It turns out that the sum of the Myerson payoffs of a player's agents in the LAF equals the position value payoffs of that player in the original game.

The plan of this note is as follows: Basic definitions and notation are given in second section. The third section introduces the link agent form of a CO-game and presents our characterization of the position value. A few remarks conclude the paper.

2. Basic definitions and notation

A (TU) game is a pair (N, v) consisting of a non-empty and finite set of players N and the coalition function $v : 2^N \to \mathbb{R}$, $v(\emptyset) = 0$. $v(K)$ is called the worth of $K \subseteq N$; subsets of N are called coalitions. The restriction of v to $N' \subseteq N$ is denoted $v|_{N'}$. A value is an operator φ that assigns payoff vectors to all games, $\varphi(N, v) \in \mathbb{R}^N$. An order of a set N is a bijection $\sigma : N \to \{1, \ldots, |N|\}$ with the interpretation that i is the $\sigma(i)$th player in σ. The set of these orders is denoted by $\Sigma(N)$. The set of players not after i in σ is denoted by $K_i(\sigma) = \{j : \sigma(j) \leq \sigma(i)\}$. The marginal contribution of i in σ is defined as $MC_i^v(\sigma) := v(K_i(\sigma)) - v(K_i(\sigma) \setminus \{i\})$. The Shapley (1953) value Sh is defined by

$$\mathrm{Sh}_i(N, v) := |\Sigma(N)|^{-1} \sum_{\sigma \in \Sigma(N)} MC_i^v(\sigma) \quad , i \in N. \tag{2.1}$$

For $K \subseteq N$, we denote by $\varphi_K(N, v, \cdot)$ the sum $\sum_{i \in K} \varphi_i(N, v, \cdot)$.

A cooperation structure for (N,v) is an undirected graph (N,L), $L \subseteq L^N := \{\{i,j\} \mid i,j \in N, i \neq j\}$. A typical element of L is written as ij or λ. The set of player i's links is denoted $L_i := \{\lambda \in L | i \in \lambda\}$. Given any graph (N,L), N splits into (maximal connected) components the set of which is denoted by $C(N,L)$; $C_i(N,L) \in C(N,L)$ denotes the component containing $i \in N$. $L|_{N'} = \{\{i,j\} \in L | i,j \in N'\}$ denotes the restriction of L to $N' \subseteq N$. A CO-game is a game together with a cooperation structure. A CO-value is an operator φ that assigns payoff vectors to all CO-games, $\varphi(N,v,L) \in \mathbb{R}^N$.

The Myerson (1977) value μ is defined by

$$\mu(N,v,L) := \mathrm{Sh}\left(v^L\right) \qquad , v^L(K) := \sum_{S \in C(K,L|_K)} v(S) \qquad , K \subseteq N. \tag{2.2}$$

The position value (Meessen 1988, Borm et al. 1992) is defined as follows. For any CO-game (N,v,L) consider the link game $\left(L, v^N\right)$ where

$$v^N\left(L'\right) = \sum_{S \in C(N,L')} v(S) \qquad , L' \subseteq L. \tag{2.3}$$

Since $v^N(\emptyset)$ may not vanish and for convenience, following Borm et al. (1992), we restrict attention to 0-normalized TU games, i.e. $v(\{i\}) = 0$ for all $i \in N$. The position value then is given by

$$\pi_i(N,v,L) = \frac{1}{2} \sum_{\lambda \in L_i} \mathrm{Sh}_\lambda\left(L, v^N\right). \tag{2.4}$$

Since μ and π are component efficient, i.e. $\mu_C(N,v,L) = \pi_C(N,v,L) = v(C)$ for all $C \in C(N,L)$, we assume that (N,L) does not contain isolated players, i.e. $|L_i| > 0$ for all $i \in N$.

3. A characterization of the position value

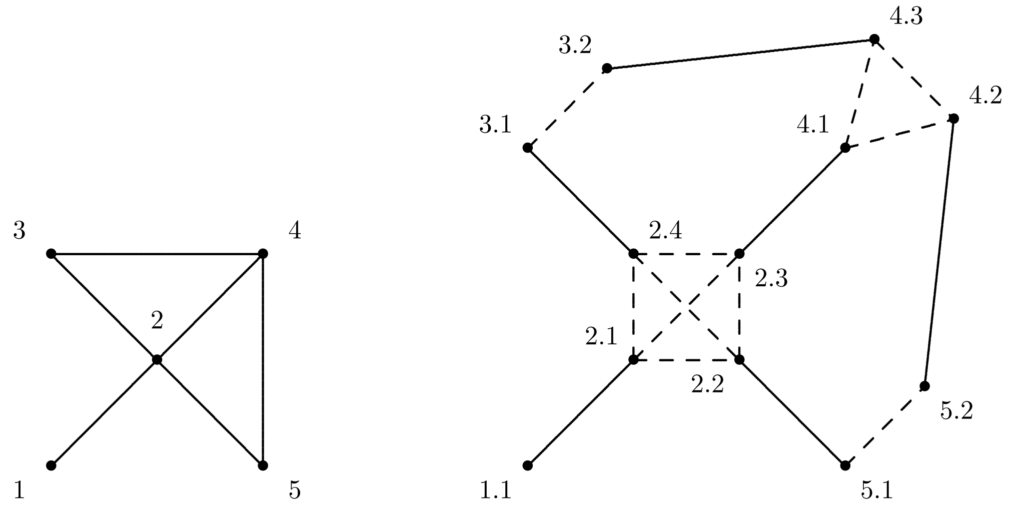
In the following, we express the position value for CO-games in terms of the Myerson value of the link agent form (LAF) of the original game. While the position value emphasizes the role of the links, the Myerson value focuses on the players. Therefore, one could think of splitting the players into separate agents which represent/control exactly one of a player's links. This is what the LAF does.

Definition 3.1. *For any CO-game $\mathcal{G} = (N,v,L)$, its* link agent form $\mathrm{LAF}(\mathcal{G}) = \left(\bar{N}, \bar{v}, \bar{L}\right)$ *is defined as follows:*

$$\bar{N} = \bigcup_{i \in N} \bar{N}(i) = \{(i,\lambda) | i \in N, \lambda \in L_i\} \qquad , \bar{N}(i) := \{i\} \times L_i \tag{3.1}$$

$$\bar{L} = \bar{L}^o \cup \bigcup_{i \in N} L^{\bar{N}(i)} \qquad , \bar{L}^o := \{\overline{ij} | ij \in L\} \qquad , \overline{ij} := \{(i,ij)(j,ij)\} \tag{3.2}$$

$$\bar{v}\left(\bar{K}\right) = v\left(N\left(\bar{K}\right)\right) \qquad , N\left(\bar{K}\right) := \left\{i | \exists \lambda \in L_i : (i,\lambda) \in \bar{K}\right\} \quad , \bar{K} \subseteq \bar{N} \tag{3.3}$$



$\mathcal{G}$ LAF($\mathcal{G}$)

FIGURE 3.1. The graph of a link agent form

The player set $\bar{N}$ comprises the link agents (i, λ) for all players $i \in N$ and all links $\lambda \in L_i$. Since we assume that there are no isolated players, all link agent sets $\bar{N}(i)$ are non-empty. The cooperation structure $\bar{L}$ contains the original links ij as the links $\overline{ij}$ in the link set $\bar{L}^o$. Further, $\bar{L}$ completely connects the set of link agents $\bar{N}(i)$ of any original player $i \in N$ via the link set $L^{\bar{N}(i)}$. By (3.3), any of a player's agents is as productive as the original player, but one of them already suffices to do the job. The former is visualized by the following example.

EXAMPLE 3.2. Figure 3.1 shows the graph of some CO-game $\mathcal{G}$ and the graph of its link agent form LAF$(\mathcal{G})$. In LAF$(\mathcal{G})$, the links which correspond to the original links in $\mathcal{G}$ are drawn as solid lines while the links which connect a player's agents are represented by dashed ones. For example, the link $\{3, 4\}$ in $\mathcal{G}$ corresponds to the link $\{3.2, 4.3\}$ in LAF$(\mathcal{G})$. Player 1 in $\mathcal{G}$ has just one link. Hence in LAF$(\mathcal{G})$, he is represented by the single agent 1.1. Player 2, for example, has four links in $\mathcal{G}$ that are represented by the agents 2.1 to 2.4 in LAF$(\mathcal{G})$ who are completely connected with each other.

Now, we can state our result.

THEOREM 3.3. *For any 0-normalized CO-game without isolated players* $\mathcal{G} = (N, v, L)$, *we have* $\pi_i(\mathcal{G}) = \mu_{\bar{N}(i)}(\text{LAF}(\mathcal{G}))$ *for all* $i \in N$.

Proof. Let $\mathcal{G}$ be as in the theorem. For any $\bar{K} \subseteq \bar{N}$ define the set of original links which this player set establishes,

$$L\left(\bar{K}\right) := \left\{ij \in L|\left(i, ij\right), \left(j, ij\right) \in \bar{K}\right\}. \tag{3.4}$$

For $(i, ij) \in \bar{K} \subseteq \bar{N}$, this implies

$$\begin{aligned}
&\bar{v}^{\bar{L}}\left(\bar{K}\right) - \bar{v}^{\bar{L}}\left(\bar{K} \setminus \{(i, ij)\}\right) \\
&\overset{(2.2)}{=} \sum_{\bar{S} \in C\left(\bar{K}, \bar{L}|_{\bar{K}}\right)} \bar{v}\left(\bar{S}\right) - \sum_{\bar{S} \in C\left(\bar{K} \setminus \{(i,ij)\}, \bar{L}|_{\bar{K} \setminus \{(i,ij)\}}\right)} \bar{v}\left(\bar{S}\right) \\
&\overset{(3.3)}{=} \sum_{\bar{S} \in C\left(\bar{K}, \bar{L}|_{\bar{K}}\right)} v\left(N\left(\bar{S}\right)\right) - \sum_{\bar{S} \in C\left(\bar{K} \setminus \{(i,ij)\}, \bar{L}|_{\bar{K} \setminus \{(i,ij)\}}\right)} v\left(N\left(\bar{S}\right)\right) \\
&= \sum_{S \in C\left(N\left(\bar{K}\right), L\left(\bar{K}\right)\right)} v\left(S\right) - \sum_{S \in C\left(N\left(\bar{K} \setminus \{(i,ij)\}\right), L\left(\bar{K} \setminus \{(i,ij)\}\right)\right)} v\left(S\right) \\
&\overset{\text{0-norm.}}{=} \sum_{S \in C\left(N, L\left(\bar{K}\right)\right)} v\left(S\right) - \sum_{S \in C\left(N, L\left(\bar{K} \setminus \{(i,ij)\}\right)\right)} v\left(S\right) \\
&\overset{(2.3)}{=} v^N\left(L\left(\bar{K}\right)\right) - v^N\left(L\left(\bar{K} \setminus \{(i, ij)\}\right)\right).
\end{aligned} \tag{3.5}$$

where the third equation holds for the following reasons: By (3.2), (i, λ) and $\left(i, \lambda'\right)$ are connected within $\bar{K}$ whenever both are contained in $\bar{K}$. Also, if (i, λ) and $\left(j, \lambda'\right)$ are connected in $\left(\bar{K}, \bar{L}|_{\bar{K}}\right)$ then by (3.4) and (3.3) i and j are connected in $\left(N\left(\bar{K}\right), L\left(\bar{K}\right)\right)$ and vice versa.

Any order $\rho \in \Sigma\left(\bar{N}\right)$ induces a unique order $\sigma^*\left(\rho\right) \in \Sigma\left(L\right)$ such that

$$\sigma^*\left(\rho\right)\left(\lambda\right) < \sigma^*\left(\rho\right)\left(\lambda'\right) \quad \Leftrightarrow \quad \max_{i \in \lambda} \rho\left(i, \lambda\right) < \max_{i \in \lambda'} \rho\left(i, \lambda'\right) \tag{3.6}$$

for all $\lambda, \lambda' \in L$. (3.3), (3.5), and (3.6) then imply $MC^{\bar{v}^{\bar{L}}}_{(i,ij)}\left(\rho\right) = 0$ if $\rho\left(i, ij\right) < \rho\left(j, ij\right)$ and $MC^{\bar{v}^{\bar{L}}}_{(i,ij)}\left(\rho\right) = MC^{v^N}_{ij}\left(\sigma^*\left(\rho\right)\right)$ if $\rho\left(j, ij\right) < \rho\left(i, ij\right)$. Since $|\lambda| = 2$ for all $\lambda \in L$, it is clear that $\sigma^*\left(\Sigma\left(\bar{N}\right)\right) = \Sigma\left(L\right)$ and that all induced orders are equally likely, and it is also clear that for all induced orders $\sigma^*\left(\rho\right)$ it is equally likely that $\rho\left(i, ij\right) < \rho\left(j, ij\right)$ or $\rho\left(j, ij\right) < \rho\left(i, ij\right)$. Hence, taking expectations over all sequences in $\Sigma\left(\bar{N}\right)$ and $\Sigma\left(L\right)$, respectively, we obtain

$$\mu_{(i,ij)}\left(\text{LAF}\left(\mathcal{G}\right)\right) = \frac{1}{2}\,\text{Sh}_{ij}\left(L, v^N\right)$$

by (2.1) and (2.2). Summing up over $\bar{N}\left(i\right)$ finally gives

$$\mu_{\bar{N}(i)}\left(\text{LAF}\left(\mathcal{G}\right)\right) = \pi_i\left(\mathcal{G}\right)$$

by (2.4) and (3.1). □

4. Conclusion

In this note, we introduced some natural modification of CO-games—the link agent form—which enabled us to express the position value in terms of the Myerson value and which can be viewed as an alternative characterization of the position value. This was achieved by shifting the focus from players to links via the LAF.

van den Nouweland et al. (1992) extended the Myerson and the position value to TU games with a conference structure (hypergraph on the player set) (CF-games and CF-values) which are extensions of the respective CO-values. So one could wonder whether the result of this note could easily be extended to CF-games. Unfortunately, this is not the case. In the proof of Theorem 3.3, it was essential that *all* links—by definition—connect exactly two players. In a hypergraph, however, the hyperlinks may connect different numbers of players. Casajus (2006) accounts for this peculiarity and introduces the hyperlink agent form of a CF-game as an extension of the LAF. Yet, the hyperlink agent form is quite technical and lacks much of the LAF's appeal.

Further, it might be worthwhile to explore whether it is possible to express the Myerson value in terms of the position value of a modification of the original CO-game that shifts the focus in the opposite direction. Yet, the Casajus (2006) construction is more technical and less natural than the LAF.

Finally, based on the LAF, one could think of the following axiom and then wonder whether there are interesting CO-values which are component efficient and invariant to player splitting.

Axiom 4.1 (Player splitting invariance, PSI). *For all 0-normalized CO-games without isolated players $\mathcal{G} = (N, v, L)$, we have*

$$\varphi_i (N, v, L) = \varphi_{\bar{N}(i)} \left(\bar{N}, \bar{v}, \bar{L}\right)$$

for all $i \in N$ where $\left(\bar{N}, \bar{v}, \bar{L}\right) = \mathrm{LAF}(\mathcal{G})$.

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