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Relativistic Quantum Field Theory — Problem Sheet 10

2 pages — Problems 10.1 to 10.3

Problem 10.1: Fermionic Fock space

a) Show that the fermionic creation/annihilation operator fulfilling

$$\{a, a^\dagger\} = \mathbb{1}$$

can be represented on $\mathbb{C}^2$. Compute the corresponding number operator $N = a^* a$ and its norm $||N|| = \sup_{||\psi||=1} ||N\psi||$. Compare with the bosonic case.

b) Verify that the fermionic creation/annihilation operators $b^\dagger(f)$, $b(g)$ with $f, g \in \mathcal{K} = L^2(\mathbb{R}^3, \widetilde{\mathrm{d}k})$ and

$$(b^\dagger(f)F)_n = \sqrt{n} P_-(f \otimes F_{n-1}),$$
$$(b(g)F)_n(\underline{k}_1, \ldots, \underline{k}_n) = \sqrt{n+1} \int \bar{f}(\underline{k}) F_{n+1}(\underline{k}, \underline{k}_1, \ldots \underline{k}_n) \widetilde{\mathrm{d}k},$$

fulfill the anti-commutation relations

$$\{b^\dagger(f), b^\dagger(g)\} = 0 = \{b(f), b(g)\}, \qquad \{b(g), b^\dagger(f)\} = \langle g|f\rangle_{\mathcal{K}}.$$

Here P_- is the projector on the antisymmetric subspace. Check that $b^\dagger(f)$ is indeed the adjoint of $b(f)$.

c) Now consider another set $d^\dagger$, d of fermionic creation/annihilation operators. Verify that if one defines their action on the tensor product $\mathcal{F} = \mathcal{F}^b \otimes \mathcal{F}^d$ of the corresponding fermionic Fock spaces as

$$b(f)(F_n \otimes G_m) = b(f)F_n \otimes G_m, \qquad d(f)(F_n \otimes G_m) = (-1)^n F_n \otimes d(f)G_m,$$

and analogously for $b^\dagger$, $d^\dagger$, then the b's and d's anti-commute, i.e.,

$$\{b(f), d(g)\} = 0 = \{b(f), d^\dagger(g)\}.$$

Problem 10.2: The charge

a) Show that the Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

is invariant under $\psi \mapsto e^{-i\alpha}\psi$ with α any constant real.

b) Show that the corresponding Noether current is given by

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

and check that it is conserved.

c) Given the field

$$\psi(x) = \sum_s \int \left(b_s(\underline{k}) u_s(\underline{k}) e^{ikx} + d_s^\dagger(\underline{k}) v_s(\underline{k}) e^{-ikx} \right) \widetilde{\mathrm{d}k},$$

Show that the charge corresponding to the current j^μ is

$$Q = \sum_s \int \left(b_s^\dagger(\underline{k}) b_s(\underline{k}) - d_s^\dagger(\underline{k}) d_s(\underline{k}) \right) \widetilde{\mathrm{d}k}.$$

d) Show that

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$$

with $D_\mu = \partial_\mu - iA_\mu$ is invariant under

$$\psi \mapsto e^{-i\alpha}\psi, \qquad A_\mu \mapsto A_\mu - \partial_\mu \alpha$$

with α being space-time dependent.

Problem 10.3: Time reversal

Time reversal T is an anti-linear, antiunitary operator (c.f. Srednicki 23 or Weinberg 2.6 for a discussion), i.e.,

$$T\lambda|\Phi\rangle = \bar{\lambda}T|\Phi\rangle, \qquad \langle T\Phi|T\Psi\rangle = \langle\Phi|\Psi\rangle.$$

a) Show that time reversal must fulfill

$$T^{-1}\underline{P}T = -\underline{P}, \qquad T^{-1}\underline{J}T = -\underline{J}$$

with $\underline{P}$ and $\underline{L}$ momentum and angular momentum.

b) On the creation operators, T should thus act as

$$T^{-1}b_s^\dagger(\underline{k})T = \zeta_s^b b_{-s}^\dagger(-\underline{k}), \qquad T^{-1}d_s^\dagger(\underline{k})T = \zeta_s^d d_{-s}^\dagger(-\underline{k}).$$

Show that this entails [Hint: Beware of the anti-linearity of T!]

$$T^{-1}\psi(x)T = \sum_s \int \widetilde{\mathrm{d}k} \left[(\zeta_{-s}^b)^* b_s(\underline{k}) u_{-s}^*(-\underline{k}) e^{ik\mathcal{T}x} + \zeta_{-s}^d d_s^\dagger(\underline{k}) v_{-s}^*(-\underline{k}) e^{-ik\mathcal{T}x} \right]$$

with

$$\mathcal{T}^\mu_\nu = \mathrm{diag}(-1, 1, 1, 1).$$

c) Conclude from

$$u_{-s}^*(-\underline{k}) = -s\mathcal{C}\gamma_5 u_s(\underline{k}), \qquad v_{-s}^*(-\underline{k}) = -s\mathcal{C}\gamma_5 v_s(\underline{k}) \tag{0.1}$$

that one may choose $\zeta_s^b = s = \zeta_s^\delta$ and obtain

$$T^{-1}\psi(x)T = D(\mathcal{T})\psi(\mathcal{T}x)$$

with

$$D(\mathcal{T}) = \mathcal{C}\gamma_5.$$

[You may also try to verify (0.1)]