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General Relativity
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Homework sheet - 8

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29. In this question, we investigate the structure of the stress energy tensor in some important examples.

(i) For the Maxwell field, the stress tensor is defined as

$$T_{\mu\nu} = -F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \ , \tag{1}$$

where $F_{\mu\nu}$ is the *field strength tensor*. In Minkowski space with inertial coordinates $(x^\mu) = (t, x, y, z)$, it is composed of the electric and magnetic fields as

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{2}$$

where $\vec{E}, \vec{B}$ are interpreted as the electric/magnetic fields for the observers following the curve with tantent $(u^\mu) = (1, 0, 0, 0)$. In terms of $F_{\mu\nu}$ the homogeneous Maxwell equations are

$$\nabla^\mu F_{\mu\nu} = 0 \ , \quad \nabla_{[\mu}F_{\alpha\beta]} = 0 \ . \tag{3}$$

In Minkowski space with metric $(g_{\mu\nu}) = \mathrm{diag}(-1, +1, +1, +1)$, recover the usual (source free) Maxwell equations in terms of $\vec{E}, \vec{B}$ by setting $\nabla_\mu \to \partial_\mu$ (inertial coordinates)!

(ii) Show that the energy density, stresses, and flux relative to u^μ are given by (here lower case Roman letters are $i, j = 1, 2, 3$ and label the x, y, z components)

$$\rho \equiv T_{00} = \frac{1}{2}(|\vec{E}|^2 + |\vec{B}|^2) \tag{4}$$

$$S_{ij} \equiv T_{ij} = -E_iE_j - B_iB_j + \delta_{ij}\rho \tag{5}$$

$$S_i \equiv T_{0i} = (\vec{E} \times \vec{B})_i. \tag{6}$$

$\vec{S}$ is also called the *Poynting vector*. It is a measure for the flux of electromagnetic energy as seen by the inertial observer. Write out the conservation equations following for ρ, S_i, T_{ij} by considering separately the cases $\nu = 0$ and $\nu = i$ in $\partial^\mu T_{\mu\nu} = 0$.

(iii) Show that the stress tensor is conserved, $\nabla^\mu T_{\mu\nu} = 0$, using the Maxwell equations (3).

(iv) For a pressureless dust fluid, the stress tensor has the form $T_{\mu\nu} = \rho u_\mu u_\nu$, where ρ is the energy density, whereas u^μ is a unit timelike vector field (meaning $u_\alpha u^\alpha = -1$) tangent to the spacetime trajectories of the dust particles. Show that the conservation law for this stress tensor is equivalent to the equations

$$u^\mu \nabla_\mu u^\nu = 0 \ , \quad \nabla^\mu(\rho u_\mu) = 0 \ .$$

Say in words what is the meaning of the first, resp. second equation!

(v) If we allow for pressure, the stress tensor of the fluid becomes $T_{\mu\nu} = \rho u_\mu u_\nu + P(g_{\mu\nu} + u_\mu u_\nu)$. Consider now Minkowski spacetime, and write $(u^\mu) = ((1 + |\vec{v}|^2)^{1/2}, \vec{v})$. For a non-relativistic fluid, we can assume that $|\vec{v}| \ll 1$, that $P \ll \rho$ and $v\partial_t P \ll |\vec{\partial} P|$ (note that the speed of light is $c = 1$ in our units). By taking $\nu = 0$ resp. $\nu = i$, show that under these approximations, the conservation equation $\partial^\mu T_{\mu\nu} = 0$ implies the conservation of mass and Euler equations:

$$\begin{aligned} &\partial_t \rho + \vec{\partial} \cdot (\rho \vec{v}) = 0 \\ &\rho\{\partial_t + (\vec{v} \cdot \vec{\partial})\}\vec{v} = -\vec{\partial} P. \end{aligned}$$

(vi) In Minkowski space, a *plane wave* is a solution of the form

$$A_\mu = \epsilon_\mu(\vec{k}) \sin(\omega t + \vec{k}\vec{x})$$

where ϵ_μ corresponds to the polarization of the wave, which has to satisfy $k^\mu \epsilon_\mu = 0$, where $(k_\mu) = (\omega, \vec{k})$, and where A_μ comprises both the electric and magnetic potentials, $(A_\mu) = (-\phi, \vec{A})$. (You may check that the field strength tensor is then $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, which automatically solves the second Maxwell equation (3).)

a) Show that $\partial_\mu A^\mu = 0$, and, using the first Maxwell equation, show that

$$\partial_\mu \partial^\mu A_\alpha = 0 \ .$$

Deduce that $k_\mu k^\mu = 0$, and whence that $\omega^2 = |\vec{k}|^2$.

b) Using eq. (1), show that the stress tensor for the wave is given by

$$T_{\mu\nu} = a^2 k_\mu k_\nu \cos^2(\omega t + \vec{k}\vec{x}) \ , \tag{7}$$

with $a^2 = \epsilon^\mu \epsilon_\mu$ the squared amplitude of the wave (why is this non-negative?)! c) Show that the time-average of this stress tensor is

$$\langle T_{\mu\nu} \rangle_t = \frac{1}{2} a^2 k_\mu k_\nu \ .$$

d) Now write $(k_\mu) = (\omega, \omega\vec{n})$ where $\vec{n}$ is a 3-dimensional vector of unit length indicating the direction in which the wave moves. We additionally take the average with respect to this vector, integrating $\vec{n}$ over the unit 2-dimensional sphere $\mathbb{S}^2$ and dividing by the total solid angle of 4π. To do this, use the identities

$$\int_{\mathbb{S}^2} n^i d^2\Omega = 0 \ , \quad \int_{\mathbb{S}^2} n^i n^j d^2\Omega = \frac{4\pi}{3} \delta^{ij} \ ,$$

and assume that a is direction independent. Show that, in components,

$$\langle T_{\mu\nu} \rangle_{t,\vec{n}} = \frac{a^2 \omega^2}{2} \operatorname{diag}\left(1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) . \tag{8}$$

Show that this has the same form as the stress tensor of a fluid with "equation of state" $P = \rho/3$ (and $(u^\mu) = (1, 0, 0, 0)$). The fact that space filled with plane waves in random directions carries a stress tensor of a perfect fluid with this equation of state plays a role in cosmology.