Prof. Stefan Hollands
Institut für Theoretische Physik
Universität Leipzig

General Relativity
Summer semester 2016
Homework sheet - 13

---

26. (Rigidly rotating rod) The purpose of this exercise is to work out the stress tensor of a rotating rod in Minkowski spacetime. The rod is assumed to rotate in the (x, y) plane about the z-axis, with angular velocity ω.

(a) We first introduce (space-fixed) cylindrical coordinates (t, r, z, ϕ) about the z-axis. Show that the Minkowski line element $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ becomes

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2. \tag{1}$$

(b) Next we introduce co-rotating cylindrical coordinates $\hat{t} = t$, $\hat{r} = r$, $\hat{z} = z$, $\hat{\phi} = \phi + \omega t$. Show that the line element becomes

$$\mathrm{d}s^2 = -\left(1 - \omega^2 \hat{r}^2\right)\mathrm{d}\hat{t}^2 - 2\hat{r}^2\omega\,\mathrm{d}\hat{\phi}\,\mathrm{d}\hat{t} + \mathrm{d}\hat{r}^2 + \hat{r}^2\,\mathrm{d}\hat{\phi}^2 + \mathrm{d}\hat{z}^2 \tag{2}$$

(c) Finally, we introduce co-rotating (body fixed) 'cartesian' coordinates $\hat{x} = \hat{r}\cos\hat{\phi}$, $\hat{y} = \hat{r}\sin\hat{\phi}$. Show that

$$\mathrm{d}s^2 = -\left(1 - \omega^2 \hat{r}^2\right)\mathrm{d}\hat{t}^2 - 2\omega\left(\hat{x}\,\mathrm{d}\hat{y} - \hat{y}\,\mathrm{d}\hat{x}\right)\mathrm{d}\hat{t} + \mathrm{d}\hat{x}^2 + \mathrm{d}\hat{y}^2 + \mathrm{d}\hat{z}^2. \tag{3}$$

(d) In the rest frame of the rod (co-rotating coordinates), the rod is assumed to be of length ℓ. Without loss of generality, we can assume that it points in the $\hat{x}$ direction, and to simplify matters, we assume that it is reflection symmetric. We make the following ansatz for the stress tensor describing such a rod:

$$T^{\hat{t}\hat{t}} = \rho(|\hat{x}|)\Theta(\ell - |\hat{x}|)\delta(\hat{y})\delta(\hat{z})\,, \tag{4a}$$

$$T^{\hat{x}\hat{x}} = -\sigma(|\hat{x}|)\Theta(\ell - |\hat{x}|)\delta(\hat{y})\delta(\hat{z})\,, \tag{4b}$$

and all other components are zero. Transform this back to cylindrical non-rotating coordinates and show that

$$T^{\hat{t}\hat{t}} = \rho(\hat{r})\Theta(\ell - \hat{r})\frac{\delta(\hat{\phi} \mod \pi)}{\hat{r}}\delta(\hat{z})\,, \tag{5a}$$

$$T^{\hat{r}\hat{r}} = -\sigma(\hat{r})\Theta(\ell - \hat{r})\frac{\delta(\hat{\phi} \mod \pi)}{\hat{r}}\delta(\hat{z})\,, \tag{5b}$$

$$\tag{5c}$$

where

$$\delta(\hat{\phi} \mod \pi) = \sum_{k \in \mathbb{N}} \delta(\hat{\phi} + k\pi) = \delta(\sin\hat{\phi})\,, \tag{6}$$

and where all other components are again zero.

(e) Transform the result now further back to the original, non-rotating (space-fixed) cylindrical coordinates (t, r, z, ϕ), and show that

$$T^{tt} = \rho(r)\Theta(\ell - r)\frac{\delta(\phi + \omega t \mod \pi)}{r}\delta(z)\,, \tag{7a}$$

$$T^{t\phi} = -\omega\rho(r)\Theta(\ell - r)\frac{\delta(\phi + \omega t \mod \pi)}{r}\delta(z)\,, \tag{7b}$$

$$T^{rr} = -\sigma(r)\Theta(\ell - r)\frac{\delta(\phi + \omega t \mod \pi)}{r}\delta(z)\,, \tag{7c}$$

$$T^{\phi\phi} = \omega^2\rho(r)\Theta(\ell - r)\frac{\delta(\phi + \omega t \mod \pi)}{r}\delta(z)\,. \tag{7d}$$

(f) Based on the formula of the previous item (or otherwise), show that conservation $\partial_\mu T^{\mu\nu} = 0$ (here ∂_μ is the covariant derivative of Minkowski spacetime, which is equal to the ordinary partial derivative in non-rotating cartesian coordinates (t, x, y, z), but which has a non-zero Christoffel symbol in the coordinates (t, r, z, ϕ). You may either compute these, or first further transform the stress tensor to the coordinates (t, x, y, z).) is equivalent to the single condition

$$\sigma(r) = \omega^2 \int_r^\ell r'\rho(r')\,\mathrm{d}r'\,. \tag{8}$$

(g) What is the physical interpretation of σ and ρ?

(h) What is the form of the stress tensor the special case $\rho(r) = \rho_0$ for $r < \ell$ and $\rho(r) = 0$ for $r \geq \ell$?

(i) Work out the quadrupole tensor for the stress tensor in the previous item. You must work in cartesian (space-fixed) coordinates.