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General Relativity
Summer semester 2016
Homework sheet - 6

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16. **Covariant derivatives on $\mathbb{S}^1$**

Consider the manifold $\mathbb{S}^1$ and a connection ∇ on it. Let $\varphi \in (-\pi, \pi)$ be a (angular) coordinate on part of $\mathbb{S}^1$ and let ∂_φ denote the coordinate derivative.

Recall that we may also view ∂_φ as a coordinate basis vector field, and we write $\mathrm{d}\,\varphi$ for the dual coordinate basis vector field. Because the manifold is one-dimensional, any covariant derivative ∇ must act on ∂_φ by

$$(\nabla \partial_\varphi) = \alpha \ \mathrm{d}\,\varphi \otimes \partial_\varphi$$

for some smooth function $\alpha(\varphi)$.

(a) Show that in the given local coordinate we have for any vector field $v = v^\varphi \partial_\varphi$ that

$$\nabla v = (\partial_\varphi v^\varphi + \alpha v^\varphi)\mathrm{d}\,\varphi \otimes \partial_\varphi.$$

(b) Write down the Christoffel symbol of the connection ∇ in the coordinate φ. Express your answer in terms of the function α.

(c) Now let $\omega = \omega_\varphi \mathrm{d}\,\varphi$ be a dual vector field. Show that

$$\nabla \omega = (\partial_\varphi \omega_\varphi - \alpha \omega_\varphi)\mathrm{d}\,\varphi \otimes \mathrm{d}\,\varphi.$$

(d) Show that the Riemann curvature of ∇ vanishes, i.e. $\nabla_\mu \nabla_\nu \omega_\rho = \nabla_\nu \nabla_\mu \omega_\rho$. (Hint: You may use the expression of the Riemann curvature in terms of the Christoffel symbol, or use the symmetries of the Riemann curvature.)

(e) Now let p be the point where $\varphi = 0$ and let $v \in T_p\mathbb{S}^1$ be the vector with $v^\varphi = 1$. Let $c : (-\pi, \pi) \to \mathbb{S}^1$ be the parametrised curve in $\mathbb{S}^1$ given by $c(\varphi) := (\cos(\varphi), \sin(\varphi))$. Find the parallel transports of v along c w.r.t. ∇ at any point on the curve.

(f) The parallel transports define a vector field on the image of the curve c. We denote this vector field again by v. Consider a change of parameter, i.e., a bijective function $\phi(s)$ and a curve $\xi(s)$ such that $\xi(s) = c(\phi(s))$. What are the parallel transports of v along ξ ?

(g) At $\varphi = 0, \dot{c}(0) = v$. Under what condition on α is the same true at every parameter value φ ?

(h) Given a connection with $\alpha = 1$ everywhere, can you find a curve $\xi(s)$ in $\mathbb{S}^1$ such that $\xi(0) = p$, and $\dot{\xi}(s) = v$ everywhere (i.e., such that the tangent vector to ξ is its own parallel transport along the curve) ?

(i) We have investigated the vector field v on the chart domain of the coordinate φ. We now wish to consider its extensions to the entire manifold. Show that v can be extended in a smooth way to the entire manifold $\mathbb{S}^1$ iff $\int_{-\pi}^{\pi} \alpha(\varphi)\mathrm{d}\,\varphi = 0$.

(j) We have seen that locally, in achart domain, the parallel transport of v to another point is independent of the choice of curve (This is because the Riemannian curvature tensor vanishes). Does the same conclusion hold globally, when the curves may not remain in a single chart ?

17. **Geodesics on the round sphere**

We consider the unit sphere $\mathbb{S}^2$ as a subset of $\mathbb{R}^3$. The Euclidean metric g^e_{ab} on $\mathbb{R}^3$ can be restricted to tangent vectors of $\mathbb{S}^2$, leading to a Riemannian metric g_{ab} on $\mathbb{S}^2$. This metric is invariant under rotations and one often uses the term *round sphere* for the Riemannian manifold $(\mathbb{S}^2, g_{ab})$.

(a) Use spherical coordinates to find expressions for $g^e_{\mu\nu}$ and $g_{\mu\nu}$.

(b) Consider a curve $s \mapsto \xi(s)$ in the round sphere. How does the geodesic equation for the Levi-Civita derivative read in spherical coordinates?

(c) Find all geodesics on the round sphere that connect the two opposite points on the equator $(\theta, \phi) = (\frac{1}{2}\pi, 0)$ and $(\theta, \phi) = (\frac{1}{2}\pi, \pi)$.
(Hint: Use symmetries to argue that we may choose $\dot{\phi} = 0$ initially. Use the geodesic equation to show that $\sin^2(\theta)\dot{\phi}$ is constant along the curve and then that $\ddot{\theta} = 0$. Take care when you cross the north or south pole.)

(d) Find all geodesics on the round sphere that connect the north pole to a fixed point on the equator. Which geodesic has the shortest length?