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General Relativity
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Homework sheet - 9

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20. In this question, we investigate the isometries of hyperbolic space, i.e. the manifold $\mathbb{H}^2 = \{(x,y) \mid y > 0\}$ with the metric $g = (dx^2 + dy^2)/y^2$, see also HW7. Recall that a diffeomorphism ψ on an n-dimensional manifold is called "isometry" if $\psi^* g = g$, or in coordinates, if

$$g_{\mu\nu}(x) = g_{\alpha\beta}(\psi(x)) \frac{\partial \psi_\alpha(x)}{\partial x_\mu} \frac{\partial \psi_\beta(x)}{\partial x_\nu} \tag{1}$$

where $x = (x_1, \dots, x_n)$ and $\psi(x) = (\psi_1(x), \dots, \psi_n(x))$ are the coordinate components, and where a sum over α, β is understood. For the case of hyperbolic space, let $x_1 = x, x_2 = y$, let $z = x + iy$, and define ψ through

$$\psi_1 = \mathrm{Re} \frac{az+b}{cz+d}\,, \quad \psi_2 = \mathrm{Im} \frac{az+b}{cz+d} \tag{2}$$

where a, b, c, d are given real numbers such that $ad - bc = 1$.

(i) Verify that ψ is an isometry of hyperbolic space.

(ii) Denote by A the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3}$$

so that $\det A = 1$. Let $\psi^{(A)}$ be the isometry defined by A as in eq. (**??**). Show that

$$\psi^{(A)} \circ \psi^{(A')} = \psi^{(AA')},$$

where A' is similarly defined in terms of a', b', c', d', and where AA' is the product of matrices. What do we learn from this about the isometry group of $\mathbb{H}^2$?

21. (Killing vectors)

(i) Recall that a Killing vector field ξ^μ is defined by the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0,$$

where $\xi_\mu = g_{\mu\nu} \xi^\nu$. Here, ∇ is as usual the Levi-Civita connection; for hyperbolic space $\mathbb{H}^2$, the corresponding connection coefficients $\Gamma^\alpha_{\mu\nu}$ were given in HW7. Show that the following three vector fields are Killing in hyperbolic space:

$$L_1 = \frac{\partial}{\partial x}, \quad L_2 = (y^2 - x^2)\frac{\partial}{\partial x} - 2xy\frac{\partial}{\partial y}, \quad L_3 = 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial x}. \tag{4}$$

(Recall that x, y refer to the coordinates of hyperbolic space, and that e.g. $\partial/\partial x$ stands for the vector field which in these coordinates has components $(1, 0)$, whereas $\partial/\partial y$ has components $(0, 1)$.) Argue that there cannot be any further, linearly independent, Killing vector fields.

(ii) Show that if ξ^μ and η^μ are two Killing fields, then their commutator, $[\xi, \eta]^\mu \equiv \xi^\alpha \nabla_\alpha \eta^\mu - \eta^\alpha \nabla_\alpha \xi^\mu$ also is Killing. Calculate the commutators of the vector fields in (i). What do you observe?

(iii) For some metric on a manifold $(M, g_{\alpha\beta})$, suppose that ξ^μ is Killing, and suppose that $\gamma(t)$ is a geodesic curve. Show that the function $f(t) = \dot{\gamma}^\mu(t)\xi_\mu(\gamma(t))$ is constant. Hint: you may start the calculation by $\dot{f} = \dot{\gamma}^\alpha \nabla_\alpha f = \ldots$. Thus, Killing vector fields give rise to constants of motion for the geodesic equation.

(iv) Now let $\gamma(t) = (x(t), y(t))$ be a curve in hyperbolic space, so that $\dot{\gamma} = (\dot{x}, \dot{y})$. Use the constants of motion f_1, f_2, f_3 obtained from L_1, L_2, L_3 to determine all possible geodesics in hyperbolic space.