Prof. Dr. S. Hollands
Prof. Dr. R. Verch
Dr. J. Zahn
Dr. M. Schmidt

UNIVERSITÄT LEIPZIG

Inst. f. Theoretische Physik
Winter Term 2017/18

Relativistic Quantum Field Theory — Problem Sheet 11

2 pages — Problem 11.1, 11.2

Problem 11.1: Polarization vectors for the electromagnetic field

In this problem $\bar{k} = (1, 0, 0, 1)$. We choose for each future directed null 4-vector p a proper orthochronous Lorentz transformation Λ_p such that $p = \Lambda_p \bar{k}$.

a) Show that any other choice $\tilde{\Lambda}_p$ must be related to the previous choice by $\tilde{\Lambda}_p = \Lambda_p \cdot \gamma$, where γ is a proper orthochronous Lorentz transformation leaving $\bar{k}$ invariant. Show that the set of such γ forms a group, the "little group" (of $\bar{k}$).

b) Show explicitly that the following matrices leave $\bar{k}$ invariant and are proper orthochronous Lorentz transformations:

$$R(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

as well as

$$T(\mathbf{a}) = \begin{pmatrix} 1 + |\mathbf{a}|^2/2 & a_1 & a_2 & -|\mathbf{a}|^2/2 \\ a_1 & 1 & 0 & -a_1 \\ a_2 & 0 & 1 & -a_2 \\ |\mathbf{a}|^2/2 & a_1 & a_2 & 1 - |\mathbf{a}|^2/2 \end{pmatrix},$$

where $\mathbf{a} = (a_1, a_2)$.

c) Verify that $R(\theta), T(\mathbf{a})$ satisfy the relations of the Euclidean group in the plane (as given in lecture).

d) Argue that a general proper orthochronous Lorentz transformation γ leaving $\bar{k}$ invariant can always be written as $\gamma = R(\theta)T(a_1, a_2)$ for suitable real θ, a_1, a_2.

e) Define for $\bar{k}$ the polarization vectors as $\epsilon^{\pm}(\bar{k}) = (0, 1, \pm i, 0)/\sqrt{2}$. For a general 4-vector p in the future lightcone, we set $\epsilon^{\pm}_{\mu}(p) = (\Lambda_p)^{\nu}{}_{\mu} \epsilon^{\pm}_{\nu}(\bar{k})$.

f) Show that $p^{\mu}\epsilon^{\pm}_{\mu}(p) = 0$. Show that any other choice $\tilde{\Lambda}_p$ leads to polarization vectors related to the previous choice by $\tilde{\epsilon}^{\pm}_{\mu}(p) = \epsilon^{\pm}_{\mu}(p) + c^{\pm}p_{\mu}$, where $c^{\pm}$ are complex constants. Hence the equivalence class of $\epsilon^{\pm}_{\mu}(p)$ in

$$\hat{V}(p) = \{v_{\mu} : p^{\mu}v_{\mu} = 0\}/\{v_{\mu} : v_{\mu} = cp_{\mu}\}$$

is independent of the precise choice of Λ_p.

g) Show that if $p' = \Lambda p$, then we can write $\Lambda = \Lambda_{p'} \gamma(\Lambda, p) \Lambda_p^{-1}$, where $\gamma(\Lambda, p)$ is an element of the little group.

h) Infer the transformation formula

$$\Lambda^\mu{}_\nu \epsilon_\mu^\pm(p) = e^{\pm i\theta(\Lambda,p)} \epsilon_\nu^\pm(\Lambda p) + c^\pm(\Lambda, p)(\Lambda p)_\nu$$

where θ is the parameter of R in the decomposition of the element $\gamma(\Lambda, p)$ of the little group associated with Λ, p as in the previous item. $c^\pm = (a_1 \pm i a_2)/\sqrt{2}$ are complex constants where a_1, a_2 are the parameters of T in the decomposition of the element $\gamma(\Lambda, p)$ of the little group associated with Λ, p. Hence argue that the *equivalence classes* of the polarization vectors (see item f)) transform covariantly under the action of the Lorentz group.

i) The polarization tensor is given by

$$\Pi_{\mu\nu}(p) = \sum_{\lambda=\pm} \overline{\epsilon_\mu^\lambda(p)} \epsilon_\nu^\lambda(p).$$

Define $\bar{l} = (1, 0, 0, -1)$ and show that $\Pi_{\mu\nu}(\bar{k}) = \eta_{\mu\nu} + (\bar{l}_\mu \bar{k}_\nu + \bar{l}_\nu \bar{k}_\mu)/2$, which projects onto the x, y plane. For a concrete choice of Λ_p write an explicit formula of $\Pi_{\mu\nu}(p)$ and show that any other choice $\tilde{\Lambda}_p$ leads to a polarization tensor related to the previous choice by $\tilde{\Pi}_{\mu\nu}(p) = \Pi_{\mu\nu}(p) +$ terms involving either p_μ or p_ν.

j) Show that $\eta^{\mu\nu} \overline{\epsilon_\mu^{\lambda'}(p)} \epsilon_\nu^\lambda(p) = \delta_{\lambda\lambda'}$. Hint: First show this for $\bar{k}$ and then use the transformation law in h).

Problem 11.2: Quantum electromagnetic field and spin-1 representation of Poincare group

a) Let

$$A_\mu(x) = \int \widetilde{dp} \sum_{\lambda=\pm} (\overline{\epsilon_\mu^\lambda(p)} a_\lambda(\vec{p}) e^{ipx} + h.c.)$$

be the quantized electromagnetic field, where the creation/annihilation operators satisfy the commutators given in the lecture, and where $p = (|\vec{p}|, \vec{p})$. Show that A_μ satisfies the Maxwell equations and the Lorentz gauge condition $\partial^\mu A_\mu = 0$.

b) The precise definition of A_μ depends on our choice of Λ_p implicit in the definition of the polarization vectors $\epsilon^\lambda(p)$. Show that any other choice $\tilde{\Lambda}_p$ leads to a quantum field $\tilde{A}_\mu$ related to the previous choice by $\tilde{A}_\mu(x) = A_\mu(x) + \partial_\mu \chi(x)$ for some χ. Hence these fields only differ by a gauge transformation. Hint: Use part f) of 11.1.

c) Show that the Maxwell field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is independent of the choice of Λ_p implicit in the definition of the polarization vectors $\epsilon^\lambda(p)$. Hint: Use part f) of 11.1.

d) The state $|p, \lambda\rangle = a_\lambda^\dagger(\vec{p})|0\rangle$ describes a single photon with helicity λ and lightlike 4-momentum $p = (|\vec{p}|, \vec{p})$. We define an action of the Poincare group by

$$U(\lambda, a)|p, \lambda\rangle = e^{-ipa} e^{i\lambda\theta(\Lambda,p)} |\Lambda p, \lambda\rangle,$$

where $\theta(\Lambda, p)$ is as in h) of problem 11.1. Show that this defines a unitary representation of the Poincare group on the subspace of 1-particle states of the Fock space of the theory.

e) If we define $U(\Lambda, a)$ on the entire Fock space by second quantization, we get a unitary representation on the bosonic Fock space spanned by (smeared) vectors of the form

$$|p_1, \lambda_1, p_2, \lambda_2, \dots\rangle = a^\dagger_{\lambda_1}(\vec{p}_1) a^\dagger_{\lambda_2}(\vec{p}_2) \dots |0\rangle.$$

Show that the quantum field A_μ transforms as

$$U(\Lambda, p)^\dagger A_\mu(x) U(\Lambda, p) = \Lambda^\nu{}_\mu A_\nu(\Lambda x + a) + \partial_\mu(\dots).$$

Hint: Use h) of problem 11.1. Thus, A_μ transforms covariantly up to a local gauge transformation.

f) Similarly, show that the field strength tensor transforms as

$$U(\Lambda, p)^\dagger F_{\mu\nu}(x) U(\Lambda, p) = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu F_{\alpha\beta}(\Lambda x + a),$$

i.e. covariantly.