General Relativity Probeklausur (mock exam) PHY-M-PWF-QFG-1

- There are **tree** pages.
- Do not forget to put your **name** and **student number** on each sheet submitted.
- In order to pass this exam, you need at least **50 marks**. The maximum total number of marks is 100. The maximum number of marks for each question is: $A = 30, B = 35, C = 35$. Make sure to present your arguments in a coherent fashion. Illegible parts of the script can be ignored.

Question A)

1. Define the Riemann tensor $R_{\alpha\beta\gamma}{}^{\delta}$ of a covariant derivative ∇_μ (without writing explicitly the Christoffel symbol).
2. State what it means for a stress tensor $T_{\mu\nu}$ to satisfy the "strong energy condition".
3. State Einstein's equation.
4. For a flat FLRW metric, $g = -dt^2+a(t)^2(dx^2+dy^2+dz^2)$, derive a condition on $a(t)$ determining when this spacetime has "particle horizons" (sometimes also called "cosmological horizons"). Supplement your explanation with a rough sketch.
5. A conformal Killing tensor $K_{\mu_1\ldots\mu_n}$ of rank $n > 1$ by definition satisfies the identity $\nabla_{(\nu}K_{\mu_1\ldots\mu_n)} = g_{(\nu\mu_1}f_{\mu_2\ldots\mu_n)}$ for some tensor field $f_{\mu_1\ldots\mu_{n-1}}$. (Parenthesis around the tensor indices indicate symmetrization.) Show that a conformal Killing tensor gives rise to a constant of motion along any null-geodesic.

Question B) The analogue of the Schwarzschild metric in 4+1 dimensions is given by

$$g = -(1-\mu/r^2)dt^2 + \frac{dr^2}{1-\mu/r^2} + r^2 d\Omega^2_{(3)} \ , \tag{1}$$

where $d\Omega^2_{(3)}$ is the maximally symmetric metric on the 3-sphere. It is given in spherical polar coordinates by

$$d\Omega^2_{(3)} = d\theta_2^2 + \sin^2\theta_2(d\theta_1^2 + \sin^2\theta_1 \ d\phi^2) \ . \tag{2}$$

The spacetime coordinates are denoted in the following as $(x^\mu) = (t, r, \theta_1, \theta_2, \phi)$. It is assumed throughout that $\mu > 0$ and that $r > \sqrt{\mu}$.

1. Show by direct calculation or otherwise that the following vector field X is Killing ($\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0$) for the metric (1):

$$X = \cos\theta_1 \frac{\partial}{\partial\theta_2} - \sin\theta_1 \cot\theta_2 \frac{\partial}{\partial\theta_1} \tag{3}$$

   You may use the identities given below under "helpful formulas". Hint: Before doing any calculations, first consider which Christoffel symbols you actually need to know. Without doing an explicit calculation, argue that also $\partial/\partial t$ and $\partial/\partial\phi$ are Killing vector fields for (1).

2. Argue that (1) admits three commuting Killing vector fields.

3. We consider a geodesic curve, denoted $(\gamma^\mu(s)) = (t(s), r(s), \theta_1(s), \theta_2(s), \phi(s))$, where an overdot means d/ds. Give an argument why it is consistent and no loss of generality to assume that $\theta_1(s) = \pi/2 = \theta_2(s)$. (This is assumed in the following.)

4. Show that $E = (1 - \mu/r^2)\dot{t}$, $L = r^2\dot{\phi}$ are conserved along the curve. What is the physical interpretation of these?

5. Show that $\dot{r}$ must satisfy $\dot{r}^2/2 + V(r) = E^2/2$, where $V(r)$ is an effective potential that you are asked to calculate in the case that the geodesic is timelike.

6. Using the previous question, show that there are no bound stable orbits for any value of L, E (a bound orbit is one where the value of r is bounded from above for all s).

7. Briefly compare your results from part 6.) to the case of particle motion in the Schwarzschild metric in *3+1 dimensions*: Is the situation qualitatively the same or different?

Question C) Consider the time-dependent mass distribution

$$T^{00}(t,x,y,z) = \frac{M}{L}\delta(z)\int_0^{L/2} d\lambda[\delta(x-\lambda\cos(\Omega t))\delta(y-\lambda\sin(\Omega t)) + \delta(x+\lambda\cos(\Omega t))\delta(y+\lambda\sin(\Omega t))], \tag{4}$$

where $M > 0, L > 0, \Omega$ are constants.

1. Identify the constants M, L, Ω by their physical meaning.

2. The quadrupole, and the reduced quadrupole tensors are defined as

$$q_{ij} = 3\int T^{00} x_i x_j d^3x \ , \quad Q_{ij} = q_{ij} - (1/3)\delta_{ij}\sum_{k=1}^{3} q_{kk}, \tag{5}$$

   where $(x_i) = (x, y, z)$. Compute these for (4). (Hint: Perform the d^3x-integral first, the $d\lambda$-integral next.)

3. The power of gravitational radiation emitted at time t is given by $(G_N = c = 1)$

$$P = \frac{1}{45} \sum_{i,j=1}^{3} \dddot{Q}_{ij}(t-R)^2 \ , \tag{6}$$

   where R is the distance to the source. Derive a formula for P in the present example using the results from part 2.

4. Use dimensional anaysis to reinstate G_N, c into your formula for P.

5. Taking $c = 3 \times 10^{10}\mathrm{cm\ s}^{-1}, G_N = 7 \times 10^{-8}\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2}, \Omega = 1 \text{ rad s}^{-1}, M = 1 \text{ kg}, L = 1 \text{ m}$, give an order of magnitude estimate for P (in Watts).

Helpful formulas: You may use the following expression for the covariant derivative associated with a metric $g_{\alpha\beta}$:

$$\nabla_\alpha v_\beta = \partial_\alpha v_\beta - \Gamma^\mu_{\alpha\beta} v_\mu \ , \quad \Gamma^\beta_{\alpha\mu} = -\frac{1}{2} g^{\beta\nu}(\partial_\nu g_{\alpha\mu} - \partial_\mu g_{\alpha\nu} - \partial_\alpha g_{\mu\nu}). \tag{7}$$