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Relativistic Quantum Field Theory — Problem Sheet 7

3 pages — Problems 7.1 to 7.2

Problem 7.1: The complex Klein-Gordon field

The complex (or charged) Klein-Gordon field is subject to the Lagrangian

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi.$$

We treat ϕ^* and ϕ as independent parameters in the variational problem.

a) Derive the equations of motion by variation w.r.t. ϕ and ϕ^*.

b) Verify the canonical momenta

$$\pi := \frac{\delta \mathcal{L}}{\delta \partial_0 \phi} = \partial_0 \phi^*, \qquad \pi^* := \frac{\delta \mathcal{L}}{\delta \partial_0 \phi^*} = \partial_0 \phi,$$

and the Hamiltonian

$$H = \int \mathrm{d}^3x \left(\dot{\phi}^* \dot{\phi} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right).$$

Verify explicitly, using the equations of motion, that the Hamiltonian is conserved, $\dot{H} = 0$.

c) Verify that the action has a global $U(1)$ symmetry, i.e., it is invariant under

$$\phi \mapsto e^{-i\alpha} \phi, \qquad \phi^* \mapsto e^{i\alpha} \phi^*,$$

with α a constant (in space-time) real parameter. Determine the corresponding Noether current j^μ. [Hint: The general procedure to determine the Noether current corresponding to a continuous symmetry is described in Section 22 of Srednicki.] Check explicitly, using the equations of motion, that the current is conserved, $\partial_\mu j^\mu = 0$. Express the corresponding charge

$$Q = \int \mathrm{d}^3x \; j^0$$

in a form similar to the Hamiltonian. What is the most simple non-trivial interaction term (polynomial in ϕ, ϕ^* of order greater than 2) that one can add to the Lagrangian without spoiling the $U(1)$ symmetry?

d) Write the field as

$$\phi(t, \underline{x}) = \int \widetilde{\mathrm{d}k} \left(a(\underline{k}) e^{ikx} + b^*(\underline{k}) e^{-ikx} \right), \tag{0.1}$$

where

$$\widetilde{\mathrm{d}k} = \frac{\mathrm{d}^3 k}{2\omega(2\pi)^3}, \qquad \omega = \sqrt{k^2+m^2}, \qquad kx = -\omega t + \underline{k}\cdot\underline{x},$$

and a, b are complex functions. Write the Hamiltonian H and the charge Q in terms of a and b. Why can one not assume that $a = b$?

We now turn to the canonical quantization of the system.

e) Given the canonical momenta derived above, verify the canonical commutation relations

$$[\phi(t,\underline{x}),\phi(t,\underline{y})] = [\dot\phi(t,\underline{x}),\dot\phi(t,\underline{y})] = [\phi(t,\underline{x}),\dot\phi(t,\underline{y})] = [\phi(t,\underline{x}),\phi^*(t,\underline{y})] = [\dot\phi(t,\underline{x}),\dot\phi^*(t,\underline{y})] = 0,$$
$$[\phi(t,\underline{x}),\dot\phi^*(t,\underline{y})] = i\delta(\underline{x}-\underline{y}).$$

(All others are obtained by taking adjoints.) Verify that these are fulfilled if a, b in (0.1) are promoted to operators fulfilling

$$[a(\underline{k}),a^*(\underline{q})] = (2\pi)^3 2\omega\delta(\underline{k}-\underline{q}), \qquad [a(\underline{k}),b(\underline{q})] = 0,$$
$$[b(\underline{k}),b^*(\underline{q})] = (2\pi)^3 2\omega\delta(\underline{k}-\underline{q}), \qquad [a(\underline{k}),b^*(\underline{q})] = 0.$$

[You may restrict consideration to two of the first 5 canonical commutation relations.]

The two sets of creation and annihilation operators can be represented on $\mathcal{H} = \mathcal{F}_a \otimes \mathcal{F}_b$, where $\mathcal{F}_{a/b}$ are two copies of the usual bosonic Fock space known from the real scalar field. These where based on the one-particle Hilbert space $L^2(\mathbb{R}^3, \widetilde{\mathrm{d}k})$, with

$$a^*(f) = \int \widetilde{\mathrm{d}k} f(\underline{k}) a^*(\underline{k}),$$
$$(a^*(f)F)_n = \sqrt{n} P_+(f \otimes F_{n-1}),$$
$$(a(f)F)_n(\underline{k}_1,\dots,\underline{k}_n) = \sqrt{n+1}\int \widetilde{\mathrm{d}k} \bar f(\underline{k}) F_{n+1}(\underline{k},\underline{k}_1,\dots,\underline{k}_n)$$

and analogously for b, b^*. We interpret a^* as creation operators for fields and b^* as creation operators for antifields.

f) Show that, up to ordering ambiguities, the Hamiltonian is positive, and Q counts the number of particles minus the number of antiparticles.

g) Charge conjugation C should act by interchanging particles and antiparticles, i.e., by

$$C^{-1}a^*(\underline{k})C = b^*(\underline{k}), \qquad C^{-1}b^*(\underline{k})C = a^*(\underline{k}).$$

Show that this implies

$$C^{-1}\phi(x)C = \phi^*(x), \qquad C^{-1}QC = -Q.$$

Verify that C is given by the flip map $C(\Phi\otimes\Psi) = \Psi\otimes\Phi$ on $\mathcal{H} = \mathcal{F}_a\otimes\mathcal{F}_b$.

h) Verify the vacuum expectation values

$$\langle 0|\phi(x)\phi(y)|0\rangle = 0, \qquad \langle 0|\phi(x)\phi^*(y)|0\rangle = \Delta_+(x-y),$$

where

$$\Delta_+(x) = \int \widetilde{\mathrm{d}k} e^{ikx}.$$

We resume the discussion at the classical level and want to "gauge" the $U(1)$ symmetry, i.e., to transform it into a local symmetry.

i) Show that the Lagrangian

$$\mathcal{L} = -D^\mu \phi^* D_\mu \phi - m^2 \phi^* \phi.$$

where

$$D_\mu \phi = (\partial_\mu - iA_\mu)\phi, \qquad D_\mu \phi^* = (\partial_\mu + iA_\mu)\phi^*$$

is invariant under the transformation

$$\phi \mapsto e^{-i\alpha}\phi, \qquad \phi^* \mapsto e^{i\alpha}\phi^*, \qquad A_\mu \mapsto A_\mu - \partial_\mu \alpha,$$

with α now being space-time dependent.

In particular, A_μ transforms as the vector potential of electromagnetism under gauge transformations. Supplemented with the Maxwell Lagrangian, this gives rise to *scalar electrodynamics*, a toy model for electrodynamics (Dirac fields coupled to Maxwell fields).

Problem 7.2: Feynman's formula

Derive the generalization

$$\frac{1}{A_1^{\alpha_1} \dots A_n^{\alpha_n}} = \frac{\Gamma(\sum_i \alpha_i)}{\prod_i \Gamma(\alpha_i)} \int_0^1 \mathrm{d}x_1 \dots \mathrm{d}x_n \; \delta(1 - \sum_i x_i) \frac{\prod_i x_i^{\alpha_i - 1}}{(\sum_i x_i A_i)^{\sum_i \alpha_i}}$$

of Feynman's formula. Hint: Use

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty \mathrm{d}t \; t^{\alpha-1} e^{-At}$$

(the definition of Γ), put indices on α, A, and take the product. Then multiply by

$$1 = \int_0^\infty \mathrm{d}s \; \delta(s - \sum_i t_i),$$

change to variables $x_i = t_i/s$, and perform the s integral.