On the construction of Hartle-Hawking-Israel states across a static bifurcate Killing horizon

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Abstract

We consider a linear scalar quantum field propagating in a spacetime of dimension $d \ge 2$ with a static bifurcate Killing horizon and a wedge reflection. Under suitable conditions (e.g. positive mass) we prove the existence of a pure Hadamard state which is quasi-free, invariant under the Killing flow and restricts to a double β_H -KMS state on the union of the exterior wedge regions, where β_H is the inverse Hawking temperature.

The existence of such a state was first conjectured by Hartle and Hawking (1976) and by Israel (1976), in the more general case of a stationary black hole spacetime. Jacobson (1994) has conjectured a similar state to exist even for interacting fields in spacetimes with a static bifurcate Killing horizon. The state can serve as a ground state on the entire spacetime and the resulting situation generalises that of the Unruh effect in Minkowski spacetime.

Our result complements a well-known uniqueness result of Kay and Wald (1991) and Kay (1993), who considered a general bifurcate Killing horizon and proved that a certain (large) subalgebra of the free field admits at most one Hadamard state which is invariant under the Killing flow. This state is pure and quasi-free and in the presence of a wedge reflection it restricts to a β_H -KMS state on the smaller subalgebra associated to one of the exterior wedge regions. Our result establishes the existence of such a state on the full algebra, but only in the static case.

Our proof follows the arguments of Sewell (1982) and Jacobson (1994), who exploited a Wick rotation in the Killing time coordinate to construct a corresponding Euclidean theory. In particular we show that for the linear scalar field we can recover a Lorentzian theory by Wick rotating back. Because the Killing time coordinate is ill-defined on the bifurcation surface we systematically replace it by a Gaussian normal coordinate. A crucial part of our proof is to establish that the Euclidean ground state satisfies the necessary analogues of analyticity and reflection positivity with respect to this coordinate.

1 Introduction

The equations that describe black hole physics have an uncanny similarity to the laws of thermodynamics. This fact was gradually realised in the 1970s, starting with the black hole area law and culminating in Hawking's discovery of black hole radiation [3, 21, 22, 23, 24, 56]. In the past few decades, much work has been devoted to investigating the fundamental physics that underlies these striking similarities, which go under the name of black hole thermodynamics [58].

A closely related research effort has developed a rigorous mathematical framework to describe quantum field theory (QFT) in general curved spacetimes in a generally covariant way [7, 25, 26].

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Largely motivated by the desire to formulate the questions of black hole thermodynamics in a precise and general setting, this area of research also has ramifications for our wider understanding of QFT and its interaction with gravity. One of the main breakthroughs in the development of generally covariant QFT was the introduction of microlocal analysis as a mathematical tool to study and characterise the singularities of *n*-point distributions of a quantum field [47]. This has led to an easier and more illuminating characterisation of the important class of Hadamard states.

In this paper, we will take advantage of these insights in QFT on curved spacetimes and apply them to one of the questions of black hole thermodynamics. Our goal is to prove the existence of a ground state for a linear scalar quantum field that propagates in a spacetime with a static black hole (or a generalisation thereof). The existence of the ground state in question was first conjectured by Hartle and Hawking [20] and by Israel [31]. They used a Wick rotation to argue that a ground state on a stationary black hole spacetime can be defined by analytic continuation from a Euclidean fundamental solution on a corresponding Riemannian manifold. Whereas Hartle and Hawking were mostly interested in this state on the physical exterior region of the black hole spacetime, Israel discussed its extension to the wedge region on the other side of the black hole. A mathematically rigorous construction of this so-called Hartle-Hawking-Israel (HHI) state in the exterior regions of Kruskal spacetime was given by Kay [35].

The HHI state was introduced to help understand the phenomenon of black hole radiation [20]. The black hole spacetime can be used to describe the end state of the collapse of a massive object and one assumes, for the sake of argument, that the quantum field will settle down in the HHI state, which is the ground state. The fact that the restriction of the HHI state to the physical exterior wedge is a thermal state at the Hawking temperature could then be interpreted as the existence of Hawking radiation. In this way, the HHI state establishes an interesting connection between black hole geometry and thermality at the Hawking temperature. Moreover, the model is much simpler than the more realistic description of [23, 56, 13], which describes the collapsing matter as a dynamical process. Unfortunately, this argument is an oversimplification, as pointed out by Kay and Wald [37]. It is sometimes difficult to imagine how a quantum field can settle down in the HHI state by any physical process. In Kruskal spacetime, for example, the HHI state shows a very high degree of correlation between the thermal radiation coming in from past infinity and the state inside the white hole region. Conversely, the absence of a HHI state would not invalidate the analysis of [23, 56, 13] that a black hole radiates thermally. To understand black hole radiation as a dynamical process, one needs a more suitable state, such as the Unruh state [55, 10]. Moreover, there are recent and reliable results indicating that Hawking radiation (as measured at future null infinity) is a global consequence of a local physical phenomenon (cf. [42]). The global arguments involving the HHI state do not seem appropriate (or even adequate) to address such local questions.

A further issue with the simplified model, which potentially undermines its accuracy as a physical approximation, is the question whether the effect of the quantum field on the metric can be neglected. In the light of the semi-classical Einstein equation, one can justify this approximation by showing that the expected (renormalised) energy density in the HHI state remains bounded, so that large back reaction effects are avoided. In the exterior regions this follows from the fact that the HHI state is Hadamard (together with the generally covariant Hadamard regularisation scheme, cf. [25]). If the state is also Hadamard near the horizon, or even just near the bifurcation surface, then this remains true throughout the future and past regions, due to the propagation of singularities (cf. [47, 12]). However, the analysis near the black hole horizon is more complicated.

The question whether the HHI state can be extended across the horizon of a black hole spacetime was first addressed in a seminal paper by Kay and Wald [37] (see also [36] for an improved result). This paper is remarkable, not only because of the uniqueness theorem that it proves, but also because the assumptions of this theorem forced the authors to introduce and refine several important notions. This includes the definition of global Hadamard states and a criterion when a quasi-free Hadamard state is pure. Furthermore, they gave a general description of the class of spacetimes with a bifurcate Killing horizon (see also [6]), which includes the non-extremal stationary black holes as well as Minkowski spacetime with the Killing field of constantly accelerated observers (as it appears in the Unruh effect [55]). The main result for a spacetime with a bifurcate Killing horizon is that a certain subalgebra of the free field algebra admits at most one state which is invariant under the Killing field and Hadamard across the Killing horizon. Moreover, if the spacetime admits a wedge reflection, then the restriction of this state to the physical exterior wedge is a thermal (KMS) state at the Hawking temperature.

Unfortunately, the existence of such a state was not proved in [37]. Besides, at a more technical level, the specification of the subalgebra featuring in the uniqueness result is somewhat subtle, as it involves the initial value problem of the Klein-Gordon equation on a null hypersurface (the so-called Goursat, or characteristic Cauchy, problem). The null hypersurface in question is a part of the Killing horizon, \mathfrak{h}_A , and Kay and Wald consider solutions on the spacetime M whose restriction to \mathfrak{h}_A is a given test-function $f \in C_0^{\infty}(\mathfrak{h}_A)$. For the existence and uniqueness of such solutions, they refer to results and techniques in [14] and they recognise in a note added in proof that such solutions may fail to be smooth across \mathfrak{h}_A . Unfortunately, the only results proved in [14] are of a local nature and they apply only to null hypersurfaces which are the future null cones of $\partial J^+(p)$ of some point p. It is to be expected that these shortcomings can be overcome by a more detailed analysis of the Goursat problem, e.g. along the lines of Hörmander's remark [27], which seems to have gone unnoticed in much of the mathematical physics literature. Such a more detailed analysis could also help to further substantiate the claim of [37] that these solutions always generate a large subalgebra of the Weyl algebra (see also footnote 2 on page 5).

Making use of the notions and results of Kay and Wald, Jacobson [32] has argued that the original construction of HHI states via a Wick rotation should work even across a bifurcate Killing horizon, at least if this Killing horizon is static. Moreover, this construction should also work for interacting QFT's. Earlier, Sewell had advanced similar arguments to define the HHI state for interacting theories on the physical exterior wedge only [51, 52]. In his sketch of a proof Jacobson constructs a Euclidean theory on the associated Riemannian manifold using path integral methods. He points out several properties of the geometry that make it plausible that this theory can be Wick rotated back to define a Lorentzian theory with a ground state. However, some doubt is cast on this claim by the fact that the analytic continuation is defined in terms of the Killing time coordinate, which is ill-defined at the bifurcation surface. A detailed investigation near the bifurcation surface is therefore necessary.

The purpose of this paper is to provide a mathematically complete and rigorous construction of the HHI state for a linear scalar field, along the lines set out by Jacobson. We will systematically replace the Killing time coordinate by a Gaussian normal coordinate and we establish that the Euclidean fundamental solution G satisfies the necessary analogues of analyticity and reflection positivity with respect to this coordinate. This will lead to an HHI state, which we show to be pure, invariant under the Killing flow and to restrict to a double β_H -KMS state in the exterior wedge regions. At present, it is unclear whether our existence proof extends to (perturbatively) interacting theories, e.g. using the arguments of [19]. We will not investigate this question in detail, nor will we consider fields with spin.

In general, analyticity of G in the Gaussian normal coordinate may only hold in an infinitesimal sense. By this we mean that the Cauchy Riemann equations hold only when restricted to a hypersurface Σ , which can be identified as a Cauchy surface for the Lorentzian spacetime. It follows that the HHI state cannot be defined directly by analytic continuation in the Gaussian normal coordinate, but we can use the Euclidean fundamental solution G to define initial data on the Cauchy surface, which in turn define the HHI state. Similarly, the Hadamard property for the HHI state across the Killing horizon does not follow from the fact that it is a boundary value of an analytic function, but it must be established by investigating the initial data on the Cauchy surface Σ . For this reason we have included detailed results on the comparison between the geometry and the Hadamard construction of both the Lorentzian spacetime and its Riemannian counterpart near the surface Σ .

Our paper is organised as follows. In Section 2 we collect all the geometric results that we need, including the analytic continuation. In Section 3 we review the necessary theory of the linear scalar field and its Wick rotation w.r.t. the Killing time parameter, which leads to double β -KMS states on the exterior wedges. Section 4 contains the details of the Hadamard construction in both the Lorentzian and the Euclidean setting. One technical lemma has been deferred to

appendix A. Section 5 combines all these ingredients to prove the existence of the HHI state across the Killing horizon and to establish its main properties, namely its purity, invariance and the β_H -KMS restriction.

2 Geometric results

A careful study of the behaviour of a quantum field near a bifurcate Killing horizon requires a detailed understanding of the differential geometry of the underlying spacetime. It is the purpose of this section to introduce the class of spacetimes that we shall study and to present their relevant features, referring the reader to the literature for proofs of known results. Because our spacetimes of interest often have an exterior region which is stationary or standard static, we refer in particular to the review [50], which describes thermal states for such spacetimes.

Our main technical tool for the purposes of this paper is contained in Subsection 2.3, where we confront the problem that the Killing time coordinate, used to define the analytic continuation in the static case, breaks down at the bifurcation surface. We circumvent this problem by introducing Gaussian normal coordinates near a suitable Cauchy surface and by proving that all the relevant geometric quantities satisfy a certain infinitesimal version of the Cauchy-Riemann equations w.r.t. these coordinates. In addition, we consider Riemannian normal coordinates, which are used to obtain the simplest coordinate expression for the Hadamard series, and we express them in terms of the Gaussian normal coordinates. These technical results will be crucial when showing that a double β -KMS state at the Hawking temperature can be extended as a Hadamard state across the Killing horizon.

Throughout this paper, we will use the following standard terminology:

Definition 2.1 By a spacetime $M = (\mathcal{M}, g_{ab})$ we will mean a smooth, oriented manifold \mathcal{M} of dimension $d \geq 2$ with a smooth Lorentzian metric g_{ab} of signature $(-+ \ldots +)$.

A Cauchy surface Σ in M is a subset $\Sigma \subset M$ that is intersected exactly once by every inextendible timelike curve in M. A spacetime is said to be globally hyperbolic when it has a Cauchy surface Σ .

We adopt the convention that a spacetime is also connected, unless stated otherwise. We are mainly interested in globally hyperbolic spacetimes and we will only consider Cauchy surfaces that are smooth, spacelike hypersurfaces [4]. A globally hyperbolic spacetime is automatically time orientable and we will always assume a choice of time orientation has been fixed. It follows that any Cauchy surface Σ inherits a natural orientation. We let h_{ij} denote the Riemannian metric on Σ induced by the Lorentzian metric g_{ab} on M.

2.1 Spacetimes with a bifurcate Killing horizon

We start with the definition of the class of spacetimes that we will consider and that encompasses in particular the most common models of black holes.

Definition 2.2 A spacetime with a bifurcate Killing horizon is a triple $M = (\mathcal{M}, g_{ab}, \xi^a)$ such that

- 1. (\mathcal{M}, g_{ab}) is a globally hyperbolic spacetime,
- 2. ξ^a is a smooth, complete Killing vector field,
- 3. $\mathcal{B} := \{x \in M | \xi^a(x) = 0\}$ is a (not necessarily connected), orientable, (d-2)-dimensional smooth submanifold of \mathcal{M} , which is called the bifurcation surface,
- 4. there exists a Cauchy surface $\Sigma \subset M$ which contains $\mathcal{B}^{,1}$

¹ \mathcal{B} is automatically a smooth submanifold of Σ .



Figure 1: The geometry of a bifurcate Killing horizon, as defined in Definition 2.2, depicted in a spacetime diagram.

By a spacetime with a stationary, resp. static, bifurcate Killing horizon we will mean a spacetime M with a bifurcate Killing horizon for which Σ can be chosen such that the Killing field ξ^a is timelike on $\Sigma \setminus \mathcal{B}$, resp. orthogonal to Σ .

Our definition of bifurcate Killing horizons coincides with that of [37], except that we allow all dimensions $d \ge 2$ and disconnected bifurcation surfaces \mathcal{B} . We refer to Figure 1 for a depiction of a generic bifurcate Killing horizon and to [37] for a more detailed description of this class of spacetimes.

Completeness of ξ^a means that the corresponding flow $\Phi: \mathbb{R} \times \mathcal{M} \to \mathcal{M}$, defined by $\Phi(0, x) = x$ and $\partial_t \Phi^a(t, x)|_{t=0} = \xi^a(x)$, yields a well-defined diffeomorphism $\Phi_t: \mathcal{M} \to \mathcal{M}$ for all $t \in \mathbb{R}$, defined by $\Phi_t(x) := \Phi(t, x)$. The fact that ξ^a is a Killing vector field means that $\Phi_t^* g_{ab} = g_{ab}$ for all $t \in \mathbb{R}$, where * denotes the pull-back. Equivalently, it can be expressed in terms of Killing's equation $\nabla_a \xi_b + \nabla_b \xi_a = 0$.

From now on, we will assume that the bifurcate Killing horizon is at least stationary. Let us fix a Cauchy surface Σ with the properties of Definition 2.2 and let n^a denote the future pointing normal vector field on Σ . We define the lapse function v and the shift vector field w^a on Σ by

$$v := -n_a \xi^a, \quad w^a := \xi^a - v n^a, \tag{1}$$

which means that $\xi^a = vn^a + w^a$ on Σ and $w^a \in T\Sigma$. We may decompose the Cauchy surface as

$$\Sigma = \mathcal{B} \cup \Sigma^+ \cup \Sigma^-.$$

where Σ^{\pm} are the sets where $\pm v > 0$. We define the following four globally hyperbolic regions of the spacetime M: the future $\mathcal{F} := I^+(\mathcal{B})$, the past $\mathcal{P} := I^-(\mathcal{B})$ and the left (-) and right (+) wedge regions $M^{\pm} := D(\Sigma^{\pm})^2$. Note in particular that Σ^{\pm} is a Cauchy surface for M^{\pm} and that we can partition M as

$$M = M^+ \cup M^- \cup \overline{\mathcal{F}} \cup \overline{\mathcal{P}},$$

²These regions are globally hyperbolic by Lemmas A.5.9 and A.5.12 of [2]. In [37] Kay and Wald prefer to define left and right wedge regions \mathcal{L} , \mathcal{R} in terms of the chronological future and past of portions of the Killing horizons of ξ^a . They then impose the restriction that $M = \overline{\mathcal{F}} \cup \overline{\mathcal{P}} \cup \mathcal{L} \cup \mathcal{R}$, from which it follows that $\mathcal{L} = M^-$ and $\mathcal{R} = M^+$. Note that in our case this restriction is not required, so the wedge regions M^{\pm} may be strictly larger than \mathcal{R}, \mathcal{L} .

where all sets are disjoint, except for $\overline{\mathcal{F}} \cap \overline{\mathcal{P}} = \mathcal{B}$. The region \mathcal{F} may contain black holes. More precisely, each connected component of \mathcal{B} gives rise to a connected component of \mathcal{F} , which, under suitable circumstances, may be a black hole (cf. [57] Sec. 12.1 for further discussion).

 Σ is a Cauchy surface on which the Killing field ξ^a is timelike or 0, and the right wedge $M^+ = (\mathcal{M}^+, g_{ab}|_{\mathcal{M}^+}, \xi^a|_{\mathcal{M}^+})$ is a (possibly disconnected) stationary spacetime, as is the left wedge $M^- = (\mathcal{M}^-, g_{ab}|_{\mathcal{M}^-}, -\xi^a|_{\mathcal{M}^-})$ if we change the sign of the Killing field to bring it in line with the existing time orientation.³ The metric of M^+ can be written in terms of local coordinates on Σ^+ and the Killing time coordinate t as

$$g_{\mu\nu} = -v^2 (dt^{\otimes 2})_{\mu\nu} + 2w_\mu \otimes_s dt_\nu + h_{\mu\nu},$$

where $h_{\mu\nu}$ is the *t*-independent Riemannian metric on Σ^+ induced by $g_{\mu\nu}$ and the lapse v and shift w^a were defined in Equation (1). By the letter ψ we will denote the diffeomorphism

$$\psi: \mathbb{R} \times (\Sigma \setminus \mathcal{B}) \to M^+ \cup M^- : (t, x) \mapsto \Phi(t, x), \tag{2}$$

where we recall that Φ is the flow of the Killing field ξ^a .

If Σ' is any other Cauchy surface containing \mathcal{B} , then $\Sigma' \setminus \mathcal{B}$ must be a Cauchy surface for $M^+ \cup M^-$ and hence ξ^a is timelike on $\Sigma' \setminus \mathcal{B}$. In other words, the stationary condition is independent of the choice of Cauchy surface containing \mathcal{B} .

To preserve the logical order of our presentation we now give the following geometric lemma, which will later reappear in Section 2.3.

Lemma 2.3 Let Σ be a smooth spacelike hypersurface in a spacetime M and let ξ^a be a timelike Killing vector field on M. Assume that w^a , defined in Equation (1), is a Killing field on (Σ, h_{ij}) , where h_{ij} is the induced metric on Σ . For a smooth curve $\gamma : [0,1] \to \Sigma$ the following statements are equivalent:

- 1. γ is a geodesic for (Σ, h_{ij}) ,
- 2. γ is a geodesic for M.

The lemma applies in particular when ξ^a is orthogonal to Σ .

Proof: The statement is local, so we may introduce local coordinates x^i on Σ and extend them to Gaussian normal coordinates on an open neighbourhood of M. Using the special form $g_{\mu\nu}dx^{\mu}dx^{\nu} = -(dx^0)^2 + h_{ij}dx^i dx^j$ of the metric in these coordinates, the geodesic equation in Mfor the curve γ reduces to the geodesic equation in (Σ, h_{ij}) plus the equation $0 = (\partial_0 h_{ij})\dot{\gamma}^i\dot{\gamma}^j$. We will show that the latter is automatically satisfied, due to the assumption on w^a . We may write $\xi_a = \xi^0 n_a + w_a$ near Σ with $n^a := (\partial_0)^a$ and consider the spatial components of Killing's equation:

$$0 = 2\nabla_{(i}\xi_{j)} = 2\xi^0 \nabla_{(i}n_{j)} + 2n_{(i}\nabla_{j)}\xi^0 + 2\nabla^{(h)}_{(i}w_{j)}.$$

Here, the last two terms vanish on Σ , because $n^i|_{\Sigma} = 0$ and w^i is a Killing field on Σ . The first term can be written using $\partial_0 h_{ij} = \mathcal{L}_{n^a} g_{ij} = 2\nabla_{(i}n_{j)}$. Since $\xi^0 \neq 0$ on Σ we find $\partial_0 h_{ij} = 0$, which proves our claim.

Remark 2.4 If the bifurcation surface of M is static, then the (possibly disconnected) spacetimes M^{\pm} are standard static globally hyperbolic spacetimes (cf. [49, 50]). If Σ is a Cauchy surface satisfying the properties of the static case of Definition 2.2, then the same is true for $\Phi_T(\Sigma)$ for any $T \in \mathbb{R}$. Conversely, given any other Cauchy surface Σ' satisfying the properties of the definition, we have $\Sigma' = \Phi_T(\Sigma)$ for some $T \in \mathbb{R}$. Indeed, for any $x \in \Sigma \setminus \mathcal{B}$ the integral curve

³To see why this is the case, one may pick an arbitrary, future pointing causal vector v^a at an arbitrary point $x \in M^+$. Let γ denote the inextendible geodesic through (x, v^a) and let $\dot{\gamma}^a$ denote its derivative. Since ξ^a is a Killing field, the inner product $\xi_a \dot{\gamma}^a$ is constant along γ (cf. [57] Proposition C.3.1). Note that γ intersects Σ at some point y ([57] Proposition 8.3.4.) and that $\dot{\gamma}$ is future pointing and causal there, so $\xi_a v^a = \xi_a \dot{\gamma}^a(y)$ is negative. By varying v^a and x it follows that ξ^a must be future pointing and timelike everywhere.

 $t \mapsto \Phi_t(x)$ is smooth and it remains timelike (by Killing's equation). Since it is inextendible (due to the completeness of ξ^a) there is a unique $T(x) \in \mathbb{R}$ such that $\Phi_{T(x)}(x) \in \Sigma'$. Now note that Σ' and $\Phi_{T(x)}(\Sigma)$ both contain x and that they are both orthogonal to $\xi^a \neq 0$. This shows that both surfaces coincide near x and hence that T(x) is locally constant on $\Sigma \setminus \mathcal{B}$.

To see that T is even globally constant, we consider a geodesic segment $\gamma : (-\epsilon, \epsilon) \to \Sigma$ in (Σ, h_{ij}) which intersects \mathcal{B} only at $\gamma(0) = p$, where the intersection is transversal. For $s \in (0, \epsilon)$ the points $\gamma(s)$ all lie in the same connected component of $\Sigma \setminus \mathcal{B}$, so there is a unique $T \in \mathbb{R}$ such that $\gamma' := \Phi_T \circ \gamma$ lies in Σ' for $s \in (0, \epsilon)$. To see that γ' lies entirely in Σ' we use the fact that γ and γ' are also geodesics in M, by Lemma 2.3. Similarly, if h'_{ij} is the induced metric on Σ' and $\tilde{\gamma}$ is the unique inextendible geodesic in (Σ', h'_{ij}) which coincides up to first-order with γ' at $s = \frac{1}{2}\epsilon$, then $\tilde{\gamma}$ is also a geodesic in M. It follows that $\tilde{\gamma}$ extends γ' , so γ' lies entirely in Σ' . Therefore, the locally constant function T on $\Sigma \setminus \mathcal{B}$ does not change value when we cross \mathcal{B} . Since Σ is connected, T must be globally constant and $\Phi_T(\Sigma) \subset \Sigma'$. The converse inclusion follows by reversing the roles of Σ and Σ' .

A useful characteristic of the bifurcation surface is its surface gravity, $\kappa > 0$ (cf.[57, 37]), which is a locally constant function on \mathcal{B} satisfying

$$\kappa^2 \equiv \frac{-1}{2} (\nabla^b \xi^a) (\nabla_b \xi_a)|_{\mathcal{B}}.$$

(This equality follows from Equation (12.5.14) in [57].) It will be convenient to know how the surface gravity can be computed from geometric objects on the Cauchy surface Σ . The following lemma answers this question.

Lemma 2.5 $\kappa^2 = h^{ij}(\partial_i v)(\partial_j v)|_{\mathcal{B}} - \frac{1}{2}h^{ij}h^{kl}(\nabla_i^{(h)}w_k)(\nabla_j^{(h)}w_l)|_{\mathcal{B}}.$

Proof: We use Gaussian normal coordinates $x^{\mu} = (x^0, x^i)$ near Σ on a neighbourhood of some arbitrary $p \in \mathcal{B}$. Combining the special form of the metric in these coordinates with the fact that $\xi^a = vn^a + w^a$ vanishes on \mathcal{B} and Killing's equation, we find

$$\begin{aligned} \kappa^2 &= \frac{-1}{2} g^{\mu\nu} g^{\rho\sigma} (\nabla_\mu \xi_\rho) (\nabla_\nu \xi_\sigma) |_{\mathcal{B}} = \frac{-1}{2} g^{\mu\nu} g^{\rho\sigma} (\partial_\mu \xi_\rho) (\partial_\nu \xi_\sigma) |_{\mathcal{B}} \\ &= \frac{-1}{2} h^{ij} h^{kl} (\partial_i \xi_k) (\partial_j \xi_l) |_{\mathcal{B}} + h^{ij} (\partial_i \xi_0) (\partial_j \xi_0) |_{\mathcal{B}} \\ &= h^{ij} (\partial_i v) (\partial_j v) |_{\mathcal{B}} - \frac{1}{2} h^{ij} h^{kl} (\nabla_i^{(h)} w_k) (\nabla_j^{(h)} w_l) |_{\mathcal{B}}. \end{aligned}$$

In the stationary case, the analysis of thermal (KMS) states of a quantum field in the right wedge M^+ and the idea of purification of such states naturally lead one to consider the case where M^- is isomorphic to M^+ , except for a reversal of the time orientation [31, 34]. We therefore introduce the following notions of wedge reflection⁴

Definition 2.6 A wedge reflection I for a spacetime M with a stationary bifurcate Killing horizon is a diffeomorphism $I: M^+ \cup M^- \cup U \to M^+ \cup M^- \cup U$ for some open neighbourhood U of \mathcal{B} , such that

- 1. I is an isometry of $M^+ \cup M^-$ onto itself, which reverses the time orientation,
- 2. $I \circ I = id$, the identity map,
- 3. I leaves \mathcal{B} pointwise fixed, and
- 4. $I^*\xi^a = \xi^a \text{ on } M^+ \cup M^-.$

⁴Our definition of a wedge reflection is slightly less restrictive than that in [37], where I is required to be a time orientation reversing isometric diffeomorphism of the entire spacetime M, which leaves \mathcal{B} pointwise fixed and satisfies $I \circ I = \text{id}$ and $I^* \xi^a = \xi^a$ everywhere.

A weak wedge reflection is a pair (Σ, ι) , where Σ is a Cauchy surface of M as in Def. 2.2 and $\iota: \Sigma \to \Sigma$ is a diffeomorphism such that

- 1. ι is an isometry for (Σ, h_{ij}) ,
- 2. $\iota \circ \iota = \mathrm{id}$,
- 3. ι leaves \mathcal{B} pointwise fixed, and
- 4. $\iota^* v = -v, \ \iota^* w^i = w^i.$

We note that $\iota(\Sigma^{\pm}) = \Sigma^{\mp}$. If the metric g_{ab} is analytic near \mathcal{B} , then the existence of I on a neighbourhood of \mathcal{B} is guaranteed [37]. We now prove the following additional results:

Proposition 2.7 Let M be a spacetime with a bifurcate Killing horizon.

- 1. All wedge reflections on M agree on $M^+ \cup M^- \cup \mathcal{B}$ and $I(M^{\pm}) = M^{\mp}$.
- 2. A Cauchy surface Σ admits at most one diffeomorphism ι such that (Σ, ι) is a weak wedge reflection.
- 3. If (Σ, ι) is a weak wedge reflection, then so is $(\Phi_T(\Sigma), \Phi_T \circ \iota \circ \Phi_{-T})$.
- 4. Given a wedge reflection I, there is a weak wedge reflection (Σ, ι) such that $\iota = I|_{\Sigma}$. In addition, if the bifurcation surface is static, we may choose Σ orthogonal to ξ .
- 5. In the stationary case, given a weak wedge reflection (Σ, ι) there exists a time orientation reversing isometric diffeomorphism $I: M^+ \cup M^- \to M^+ \cup M^-$ such that $I^*\xi^a = \xi^a$, $I \circ I = id$ and $I|_{\Sigma \setminus \mathcal{B}} = \iota|_{\Sigma \setminus \mathcal{B}}$. If w^i is a Killing field for (Σ, h_{ij}) near \mathcal{B} , then we may extend I to a wedge reflection. This applies in particular in the static case.

Proof: If I is a wedge reflection on M and $p \in \mathcal{B}$, then the derivative dI_p is an isomorphism of the tangent space T_pM , which is isometric (by continuity). dI_p acts trivially on tangent vectors of $T_p\mathcal{B}$ and the orthogonal complement is spanned by two future pointing null vectors l^a, m^a . Note that ξ^a is timelike on a neighbourhood of p in $M^+ \cup M^-$, where it is future pointing on M^+ and past pointing on M^- (cf. Figure 1). Because I reverses the time orientation, it also reverses M^+ and M^- on a neighbourhood of p. Together with the fact that $I \circ I = \text{id this implies that } dI_p(l^a) = -l^a$ and $dI_p(m^a) = -m^a$.

If I and I' are two wedge reflections of M, then $\psi := I' \circ I$ is a diffeomorphism of $M^+ \cup M^- \cup U$ into M, for some neighbourhood U of \mathcal{B} . Note that $d\psi$ acts as the identity on $TM|_{\mathcal{B}}$ and that it is isometric on $\overline{M^+ \cup M^-}$. We may use the exponential map to show that $\psi = \mathrm{id}$ on $(M^+ \cup M^-) \cap V$ for some open neighbourhood V of \mathcal{B} . Because $M^+ \cup M^- \cup \mathcal{B}$ is connected, we may continue the result $\psi = \mathrm{id}$ to this entire set, so that I = I' on $M^+ \cup M^- \cup \mathcal{B}$. The fact that $I(M^{\pm}) = M^{\mp}$ will follow from statement 4 and the facts that $M^{\pm} = D(\Sigma^{\pm})$ and $\iota(\Sigma^{\pm}) = \Sigma^{\mp}$.

Now let Σ be any Cauchy surface as in Def. 2.2. If (Σ, ι) is a weak wedge reflection and $p \in \mathcal{B}$, then $\iota(p) = p$ and $d\iota_p$ acts on $T_p\Sigma$ as the orthogonal reflection (w.r.t. h_{ij}) in the linear subspace $T_p\mathcal{B}$. The uniqueness of ι is then shown by the same argument as in the previous paragraph. The fact that $(\Phi_T(\Sigma), \Phi_T \circ \iota \circ \Phi_{-T})$ is also a weak wedge reflection is straightforward.

Now suppose that I is a wedge reflection and Σ' is any Cauchy surface as in Def. 2.2. By the results of [5] there exists a smooth function T' on M whose gradient $\nabla_a T'$ is everywhere timelike and past pointing and such that $\Sigma' = (T')^{-1}(0)$. Now set $T := T' - I^*T'$, which is again a smooth function with a past pointing, timelike gradient. $\Sigma := T^{-1}(0)$ is a smooth, spacelike hypersurface. On every inextendible timelike curve γ in M we can find points p_{\pm} such that $\pm T'(p_{\pm}) > 0$ and $\mp T'(I(p_{\pm})) > 0$. Hence $\pm T(p_{\pm}) > 0$, so that γ must intersect Σ and Σ is a Cauchy surface. Furthermore, $I(\Sigma) = \Sigma$, so that $\iota := I|_{\Sigma}$ is well-defined. It is immediately verified that (Σ, ι) is a weak wedge reflection.

In the static case, the Cauchy surface Σ constructed above may fail to be orthogonal to ξ^a . However, if Σ is any Cauchy surface orthogonal to ξ^a , as in Def. 2.2, then $(\Phi_T \circ I)(\Sigma) = \Sigma$ for some $T \in \mathbb{R}$, by Remark 2.4. On T_pM with $p \in \mathcal{B}$ we may consider the linear isomorphism $d(\Phi_T \circ I)_p = (d\Phi_T)_p \circ dI_p$, which acts trivially on vectors in $T_p\mathcal{B}$. On the normal vector r^a to \mathcal{B} in Σ we have $dI_p(r^a) = -r^a$ by the first paragraph of this proof and by considering the action of Φ_T we see that $d(\Phi_T \circ I)_p(r^a)$ can only lie in $T_p\Sigma$ if and only if T = 0. It follows that we must have $I(\Sigma) = \Sigma$, so taking $\iota := I|_{\Sigma}$ we find the weak wedge reflection (Σ, ι) with Σ orthogonal to ξ .

Finally, let (Σ, ι) be a weak wedge reflection. If the bifurcation surface is stationary, we can define I on $M^+ \cup M^-$ by

$$I \circ \psi(t, x) := \psi(t, \iota(x))$$

with ψ as in Equation (2). It is clear that $I \circ I = id$, so I is a diffeomorphism, and $I|_{\Sigma \setminus \mathcal{B}} = \iota|_{\Sigma \setminus \mathcal{B}}$ by construction. Furthermore, $I^*\xi^a = \xi^a$ and $I(M^{\pm}) = M^{\mp}$, so I must reverse the time orientation.

To see that I is isometric we fix a $p \in \Sigma \setminus \mathcal{B}$ and we note that $I^*(n^a) = I^*(\frac{1}{v}(\xi^a - w^a)) = \frac{1}{-v}(\xi^a - w^a) = -n^a$, because $I^*(w^a) = \iota^*(w^a) = w^a$. Decomposing any tangent vector $\nu^a \in T_pM$ as $\nu^a = \alpha n^a + \mu^a$ with $\mu^a \in T_p\Sigma$ we find $I^*(\nu^a) = -\iota^*(\alpha)n^a + \iota^*(\mu^a)$. As ι is an isometry of $T_p\Sigma$, it follows that the same is true for I on T_pM . Because I commutes with the isometries Φ_t , it must then be isometric on $M^+ \cup M^-$.

Let $N\mathcal{B} \subset TM|_{\mathcal{B}}$ be the normal bundle to \mathcal{B} in M. There is a neighbourhood V of the zero section of this bundle on which the exponential map is a diffeomorphism $V \simeq \exp(V) =: \tilde{V}$. Without loss of generality we may assume that V has a convex intersection with each fibre of $N\mathcal{B}$ and that V = -V, where -1 is the fibre-wise multiplication by -1 on $N\mathcal{B}$. Then $I' := \exp \circ -1 \circ \exp^{-1}$ is a diffeomorphism of \tilde{V} onto itself. Any wedge reflection must coincide with I' on a neighbourhood of \mathcal{B} in $M^+ \cup M^-$. Conversely, if I' coincides with I on such a neighbourhood, then we can extend I to a wedge reflection. Now, given any $x \in \tilde{V} \cap \Sigma$, let $\gamma(s) := \exp_p^{(h)}(sv)$ be the geodesic in $(\tilde{V} \cap \Sigma, h_{ij})$ from $I(x) := \gamma(-1)$ to $x = \gamma(1)$. If w^i is a Killing field for (Σ, h_{ij}) on $\tilde{V} \cap \Sigma$ we know from Lemma 2.3 that γ is also a geodesic in M, which entails that $I'(\gamma(s)) = \gamma(-s)$. Therefore I and I' coincide on $\tilde{V} \cap \Sigma$. Because both commute with the flow of ξ^a they even coincide on a neighbourhood of \mathcal{B} in $M^+ \cup M^-$, so we may extend I by I' to $M^+ \cup M^- \cup \tilde{V}$, making it into a wedge reflection.

Note in particular that in the static case, a wedge reflection is equivalent to a weak wedge reflection (Σ, ι) with Σ orthogonal to ξ^a . Furthermore, for any two weak wedge reflections (Σ, ι) and (Σ', ι') with both Σ and Σ' orthogonal to ξ^a we must have $\Sigma' = \Phi_T(\Sigma)$ and $\iota' = \Phi_T \circ \iota \circ \Phi_{-T}$ for some $T \in \mathbb{R}$. Hence, both weak wedge reflections give rise to the same map I on $M^+ \cup M^-$. (Whether an equivalence of weak and strong wedge reflections holds in the general stationary case is unclear.)

2.2 Complexification beyond the horizon and the Hawking temperature

If M is a spacetime with a static bifurcate Killing horizon, then M^+ is a (possibly disconnected) standard static spacetime and we may define complexifications and Riemannian manifolds with a compactified imaginary time variable (cf. [50]). For R > 0 we define the cylinder

$$\mathcal{C}_R := \mathbb{C}/\sim, \quad z \sim z' \Leftrightarrow z - z' \in 2\pi i R \mathbb{Z}.$$

Under this equivalence relation, the imaginary axis of \mathbb{C} becomes compactified to the circle \mathbb{S}_R of radius R. The complexification $(M^+)_R^c$ is then defined as a real manifold, endowed with a symmetric, complex-valued tensor field:

$$(M^+)_R^c = \mathcal{C}_R \times \Sigma^+,$$
$$(g_R^c)_{\mu\nu} = -v^2 (dz^{\otimes 2})_{\mu\nu} + h_{\mu\nu}$$

where v and $h_{\mu\nu}$ are independent of the coordinate $z = t + i\tau$ on C_R . Using the diffeomorphism ψ of Equation (2), restricted to $\mathbb{R} \times \Sigma^+$, we can embed M^+ into $(M^+)_R^c$ as the $\tau = 0$ surface. $(g_R^c)_{\mu\nu}$ is the analytic continuation of $g_{\mu\nu}$ in z. We may also consider the associated Riemannian manifold, endowed with the pull-back metric:

$$M_R^+ := \left\{ (z, x) \in (M^+)_R^c | t = 0 \right\},\$$



Figure 2: The embedding of M^- into $(M^+)^c_R$. Depicted are M^+ , some integral curves of the Killing field ξ^a , and their complexifications, which are compactified to circles. $(M^+)^c_R$ is obtained by rotating M^+ around a vertical axis through \mathcal{B} and $M^- \simeq M^+$ is embedded on the opposite side of the circle. Note, however, that the isomorphism $M^- \simeq M^+$ reverses the time orientation when compared to M^- in Figure 1.

$$(g_R)_{\mu\nu} = v^2 (d\tau^{\otimes 2})_{\mu\nu} + h_{\mu\nu}.$$

Note that $M_R^+ \simeq \mathbb{S}_R \times \Sigma^+$ as a manifold. We can identify the surface $\Sigma^+ \simeq M^+ \cap M_R^+$ in $(M^+)_R^c$ also with the $\tau = 0$ surface in M_R^+ . Furthermore, M_R^+ has a Killing field $(\xi_R)^a \partial_a = \partial_\tau$, which can be viewed as the analytic continuation of $\xi^a \partial_a = \partial_t$.

If M has a wedge reflection I, and hence a weak wedge reflection (Σ, ι) , we can extend the embedding of M^+ into $(M^+)^c_B$ to an embedding

$$\chi: M^{+} \cup M^{-} \to (M^{+})_{R}^{c}: \begin{cases} \chi \circ \psi(t, x) = (t, x) & \psi(t, x) \in M^{+} \\ \chi \circ \psi(t, x) = (t + i\pi R, \iota(x)) & \psi(t, x) \in M^{-} \end{cases},$$
(3)

where we used the diffeomorphism ψ of Equation (2). In other words, if $x \in M^-$, then $\psi(x)$ is obtained by composing the wedge reflection I, the embedding of M^+ into $(M^+)^c_R$ and a rotation over the angle π . (See Figure 2.)

 χ restricts to an embedding $\mu := \chi|_{\Sigma \setminus \mathcal{B}}$ of $\Sigma \setminus \mathcal{B}$ into the Riemannian manifold M_R^+ , so that Σ^- is embedded as the $\tau = \pi R$ hypersurface. We now wish to consider whether this embedding can be extended to an embedding $\overline{\mu}$ of all of Σ into some extension $\overline{M_R^+}$ of the manifold M_R^+ , and whether the Riemannian metric $(g_R)_{ab}$ can be extended to $\overline{M_R^+}$ as well.

A suitable extension $\overline{M_R^+}$ of M_R^+ can readily be obtained by a standard gluing technique. To see how this works, we let $\pi_{N\mathcal{B}}: N\mathcal{B} \to \mathcal{B}$ denote the normal bundle of \mathcal{B} in Σ with zero section \mathcal{Z} . Note that $N\mathcal{B} \subset T\Sigma|_{\mathcal{B}}$ and since both Σ and \mathcal{B} are orientable, $N\mathcal{B} \simeq \mathcal{B} \times \mathbb{R}$ is a trivial bundle. We may introduce the normal vector field n^a to \mathcal{B} in Σ , which points towards Σ^+ . This determines an orthonormal frame and an orientation on $N\mathcal{B}$. There is a neighbourhood $U \subset N\mathcal{B}$ of \mathcal{Z} on which the exponential map $\exp: U \to \Sigma$ defines a diffeomorphism. Next we consider the bundle $\mathcal{B} \times \mathbb{R}^2$ with the canonical Euclidean inner product in each fibre and a fixed orthonormal frame. We introduce the subbundles

$$X := \{ (x, v) \in \mathcal{B} \times \mathbb{R}^2 | (x, |v|n^a) \in U \}, \quad \dot{X} := \{ (x, v) \in X | v \neq 0 \}$$

and we may embed \dot{X} into M_R^+ by

$$\eta: \dot{X} \to M_{R}^{+}: (x, re^{i\phi}) \mapsto (R\phi, \exp_{x}(r)),$$

where ϕ is defined with respect to the fixed orthonormal frame. The extended spacetime can then be defined by gluing X against M_R^+ along \dot{X} , i.e.

$$\overline{M_R^+} := (M_R^+ \cup X) / \sim,$$

where \sim indicates that we identify the domain and range of η .

On $\exp(U \setminus \mathcal{Z})$ the embedding μ is given by $\eta^{-1} \circ \mu(\exp_x(s)) = (x, (s, 0))$. This may be checked separately for the cases s > 0 and s < 0, using the properties of ι , which imply $\exp_x(-s) = \iota(\exp_x(s))$ for $x \in \mathcal{B}$. The extension $\overline{\mu} : \Sigma \to \overline{M_R^+}$ can then be defined by taking $\overline{\mu}(\exp_x(s)) = (x, (s, 0))$ also when s = 0.

Now that we have defined the extended manifold $\overline{M_R^+}$ we wish to investigate whether the Riemmannian metric $(g_R)_{ab}$ on M_R^+ can be extended too. This is where a particular value of the radius R is singled out, which corresponds to the Hawking temperature (cf. [32, 16] and references therein).

Lemma 2.8 The components of the metric $(g_R)_{ab}$ can be extended to $\overline{M_R^+}$ as bounded functions. A continuous extension $(\bar{g}_R)_{ab}$ exists if and only if $R \equiv \kappa^{-1}$, in which case the extension is even smooth.

Proof: To prove this lemma, we work in suitably chosen local coordinates. First, we introduce local coordinates x^i , $i = 2, \ldots, d-1$ on \mathcal{B} and we let r denote the Gaussian normal coordinate near \mathcal{B} on Σ , with r > 0 on Σ^+ . As before, we let t denote the Killing time on M^+ , so for some $\rho > 0$ the local coordinates $(t, r, x^i) \in \mathbb{R} \times (0, \rho) \times \mathcal{B}$ describe an open region in M^+ whose boundary in M contains \mathcal{B} . After complexification and restriction to the Riemannian manifold, we have local coordinates $(\tau, r, x^i) \in \mathbb{S}_R \times (0, \rho) \times \mathcal{B}$. Expressed in these local coordinates the Riemannian metric h_{ij} on Σ takes the form

$$h_{ij} = (dr^{\otimes 2})_{ij} + k_{ij}(r, x^l),$$

where $k_{ij}(r, x^l)$ denotes the Riemannian metric induced on \mathcal{B} . Correspondingly, the Lorentzian metric on M^+ and the Riemannian metric on M^+_R take the form

$$g_{\mu\nu} = -v^2 (dt^{\otimes 2})_{\mu\nu} + (dr^{\otimes 2})_{\mu\nu} + k_{\mu\nu}(r, x^i),$$

$$(g_R)_{\mu\nu} = v^2 (d\tau^{\otimes 2})_{\mu\nu} + (dr^{\otimes 2})_{\mu\nu} + k_{\mu\nu}(r, x^i).$$
(4)

Changing coordinates $(\tau, r) \to (X, Y)$ with

$$X := r \cos\left(\frac{\tau}{R}\right), \quad Y := r \sin\left(\frac{\tau}{R}\right)$$

the metric on M_R^+ takes the form

$$(g_R)_{\mu\nu} = (1 - \alpha Y^2)(dX^{\otimes 2})_{\mu\nu} + (1 - \alpha X^2)(dY^{\otimes 2})_{\mu\nu} + \alpha XY(dX \otimes dY + dY \otimes dX)_{\mu\nu} + k_{\mu\nu}(X, Y, x^i),$$

where the function α is defined by $\alpha := r^{-2} - r^{-4}R^2v^2$.

By construction of the Gaussian normal coordinate r we have $\iota^* r = -r$. As $\iota^* k_{ij} = k_{ij}$ it follows that k_{ij} is an even tensor in r and its Taylor expansion around r = 0 only contains even powers. This suffices to show that k_{ij} depends smoothly on X and Y, since $r^2 = X^2 + Y^2$. Hence, k_{ij} extends smoothly to all of $\overline{M_R^+}$.

The functions $X^2 \alpha$, $Y^2 \alpha$ and $XY \alpha$ remain bounded near the set \mathcal{B} , where r = 0, but if we take the limit $r \to 0^+$ we find that e.g. $X^2 \alpha$ approaches a value that may in general depend on τ as well as x^i . To eliminate this dependence and to get a continuous extension, it is necessary and sufficient to impose

$$\lim_{r \to 0^+} r^2 \alpha(r, x^i) = 0$$

In order to analyse this limit, we first prove that $v(r, x^i) = \kappa r + r^3 \beta(r, x^i)$ for some smooth β near \mathcal{B} . To see this, we use a Taylor expansion around r = 0. As $\iota^* v = -v$ we cannot have any even terms in r, so the constant and second-order terms vanish. Since $\nabla_a^{(h)} v$ and $\nabla_a^{(h)} r$ are both normal to \mathcal{B} the first-order term is fixed by Lemma 2.5. The term with β is just the remainder.⁵ Now the vanishing of the limit above simply means

$$R^{-1} \equiv \lim_{r \to 0^+} \frac{v(r, x^i)}{r} = \kappa.$$

To see that the extension $(\bar{g}_R)_{ab}$ is even smooth when this holds, we note that α takes the form $\alpha = -2R\beta - r^2R^2\beta^2$ near \mathcal{B} , which is smooth.

In order to satisfy the condition of Lemma 2.8 it is necessary for the surface gravity κ to be constant, because R is constant too. If \mathcal{B} is connected this is no additional assumption, but in general it may fail (cf. [37] for further discussion and examples). Anticipating the relation between the radius R and the temperature, we define the Hawking radius to be

$$R_H := \kappa^{-1}$$

whenever κ is constant.

The Killing field $\xi_R^{\mu} \partial_{\mu} = \partial_{\tau} = \frac{1}{R} (X \partial_Y - Y \partial_X)$ always admits a smooth extension to $\overline{M_R^+}$, which we will denote by $\overline{\xi_R}^{\mu}$. Furthermore, we wish to record the following lemma, whose proof is closely related to that of Lemma 2.8:

Lemma 2.9 If $V \in C^{\infty}(\Sigma)$ satisfies $\iota^* V = V$ and R > 0, then there exists a unique smooth extension W of V to $\overline{M_R^+}$ such that $\overline{\xi_R}^{\mu} \partial_{\mu} W = 0$.

Proof: On $M_R^+ \simeq \mathbb{S}^1 \times \Sigma^+$ there is exactly one smooth function W such that $\xi_R^{\mu} \partial_{\mu} W = 0$ and $W|_{\Sigma^+} = V|_{\Sigma^+}$. It is given by $W(\tau, x^i) = V(x^i)$. Note that $W(\pi R, x^i) = V(x^i) = V(\iota(x^i))$, so $W|_{\Sigma^-} = V|_{\Sigma^-}$ too. We now define $W|_{\mathcal{B}} := V|_{\mathcal{B}}$ and it remains to prove that W is smooth. For this purpose we use again local coordinates (τ, r, x^i) and (X, Y, x^i) near \mathcal{B} , as in the proof of Lemma 2.8. We have $W(X, Y, x^i) = V(\sqrt{X^2 + Y^2}, x^i)$, so W is continuous at \mathcal{B} . Moreover, the Taylor series of V at r = 0 is even in r, because $\iota^* V = V$. V therefore only depends on r^2 and W depends only on X and Y through $X^2 + Y^2$, so the extension is smooth.

2.3 Analytic continuation beyond the horizon

The Killing time coordinate on M^+ is used to define the complexification $(M^+)_R^c$ and the Riemannian manifold M_R^+ , but it becomes a bad choice of coordinate near the boundary of $M^+ \subset M$. This is particularly inconvenient when we wish to study the behaviour near the bifurcation surface \mathcal{B} . For that reason, we now consider Gaussian normal coordinates instead and study their properties regarding the complexification procedure above. Furthermore, we will consider Riemannian normal coordinates, which are the most convenient choice of coordinates when describing the Hadamard parametrix construction in Section 4 below. In order to investigate this construction in the light of our complexification procedure, we will also establish some results on the relation between Riemannian and Gaussian normal coordinates.

We consider a spacetime M with a static bifurcate Killing horizon, with a wedge reflection and with a surface gravity $\kappa > 0$ which is globally constant. Let x^i denote local coordinates on a neighbourhood U in a Cauchy surface Σ with the properties of Definition 2.2. We let $x^{\mu} = (x^0, x^i)$ denote corresponding Gaussian normal coordinates on a portion V of M containing U. Furthermore, we will write $M' := \overline{M_{R_H}^+}$ and we let $(x')^{\mu} = ((x')^0, (x')^i)$ be Gaussian normal coordinates on a region $V' \subset M'$, containing $U' := \overline{\mu}(U)$, such that $x^i = \overline{\mu^*}(x')^i$. We choose the Gaussian normal coordinates in such a way that ∂_{x^0} and $\partial_{(x')^0}$ point in the same direction as $\pm \partial_t$ and $\pm \partial_{\tau}$ on Σ^{\pm} and $\overline{\mu}(\Sigma^{\pm})$, respectively. This determines them uniquely.

 $^{{}^{5}\}beta$ is smooth by e.g. [38] Ch.13, §6 and Theorem 8.1.

Remark 2.10 The results of this subsection focus specifically on the case of the Hawking radius, M', but analogous results hold for M_R^+ with any R > 0, when U is a coordinate neighbourhood of $\Sigma \setminus \mathcal{B}$.

Proposition 2.11 Expressing the metrics g_{ab} and $(\bar{g}_R)_{ab}$ in these Gaussian normal coordinates as

$$g_{\mu\nu}dx^{\mu}dx^{\nu} = -(dx^0)^{\otimes 2} + h_{ij}dx^i dx^j, \quad (\bar{g}_R)_{\mu\nu} = (d(x')^0)^{\otimes 2} + h'_{ij}d(x')^i d(x')^j,$$

we have for $1 \leq i, j \leq d-1$ and $n \geq 0$:

$$\partial_{0}^{2n} h_{ij}|_{U} = i^{2n} \overline{\mu}^{*} \left((\partial_{0}')^{2n} h_{ij}'|_{U'} \right),$$

$$\partial_{0}^{2n+1} h_{ij}|_{U} = i^{2n+1} \overline{\mu}^{*} \left((\partial_{0}')^{2n+1} h_{ij}'|_{U'} \right) = 0.$$
(5)

Proof: On $U \cap \Sigma^+$ and $U' \cap \overline{\mu}(\Sigma^+) = \mu(U \cap \Sigma^+)$ this follows directly from Proposition 3.3 in [50]. The same is then seen to be true on $U \cap \Sigma^-$, after applying the isomorphism I to $M^- \cup M^+$ and the isomorphism $(\tau, x) \mapsto (i\pi R + \tau, x)$ to M'. The result extends by continuity to $U \cap \mathcal{B}$ and $\overline{\mu}(U \cap \mathcal{B})$.

In [50] we argued that Equation (5) on Σ^+ can be interpreted as an infinitesimal analytic continuation in the Gaussian normal coordinates. Proposition 2.11 shows that this infinitesimal analytic continuation still works fine across the bifurcation surface \mathcal{B} , where the Killing time coordinate is no longer a good coordinate.

The information of Proposition 2.11 allows us to prove analogous statements for various objects which can be constructed from the metric:

Corollary 2.12 Expressing the Killing fields, metric, inverse metric, Christoffel symbol and Riemann curvature of g_{ab} and $(\bar{g}_R)_{ab}$ in Gaussian normal coordinates we have for all $n \ge 0$:

$$\begin{aligned}
\partial_{0}^{n} g^{\mu\nu}|_{U} &= i^{n+c} \overline{\mu}^{*} \left((\partial_{0}^{\prime})^{n} \overline{g}_{R}^{\mu\nu}|_{U^{\prime}} \right), \\
\partial_{0}^{n} (\Gamma_{\nu\rho}^{\mu})|_{U} &= i^{n+c} \overline{\mu}^{*} \left((\partial_{0}^{\prime})^{n} (\overline{\Gamma}_{R})_{\nu\rho}^{\mu}|_{U^{\prime}} \right), \\
\partial_{0}^{n} (R_{\mu\nu\rho}^{\alpha})|_{U} &= i^{n+c} \overline{\mu}^{*} \left((\partial_{0}^{\prime})^{n} (\overline{R}_{R})_{\mu\nu\rho}^{\alpha}|_{U^{\prime}} \right), \\
\partial_{0}^{n} \xi^{\mu}|_{U} &= i^{n+c+1} \overline{\mu}^{*} \left((\partial_{0}^{\prime})^{n} (\overline{\xi}_{R})^{\mu}|_{U^{\prime}} \right)
\end{aligned} \tag{6}$$

where c is the number of lower indices equal to zero, minus the number of upper indices equal to zero.

Whereas the left-hand side of all these equations is always real, the right-hand side is real or purely imaginary, depending on whether n + c is even or odd. In this way we see that the expressions on both sides vanish when n + c is odd.

Proof: The first statement is obvious when one or both of the indices are 0, because the inverse metric components are then constantly 0, 1 or -1. For the remaining indices this can be proven by induction by taking normal derivatives of the equality $\delta^i_{\ j} = h^{il}h_{jl}$ and its Euclidean counterpart and using the results of Proposition 2.11.

The Christoffel symbol $\Gamma^{\mu}_{\nu\rho}$ vanishes when at least two of the indices μ, ν, ρ are zero, since $\partial_{\mu}g_{00} = 0$. The analogous statement in the Euclidean case is also true. For the remaining choices of indices we can express the Christoffel symbol in terms of h_{ij} and its inverse, so the result follows from Proposition 2.11 and the first line of Equation (6) in a straightforward manner. The claim for the Riemann curvature follows from its expression in terms of the Christoffel symbols.

Finally we note that the Killing fields are uniquely determined by their initial values on Σ and Killing's Equation. In particular, $\partial_0 \xi^0 \equiv 0$ and hence $\partial_0^n \xi^0|_U = 0$ when $n \geq 1$ and similarly for $\overline{\xi_R}^0$. Since $\xi^0|_U = v = \overline{\mu}^*(\overline{\xi_R}^0|_{U'})$ this proves the claim for $\mu = 0$. For a detailed proof concerning the spatial components we refer to the proof of Proposition 3.3 in [50].

The following corollary is a related result on the geometry of the Cauchy surface Σ (see also Lemma 2.3):

Corollary 2.13 For a smooth curve $\gamma:[0,1] \rightarrow \Sigma$ the following statements are equivalent:

- 1. γ is a geodesic in (Σ, h_{ij}) ,
- 2. γ is a geodesic in M,
- 3. $\overline{\mu} \circ \gamma$ is a geodesic in M'.

The proof is the same as for Corollary 3.13 in [50].

To extend the comparison of the geometry near Σ in M and $\overline{\mu}(\Sigma)$ in M' further we will now consider Riemannian normal coordinates. These can be defined locally on any pseudo-Riemannian manifold and for the purposes of defining them we will consider this general setting.

Let O be a convex normal neighbourhood of a pseudo-Riemannian manifold $N = (\mathcal{N}, g_{ab})$. We may introduce local coordinates on $O^{\times 2}$ as follows. Define the embedding $O^{\times 2} \to TO$ by $(x, y) \mapsto (\exp_y^{-1}(x), y)$, where \exp_y is the exponential map, which defines a diffeomorphism from a neighbourhood of $0 \in T_y N$ onto O. Next, we introduce an arbitrary frame $(e_\mu)^a$ of TO to identify $TO \simeq \mathbb{R}^d \times O$, with standard Cartesian coordinates \tilde{v}^μ on \mathbb{R}^d and arbitrary coordinates \tilde{y}^μ on O. The composition of these two maps is an embedding $\rho: O^{\times 2} \to \mathbb{R}^d \times O$. The desired coordinates on $O^{\times 2}$ are then given by

$$(v^{\mu}, y^{\nu}) = \rho^*(\tilde{v}^{\mu}, \tilde{y}^{\nu}).$$
(7)

For any fixed $y \in O$, the coordinates v^{μ} are Riemannian normal coordinates on O, centred on yand satisfying $g_{\mu\nu}(0) = (e_{\mu})^a(y)(e_{\nu})_a(y)$. With a slight abuse of language we will also refer to the coordinates (v, y) as Riemannian normal coordinates on $O^{\times 2}$.

We now return to the geometry of spacetimes with a static bifurcate Killing horizon. For any point $y \in \Sigma$ we can choose convex normal neighbourhoods $V \subset M$ and $V' \subset M'$ such that $y \in V$ and $\overline{\mu}(y) \in V'$. The sets V and V' do not contain any pair of points which are conjugate along the unique geodesic that connects them (cf. [43] Proposition 10.10 and the comments below it). We may also choose a convex normal neighbourhood $U \subset \Sigma$ containing y and such that $U \subset V \cap \Sigma$ and $U' := \overline{\mu}(U) \subset V'$.

We let x^{μ} and y^{μ} be Gaussian normal coordinates on a neighbourhood of U and we let (v^{μ}, y^{μ}) be Riemannian normal coordinates on $U^{\times 2}$, defined using the frame $\partial_{y^{\mu}}$ associated to the coordinates y^{μ} . Similarly, let $(x')^{\mu}$, $(y')^{\mu}$ be Gaussian normal coordinates near U' such that $x^{i} = \overline{\mu}^{*}(x')^{i}$ and $y^{i} = \overline{\mu}^{*}(y')^{i}$ on U and let $((v')^{\mu}, (y')^{\mu})$ be Riemannian normal coordinates defined using the frame $\partial_{(y')^{\mu}}$.

Proposition 2.14 On $U^{\times 2}$ we have, in the coordinates introduced above:

$$\begin{split} \partial_{x^{0}}^{k} \partial_{y^{0}}^{l} v^{j} &= i^{k+l} (\overline{\mu}^{\times 2})^{*} \left(\partial_{(x')^{0}}^{k} \partial_{(y')^{0}}^{l} (v')^{j} \right), \\ \partial_{x^{0}}^{k} \partial_{y^{0}}^{l} v^{0} &= i^{k+l-1} (\overline{\mu}^{\times 2})^{*} \left(\partial_{(x')^{0}}^{k} \partial_{(y')^{0}}^{l} (v')^{0} \right). \end{split}$$

Proof: For $x, y \in V$, $v^{\mu}(x, y) \in T_y V$ is the unique vector such that $[0, 1] \ni t \mapsto \exp_y(tv^{\mu}(x, y))$ is the unique geodesic in V from y to x, where the index μ refers to the frame $\partial_{y^{\mu}}$. For $x, y \in U$ we note that $v^0 \equiv 0$, by Corollary 2.13. Similarly, $(v')^0 \equiv 0$ on $(U')^{\times 2}$. Furthermore, the relations $x^i = \overline{\mu}^*(x')^i$ and $y^i = \overline{\mu}^*(y')^i$ on U and the fact that $\overline{\mu}$ is an isometry entail that

$$v^j = (\overline{\mu}^{\times 2})^* (v')^j$$

on $U^{\times 2}$, which proves the desired equality in the absence of normal derivatives.

Let us now fix $x, y \in U$ and write $x = (0, x^i)$ and $y = (0, y^i)$. For sufficiently small s the curves $\gamma_0(s)$ and $\gamma_1(s)$ in V, defined in Gaussian normal coordinates by $\gamma_0^{\mu}(s) := (s, y^i)$ and $\gamma_1^{\mu}(s) := (s, x^i)$, are geodesics with tangent vector n^{μ} at y, resp. x. For some sufficiently small $\epsilon > 0$ we may then define the map $\gamma^{\mu}: (-\epsilon, \epsilon)^{\times 2} \times [0, 1] \rightarrow V$ such that $t \mapsto \gamma^{\mu}(r, s, t)$ is the unique geodesic in V between $\gamma_0(r)$ and $\gamma_1(s)$. Note that γ is uniquely determined by the choice of x, y and that $v^{\mu}(\gamma_1(s), \gamma_0(r)) = \partial_t \gamma^{\mu}(r, s, t)|_{t=0}$.

We will now derive an equation for $X_{k,l}^{\mu}(t) := \partial_r^k \partial_s^l \gamma^{\mu}(0,0,t)$ for all $k, l \ge 0$, in analogy with the Jacobi equation (also known as the geodesic deviation equation). We start with the geodesic equation for fixed r, s:

$$\partial_t^2 \gamma^\mu + \Gamma^\mu_{\nu\rho}(\gamma) \partial_t \gamma^\nu \partial_t \gamma^\rho = 0.$$

Taking partial derivatives with respect to r and s and evaluating on r = s = 0 then yields:

$$0 = \partial_t^2 X_{k,l}^{\mu} + \sum_{k'=0}^k \sum_{k''=0}^{k'} \sum_{l'=0}^l \sum_{l''=0}^{l'} \binom{k}{k'} \binom{k'}{k''} \binom{l}{l'} \binom{l'}{l'} \binom{l'}{l''} \binom{l'}{l''} \binom{l'}{l''} \binom{k''}{\ell''} \frac{\lambda_{l''=0}^{k''}}{\lambda_s^{\mu} \Gamma_{\nu\rho}^{\mu}(\gamma)} \frac{\lambda_{l'} X_{k'-k'',l'-l''}}{\lambda_s^{\mu} \lambda_{k-k',l-l'}}.$$
(8)

Similarly, we consider the map $(\gamma')^{\mu}: (-\epsilon, \epsilon)^{\times 2} \times [0, 1] \to V'$ such that $t \mapsto (\gamma')^{\mu}(r, s, t)$ is the geodesic between $(\gamma')^{\mu}(r,0,0)$ and $(\gamma')^{\mu}(0,s,1)$, where $r \mapsto (\gamma')^{\mu}(r,s,0)$ and $s \mapsto (\gamma')^{\mu}(r,s,1)$ are geodesics through x and y with tangent vector n^{μ} . Defining $(X')_{k,l}^{\mu}(t) := \partial_r^k \partial_s^l(\gamma')^{\mu}(0,0,t)$ in Gaussian normal coordinates one derives the equation

in analogy to Equation (8). Define $Y_{k,l}^{\mu} := X_{k,l}^{\mu} - i^{k+l+c} (X')_{k,l}^{\mu}$ as a function of t, where c = -1 if $\mu = 0$ and c = 0 else. We will prove by induction over N = k + l that $Y_{k,l}^{\mu} \equiv 0$. If k = l = 0 we have

$$Y_{0,0}^j = \gamma^j - (\gamma')^j \equiv 0, \quad Y_{0,0}^0 = \gamma^0 + i(\gamma')^0 = 0 + i0 = 0.$$

Now assume that the claim holds for all (k', l') with $k' + l' \leq N$ for some $N \geq 0$ and consider k, lwith k + l = N + 1. We may use Equations (8,9) to write

If we use the chain rule to expand the derivatives acting on the Christoffel symbols, then any normal derivative acting on $\Gamma^{\mu}_{\nu\rho}$ is accompanied by a factor X⁰. By the induction hypothesis and Corollary 2.12 we therefore see that all terms in the sum vanish, except those involving $X_{k,l}^{\mu}$ and $(X')_{k,l}^{\mu}$ with k+l=N+1. This leads to

$$\begin{aligned} 0 &= \partial_t^2 Y_{k,l}^{\mu} + 2\Gamma_{\nu\rho}^{\mu}(\gamma)\partial_t X_{0,0}^{\nu}\partial_t X_{k,l}^{\rho} - 2i^{k+l+c}(\bar{\Gamma}_R)_{\nu\rho}^{\mu}(\gamma)\partial_t (X')_{0,0}^{\nu}\partial_t (X')_{k,l}^{\rho} \\ &+ (\partial_{\alpha}\Gamma_{\nu\rho}^{\mu})(\gamma)\partial_t X_{0,0}^{\nu}\partial_t X_{0,0}^{\rho} X_{k,l}^{\alpha} - i^{k+l+c}(\partial_{\alpha}'(\bar{\Gamma}_R)_{\nu\rho}^{\mu})(\gamma)\partial_t (X')_{0,0}^{\nu}\partial_t (X')_{0,0}^{\rho}(X')_{k,l}^{\alpha} \\ &= \partial_t^2 Y_{k,l}^{\mu} + 2\Gamma_{\nu\rho}^{\mu}(\gamma)\partial_t X_{0,0}^{\nu}\partial_t Y_{k,l}^{\rho} + (\partial_{\alpha}\Gamma_{\nu\rho}^{\mu})(\gamma)\partial_t X_{0,0}^{\nu}\partial_t X_{0,0}^{\rho} Y_{k,l}^{\alpha}, \end{aligned}$$

which is the Jacobi equation for the vector field $Y_{k,l}^{\mu}$ on $\gamma(0,0,.)$.⁶ The values of $Y_{k,l}^{\mu}$ at the endpoints x and y of the geodesic $\gamma(0,0,.)$ are easily determined by the fact that $\gamma(r,s,0) =$

$$D_{t}^{2}Y^{\mu} = -R_{\nu\alpha\beta}^{\ \ \mu}\partial_{t}X_{0,0}^{\alpha}\partial_{t}X_{0,0}^{\beta}Y^{\nu},$$

cf. [57] Eq. (3.3.18).

⁶This equation is more commonly written using the covariant derivative $D_t Y^{\mu} := \partial_t Y^{\mu} + \Gamma^{\mu}_{\nu\rho}(\gamma) \partial_t X^{\nu}_{0,0} Y^{\rho}$, in terms of which the Jacobi equation reads

 $\gamma_0(r) = (r, \gamma_0^i(0))$ and $\gamma(r, s, 1) = \gamma_1(s) = (s, \gamma_1^i(0))$ and similarly for the Euclidean case. Taking derivatives with respect to r and s one easily finds that $Y_{k,l}^{\mu}(x) = Y_{k,l}^{\mu}(y) = 0$ for all $k + l \ge 0$ and all μ . Recall that the points x and y are not conjugate along the unique geodesic $\gamma(0, 0, .)$ in V that connects them, so the Jacobi vector field $Y_{k,l}^{\mu}$ which vanishes at the boundaries must vanish identically. Hence, $Y_{k,l}^{\mu} = 0$ for all k, l. This result on $Y_{k,l}^{\mu} = 0$ implies the proposition.

In our discussion of the Hadamard series, it will be convenient to consider Riemannian normal coordinates based on an orthonormal frame $(e_{\mu})^a$, rather than a coordinate frame. We will now discuss the modifications that this entails for the above results. We may first choose an orthonormal frame $(e_i)^a$ of TU, with a corresponding frame $(e'_i)^a := \overline{\mu}_*(e_i)^a$ of TU'. These frames can be extended to orthonormal frames of $TM|_U$ and $TM'|_{U'}$, respectively, by including the normal vector field $e^a_0 := n^a$, resp. $(e'_0)^a := n^a$. Furthermore, the frames can be extended to a neighbourhood of U, resp. U', by parallel transporting them along the geodesics whose tangent vectors are e_0 on U, resp. e'_0 on U'.

Using these orthonormal frames we have

Lemma 2.15 Expressing all components and derivatives in terms of the Gaussian normal coordinates x^{μ} , resp. $(x')^{\mu}$, the orthonormal frames $(e_{\alpha})^{\mu}$ and $(e'_{\alpha})^{\mu}$ satisfy

$$\begin{aligned} (e_0)^{\mu} &= \delta_0^{\mu} &= (e'_0)^{\mu} \\ (e_i)^0 &= 0 &= (e'_i)^0 \\ \partial_{x^0}^k (e_\alpha)^{\mu} &= i^k \overline{\mu}^* (\partial_{(x')^0}^k (e'_\alpha)^{\mu}) \end{aligned}$$

on U for all $k \geq 0$.

Proof: By definition we have $e_0 = \partial_{x^0}$, which means that $(e_0)^i \equiv 0$ and $(e_0)^0 \equiv 1$. Similarly, $(e'_0)^i \equiv 0$ and $(e'_0)^0 \equiv 1$, from which the statement for $\alpha = 0$ follows. For $\alpha > 0$ the vanishing of $(e_\alpha)^0$ and $(e'_\alpha)^0$ follows from the orthonormality of the frames. Furthermore, the last equality holds for k = 0, by definition of e'_i in terms of e_i and by the fact that $x^i = \overline{\mu}^*(x')^i$. The extension away from U, resp. U', is then defined by the parallel transport, which is expressed by the equations

$$\partial_{x^0}(e_i)^{\mu} = -\frac{1}{2}(e_i)^{\nu}g^{\mu\rho}\partial_{x^0}g_{\nu\rho},$$

resp.

$$\partial_{(x')^0} (e'_i)^{\mu} = -\frac{1}{2} (e'_i)^{\nu} (\bar{g}_R)^{\mu\rho} \partial_{x^0} (\bar{g}_R)_{\nu\rho},$$

where we used the fact that the relevant components of the Christoffel symbols simplify in Gaussian normal coordinates. For the components $(e_i)^0$ and $(e'_i)^0$ the right-hand side vanishes identically, so these components vanish identically. For the other components we may prove the desired equality by induction over $k \ge 0$, by applying (k-1) normal derivatives on both sides and noting that the factors of *i* are due to Proposition 2.11, Corollary 2.12 and the induction hypothesis.

When using the frames e_{α} and e'_{α} to define Riemannian normal coordinates (w^{μ}, y^{μ}) and $((w')^{\mu}, (y')^{\mu})$, the corresponding statement of Proposition 2.14 remains valid. To see this, we introduce the dual frames $(f^{\alpha})_{\mu} := g_{\mu\nu}\eta^{\alpha\beta}(e_{\beta})^{\nu}$ of $(e_{\alpha})^{\mu}$ and similarly for $(e'_{\alpha})^{\mu}$. Note that $(f^{\alpha})_{\mu}(e_{\beta})^{\mu} = \delta^{\alpha}{}_{\beta}$ and $(f^{\alpha})_{\mu}(e_{\alpha})^{\nu} = \delta^{\nu}{}_{\mu}$. (The first follows directly from the fact that the $(e_{\alpha})^{\mu}$ are orthonormal. The second follows from the fact that the $(e_{\alpha})^{\mu}$ are a frame, because it holds when contracted with any $(e_{\beta})^{\mu}$.) We may now write $w^{\mu}(x, y) = (f^{\mu})_{\nu}(y)v^{\nu}(x, y)$. Using the definition of the dual frame and Lemma 2.15 it follows that the desired equalities for w^{μ} and $(w')^{\mu}$ are equivalent to those of Proposition 2.14. This proves

Proposition 2.16 On $U^{\times 2}$ we have, in the coordinates introduced above:

$$\begin{array}{lll} \partial_{x^{0}}^{k}\partial_{y^{0}}^{l}w^{j} & = & i^{k+l}(\overline{\mu}^{\times 2})^{*}\left(\partial_{(x')^{0}}^{k}\partial_{(y')^{0}}^{l}(w')^{j}\right),\\ \partial_{x^{0}}^{k}\partial_{y^{0}}^{l}w^{0} & = & i^{k+l-1}(\overline{\mu}^{\times 2})^{*}\left(\partial_{(x')^{0}}^{k}\partial_{(y')^{0}}^{l}(w')^{0}\right). \end{array}$$

To close this section, we consider the squared geodesic distance of a pseudo-Riemannian manifold, which is also known as Synge's world function in the Lorentzian case. It is defined as

$$\sigma(x,y) := \frac{1}{2} \|\exp_y^{-1}(x)\|_{g(y)}^2,$$

and in general it may take both positive and negative values. In the Riemannian normal coordinates v^{μ} (defined using the frame $\partial_{y^{\mu}}$) it takes the form

$$\sigma(v,y) = \frac{1}{2} (\exp_y^* g)_{\mu\nu}(0) v^{\mu} v^{\nu}.$$

As the map $t \mapsto \exp_y(tv^{\mu})$ is a geodesic, by definition of the exponential map, one may use the geodesic equation and a partial integration to show that

$$\sigma(v,y) = \frac{1}{2} \int_0^1 (\exp_y^* g)_{\mu\nu}(tv) v^{\mu} v^{\nu} dt$$

In other words, $\sigma(x, y)$ is the length squared of the unique geodesic in V which connects x to y in unit parameter time. We therefore have $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in V$ and one can also show that

$$(\exp_y^* g)_{\mu\nu}(0)v^{\nu} = \partial_{\mu}\sigma(v, y) = (\exp_y^* g)_{\mu\nu}(v)v^{\nu},$$

$$(\exp_y^* g)^{\mu\nu}(v)\partial_{\nu}\sigma(v, y) = v^{\mu} = (\exp_y^* g)^{\mu\nu}(0)\partial_{\nu}\sigma(v, y),$$

$$2\sigma(v, y) = (\exp_y^* g)^{\mu\nu}(v)\partial_{\mu}\sigma(v, y)\partial_{\nu}\sigma(v, y),$$

$$\sigma(0, y) = \partial_{\mu}\sigma(0, y) = 0,$$

$$\partial_{\mu}\partial_{\nu}\sigma(0, y) = (\exp_y^* g)_{\mu\nu}(0),$$
(10)

where all derivatives refer to the coordinates v^{μ} .

A comparison of σ in the Euclidean and Lorentzian case yields:

Corollary 2.17 Let σ be Synge's world function on V and let $\bar{\sigma}_R$ be the squared geodesic distance on V'. For all $k, l \geq 0$ we have

$$\partial_{x^0}^k \partial_{y^0}^l \sigma = i^{k+l} (\overline{\mu}^{\times 2})^* (\partial_{(x')^0}^k \partial_{(y')^0}^l \overline{\sigma}_R)$$

on $U^{\times 2}$.

Proof: This is a direct consequence of Propositions 2.14 and 2.11 and the fact that

$$\sigma = -\frac{1}{2}(v^0)^2 + \frac{1}{2}h_{ij}v^i v^j, \quad \bar{\sigma}_R = -\frac{i^2}{2}((v')^0)^2 + \frac{1}{2}h'_{ij}(v')^i(v')^j,$$

where h_{ij} and h'_{ij} are evaluated at y and y', respectively.

3 The linear scalar quantum field

In this section, we introduce the linear scalar field, its quantisation in a spacetime with a bifurcate Killing horizon and the class of quasi-free Hadamard states. We apply the initial value formulation of the field equation to two-point distributions, which yields a convenient setting to discuss the local aspects of the Wick rotation in the static case. We also briefly review how a Wick rotation can be used to obtain double β -KMS states in the disconnected spacetime $M^+ \cup M^-$ (and hence β -KMS states in M^+). For the purpose of this Wick rotation, we use global methods as in [50], which complement the local description that is used throughout most of this paper.

As a matter of convention, we will identify distribution densities on M, M_R^+ , Σ etc. with distributions, using the respective volume forms $d\text{vol}_g$, $d\text{vol}_{g_R}$ and $d\text{vol}_h$. To unburden our notation we will often leave the volume form implicit, which should not lead to any confusion. However, we point out that the volume form is important when restricting to submanifolds, because in that case a change in volume form is involved.

3.1 Initial value formulation of the linear scalar field

We recall that it is well understood how to quantise a linear scalar field on any globally hyperbolic spacetime M (cf. e.g. [1, 11, 7, 2]). At the classical level the theory is described by the (modified) Klein-Gordon operator

$$K := -\Box + V,$$

where the real-valued function $V \in C^{\infty}(M, \mathbb{R})$ serves as a potential. In any globally hyperbolic spacetime, the operator K has unique advanced (-) and retarded (+) fundamental solutions E^{\pm} and we define $E := E^{-} - E^{+}$. We describe the quantum theory by the Weyl C^{*} -algebra \mathcal{A} , generated by the operators W(f) with $f \in C_{0}^{\infty}(M, \mathbb{R})$ satisfying the Weyl relations

- 1. $W(f)^* = W(-f),$
- 2. W(Kf) = I,
- 3. $W(f)W(f') = e^{\frac{-i}{2}E(f,f')}W(f+f').$

Note that W(f) and W(f') are linearly dependent if and only if they are equal, which is the case if and only if $f' \in f + KC_0^{\infty}(M, \mathbb{R})$.⁷

An algebraic state ω on the Weyl algebra \mathcal{A} gives rise to a representation π_{ω} of the algebra on a Hilbert space \mathcal{H}_{ω} by the GNS-construction. We will mostly consider states for which the maps

$$\omega_n(f_1,\ldots,f_n) := (-i)^n \partial_{s_1} \cdots \partial_{s_n} \omega(W(s_1f_1) \cdots W(s_nf_n))|_{s_1=\ldots=s_n=0}$$

are distributions on $M^{\times n}$ for all $n \ge 1$: the *n*-point distributions. In fact, our primary interest is in quasi-free states, for which all *n*-point distributions can be expressed in terms of the two-point distribution via Wick's Theorem. We mention without proof the following well-known result:

Proposition 3.1 The two-point distribution $\omega_2 \in \mathcal{D}(M^{\times 2})$ of any state ω has the following properties:

- 1. $\omega_2(x,y)$ solves the Klein-Gordon equation in both variables,
- 2. $2\omega_{2-}(x,y) := \omega_2(x,y) \omega_2(y,x) = iE(x,y),$
- 3. $\omega_2(\overline{f}, f) \ge 0$ for all $f \in C_0^{\infty}(M)$.

Furthermore, any distribution ω_2 with these properties is the two-point distribution of a unique quasi-free state.

For quasi-free states it only remains to analyse the distributions ω_2 with these three properties. Equivalently, we can study one-particle structures:

Definition 3.2 A one-particle structure for K on M is a pair (p, \mathcal{H}) , which consists of a Hilbert space \mathcal{H} and an \mathcal{H} -valued distribution p on M such that

- 1. p has a dense range,
- 2. p(Kf) = 0 for all $f \in C_0^{\infty}(M)$,
- 3. $\langle p(\bar{f}), p(f') \rangle \langle p(\bar{f}'), p(f) \rangle = iE(f, f').$

⁷Proof: if $W(f) = \lambda W(f')$ for some $\lambda \in \mathbb{C}$, then we may use the fact that $W(-f) = W(f)^{-1}$ and compute for all $\chi \in C_0^{\infty}(M, \mathbb{R})$:

$$I = \lambda W(\chi) W(f') W(-f) W(-\chi) = \lambda e^{-iE(\chi, f'-f) + \frac{i}{2}E(f', f)} W(f'-f).$$

Comparing a general χ with $\chi = 0$ we can eliminate the Weyl operators to find $1 = e^{-iE(\chi, f'-f)}$ for all χ , which means that E(f'-f) = 0. By a standard result [11] it follows that $f' - f \in KC_0^{\infty}(M, \mathbb{R})$, which in turn implies W(f) = W(f - f')W(f') = W(f').

The bijective relationship between one-particle structures and two-point distributions is given by

$$\omega_2(f, f') = \langle p(\bar{f}), p(f') \rangle. \tag{11}$$

Note that any two-point distribution ω_2 determines a one-particle Hilbert space \mathcal{K}_{ω_2} , which is defined as the Hilbert space completion of $C_0^{\infty}(M)$ after dividing out the linear space of vectors of zero norm in the semi-definite inner product $\langle f, h \rangle := \omega_2(\overline{f}, h)$. The map $\mathcal{K}_{\omega_2}: C_0^{\infty}(M) \to \mathcal{K}_{\omega_2}$ defined by $\mathcal{K}_{\omega_2}(f) := [f]$ is a Hilbert space-valued distribution (cf. [54]), which may be interpreted as $\mathcal{K}_{\omega_2}(f) = \pi_{\omega}(\Phi(f))\Omega_{\omega}$, where $\Omega_{\omega} \in \mathcal{H}_{\omega}$ is the GNS-vector in the GNS-representation π_{ω} of the quasi-free state ω determined by ω_2 .

Let us now recall the initial value formulation of the Klein-Gordon equation in a globally hyperbolic spacetime M on a Cauchy surface $\Sigma \subset M$ with future pointing normal vector field n^a . If ω_2 is the two-point distribution of a state, then it is completely determined by its initial data⁸ on $\Sigma^{\times 2}$, namely

$$\begin{split} \omega_{2,00} &:= \omega_2|_{\Sigma^{\times 2}}, \qquad \omega_{2,01} := (1 \otimes n^a \nabla_a) \omega_2|_{\Sigma^{\times 2}} \\ \omega_{2,10} &:= (n^a \nabla_a \otimes 1) \omega_2|_{\Sigma^{\times 2}}, \qquad \omega_{2,11} := (n^a \nabla_a \otimes n^b \nabla_b) \omega_2|_{\Sigma^{\times 2}}. \end{split}$$

These distributional restrictions are well-defined by a microlocal argument and for their definition we treat ω_2 as a distribution, not a distribution density. To see how these initial data determine ω_2 we let $f, f' \in C_0^{\infty}(M)$ and we introduce the initial data $f_0 := Ef|_{\Sigma}, f_1 := n^a \nabla_a Ef|_{\Sigma}$ and similarly for f'. By a standard computation (analogous to Lemma A.1 of [11]) one may show that

$$\omega_2(f, f') = \sum_{m,n=0}^{1} (-1)^{m+n} \omega_{2,mn}(f_{1-m}, f'_{1-n}), \qquad (12)$$

where we used the fact that ω_2 is a distributional bi-solution to the Klein-Gordon equation. (Recall that the volume forms of M, respectively Σ , are implicit on the left, respectively right-hand side of this equation.)

There is a preferred class of states, called Hadamard states, which are characterised by the fact that their two-point distribution has a singularity structure at short distances that is of the same form as that of the Minkowski vacuum state. To put it more precisely, ω_2 is of Hadamard form if and only if [47]

$$WF(\omega_2) = \{ (x,k;y,l) \in T^*M^{\times 2} | l \neq 0 \text{ is future pointing and lightlike and } (y,l)$$
(13)
generates a geodesic γ which goes through x with tangent vector $-k \}.$

By the Propagation of Singularities Theorem and the fact that ω_2 solves the Klein-Gordon equation in both variables it suffices to check the condition in Equation (13) on a Cauchy surface Σ :

$$WF(\omega_2)|_{\Sigma} \subset \left\{ (x, -k; x, k) | (x, k) \in V^+ M|_{\Sigma} \right\},\$$

where V^+M denotes the fibre bundle of future pointing covectors. Unfortunately, it is somewhat complicated to see whether a state ω_2 is Hadamard by inspecting its initial data on a Cauchy surface Σ . The initial data of ω_2 should be smooth away from the diagonal in $\Sigma^{\times 2}$, so it suffices to characterise the singularities near the diagonal.⁹ However, for the singularities near the diagonal we are not aware of any general argument that avoids the use of the Hadamard parametrix construction, which involves the Hadamard series after which Hadamard states are named.¹⁰ We will explain this construction in detail for both the Lorentzian and Euclidean setting in Section 4 below.

 $^{^{8}}$ To analyse the singularities and restrictions of distributions we freely make use of basic notions and results from microlocal analysis, referring the reader to [29] for details.

⁹Conversely, if ω_2 has the correct singularity structure near the diagonal on Σ , then it follows essentially from [48] and the propagation of singularities that ω_2 is Hadamard and hence smooth away from the diagonal in $\Sigma^{\times 2}$.

¹⁰The recent work [18] presents a more elegant procedure, but it makes additional assumptions on the Cauchy surface that we wish to avoid.

3.2 Double β -KMS states on $M^+ \cup M^-$ in the stationary case

We consider the Klein-Gordon equation on a spacetime M with a stationary bifurcate Killing horizon. Because the right wedge M^+ is a (possibly disconnected) stationary, globally hyperbolic spacetime we can apply the analysis of [50] to obtain ground and β -KMS states under suitable circumstances. We will briefly review these results and show how they can be extended to the disconnected spacetime $M^+ \cup M^-$.

In order to apply the results of [50], we assume that the potential V is stationary and positive on the right wedge:

$$\xi^a \nabla_a V|_{M^+} = \partial_t V|_{M^+} \equiv 0, \quad V|_{M^+} > 0.$$

On M^+ the Klein-Gordon operator can be written in terms of the Killing time coordinate t and the induced metric h_{ij} on the Cauchy surface Σ^+ :

$$v^{\frac{3}{2}}Kv^{\frac{1}{2}} = \partial_t^2 + D\partial_t + C,$$

$$D := -(\nabla_i^{(h)}w^i + w^i\nabla_i^{(h)}),$$

$$C := -v^{\frac{1}{2}}\nabla_i^{(h)}(vh^{ij} - v^{-1}w^iw^j)\nabla_j^{(h)}v^{\frac{1}{2}} + Vv^2.$$
(14)

The operator K is a symmetric operator on $L^2(M^+)$ defined on the dense domain $C_0^{\infty}(M^+)$.

We now formulate the fundamental result on ground and β -KMS states on M^+ ([50] Theorems 5.1 and 6.2, which may be generalised to spacetimes which are not necessarily connected). For an overview of further properties of the ground and β -KMS states, we refer to [50] and references therein.

Theorem 3.3 Consider a linear scalar field on M^+ with a stationary potential V such that V > 0.

- 1. There exists a unique extremal ground state ω^0 with a well-defined, vanishing one-point distribution.
- 2. For every $\beta > 0$ there exists a unique extremal β -KMS state $\omega^{(\beta)}$ with a well-defined, vanishing one-point distribution.

All these states are quasi-free and Hadamard.

Remark 3.4 Other ground and β -KMS states can be obtained as follows. Firstly, one may replace the quantum field $\Phi(x)$ by $\Phi(x) + \phi(x)I$ (a gauge transformation of the second kind), where $\phi(x)$ is a real-valued, Killing time independent (weak) solution of the Klein-Gordon equation, if such solutions exist. More precisely, we replace W(f) by $e^{i\phi(f)}W(f)$, where ϕ is interpreted as a distribution density. This defines an automorphism of the Weyl algebra and the pull-back of the states in Theorem 3.3 under this isomorphism remain extremal ground, resp. β -KMS states. Furthermore, one may take mixtures of such ground or β -KMS states to obtain non-extremal ones. It can be shown that all ground and β -KMS states are of this form [50] and that their twopoint distributions ω_2 majorise those of Theorem 3.3 (i.e. $\omega_2(\overline{f}, f) \ge \omega_2^{(\beta)}(\overline{f}, f)$ and similarly for ground states). If any solutions $\phi(x)$ exist at all, the corresponding ground and β -KMS states are often discarded, because the one-point distribution $\phi(x)$ grows exponentially near spatial infinity. Restricting attention e.g. to tempered n-point distributions in Minkowksi spacetime one disqualifies all states other than the ones in Theorem 3.3.

We will now describe how the one-particle structure $(p_{(\beta)}, \mathcal{H}_{(\beta)})$, which gives rise to the twopoint distribution $\omega_2^{(\beta)}$ of the β -KMS state on M^+ , can be obtained from the classical Hilbert space of finite energy solutions (cf. [33]).

We let \mathcal{H}_e be the Hilbert space of finite energy solutions ϕ of the Klein-Gordon equation on M^+ , where the norm is given by the square root of the energy. \mathcal{H}_e contains a dense subset of spacelike compact, smooth solutions, whose energy may be obtained by integrating the energy density over any Cauchy surface (cf. [50]). Complex conjugation on these spacelike compact solutions preserves the energy, so it can be extended to a complex conjugation C on \mathcal{H}_e (i.e. a complex anti-linear involution). There is an \mathcal{H}_e -valued distribution

$$p_{cl}: C_0^\infty(M^+) \to \mathcal{H}_e: f \mapsto Ef,$$

which satisfies $Cp_{cl}(f) = p_{cl}(\bar{f})$ and solves the Klein-Gordon equation in the sense that $p_{cl}(Kf) = 0$ for all $f \in C_0^{\infty}(M^+)$. The Killing time evolution is implemented on \mathcal{H}_e by a strongly continuous unitary group e^{itH_e} , where the Hamiltonian H_e is an invertible self-adjoint operator. We note that the range of p_{cl} is a core for all powers of H_e and for $|H_e|^{-1}$ (cf. [50] Thm. 4.2) and we let P_{\pm} denote the spectral projections onto the positive and negative spectrum of H_e .

The one-particle structure $(p_{(\beta)}, \mathcal{H}_{(\beta)})$ can now be expressed as (cf. [50] Thm. 4.3):

$$p_{(\beta)}(f) := \sqrt{2}P_{-}|H_{e}|^{-\frac{1}{2}}(I - e^{-\beta|H_{e}|})^{-\frac{1}{2}}p_{cl}(f)$$

$$\oplus \sqrt{2}P_{+}|H_{e}|^{-\frac{1}{2}}e^{-\frac{\beta}{2}|H_{e}|}(I - e^{-\beta|H_{e}|})^{-\frac{1}{2}}p_{cl}(f)$$

which is a distribution on M^+ with values in the Hilbert space $P_-\mathcal{H}_e \oplus P_+\mathcal{H}_e \simeq \mathcal{H}_e$. Note that $p_{(\beta)}$ has a dense range, so $H_{(\beta)} = \mathcal{H}_e$.¹¹ The Killing time evolution is now implemented by $H = |H_e| \oplus -|H_e| = -H_e$. A similar, but simpler, description holds for the ground state.

We now assume that M admits a wedge reflection and we wish to extend the states above from M^+ to the union $M^+ \cup M^-$. More precisely, in this section we will only assume that there is an isometric, involutive diffeomorphism I of $M^+ \cup M^-$ which reverses the time orientation and which satisfies $I^*\xi^a = \xi^a$. This assumption is even weaker than the existence of a weak wedge reflection, but it suffices for the purposes of this section, because we are not yet investigating extensions across the Killing horizon. Note that a Cauchy surface Σ^+ of M^+ maps to a Cauchy surface $\Sigma^- := I(\Sigma^+)$ of M^- .

The quotient space $C_0^{\infty}(M^+ \cup M^-, \mathbb{R})/KC_0^{\infty}(M^+ \cup M^-, \mathbb{R})$ is a symplectic space with the symplectic form E. If we also assume

$$I^*V = V,$$

then it naturally carries the structure of a double linear dynamical system, in the sense of [33]. This means that it is a direct sum of two symplectic spaces,

$$(C_0^{\infty}(M^+,\mathbb{R})/KC_0^{\infty}(M^+),\mathbb{R})\oplus (C_0^{\infty}(M^-,\mathbb{R})/KC_0^{\infty}(M^-,\mathbb{R})),$$

each of which is preserved under the Killing time evolution, and there is a linear involution, namely I^* , which maps the symplectic subspace of M^+ onto that of M^- and vice versa, which commutes with the Killing time evolution and which is anti-symplectic in the sense that $E(I^*f, I^*f') = -E(f, f')$. To see how this last property of I^* arises we only need to fix a Cauchy surface Σ^+ of M^+ and to express the symplectic form E in terms of initial data on $\Sigma := \Sigma^+ \cup I(\Sigma^+)$:

$$E(f, f') = \int_{\Sigma} Ef \cdot n^a \nabla_a Ef' - Ef' \cdot n^a \nabla_a Ef.$$
⁽¹⁵⁾

To compute $E(I^*f, I^*f')$, the data of EI^*f and EI^*f' are expressed as the pull-backs of the data of Ef and Ef' by $I|_{\Sigma}$, where the normal derivatives get an additional sign, because I reverses the time orientation.

The Weyl algebra of the scalar quantum field on $M^+ \cup M^-$ is the spatial tensor product of the algebras on M^+ and M^- and the wedge reflection I gives rise to a complex anti-linear involution $\tau_I : zW(f) \mapsto \overline{z}W(I^*f)$, which preserves products and the *-operation and which commutes with

¹¹Proof: Given any $\psi \in \mathcal{H}_e$ we define $\psi_{\pm} := P_{\pm}\psi$. For a dense set of such ψ the vector $\tilde{\psi} := |H_e|^{\frac{1}{2}}(I - e^{-\beta|H_e|})^{\frac{1}{2}}(\psi_- \oplus e^{\frac{\beta}{2}|H_e|}\psi_+)$ is well-defined. Because the range of p_{cl} is a core for $|H_e|^{-\frac{1}{2}}(I - e^{-\beta|H_e|})^{-\frac{1}{2}}$, we can find a sequence $f_n \in C_0^{\infty}(M^+)$ such that $p_{cl}(f_n)$ converges to $\tilde{\psi}$ and $|H_e|^{-\frac{1}{2}}(I - e^{-\beta|H_e|})^{-\frac{1}{2}}p_{cl}(f_n)$ converges to $\psi_- \oplus e^{\frac{\beta}{2}|H_e|}\psi_+$. Because $e^{-\frac{\beta}{2}|H_e|}$ is bounded it follows that $p_{(\beta)}(f_n)$ converges to $\sqrt{2}\psi$ and therefore that $p_{(\beta)}$ has a dense range.

the Killing time evolution. We will call a state ω on $M^+ \cup M^-$ a double β -KMS state when its restriction to M^+ is a β -KMS state $\omega^{(\beta)}$ and when

$$\omega(W(f^+ + f^-)) = F_{W(I^* f^-), W(f^+)}\left(\frac{i\beta}{2}\right)$$
(16)

for all $f^{\pm} \in C_0^{\infty}(M^{\pm})$, where $F_{W(I^*f^-)W(f^+)}$ is the bounded continuous function on $\overline{\mathcal{S}}_{\beta} := \{z \in \mathbb{C} | \operatorname{Im}(z) \in [0,\beta] \}$ which is holomorphic on its interior and which satisfies $F_{W(I^*f^-)W(f^+)}(t) = \mathbb{C} | \mathbb$ $\omega^{(\beta)}(W(I^*f^-)W(\Phi_{-t}(f^+)))$ and $F_{W(I^*f^-)W(f^+)}(t+i\beta) = \omega^{(\beta)}(W(\Phi_{-t}(f^+))W(I^*f^-))$. This function exists by the definition of β -KMS states. We note that any double β -KMS state is invariant under the wedge reflection in the sense that

$$\omega(\tau_I(A)) = \overline{\omega(A)},\tag{17}$$

because $F_{W(I^*f^-),W(f^+)}\left(\frac{i\beta}{2}\right) = F_{W(f^+),W(I^*f^-)}\left(\frac{i\beta}{2}\right)$. For any $\beta > 0$ the one-particle structure $(p_{(\beta)}, \mathcal{H}_e)$ on M^+ also determines a double β -KMS one-particle structure in the sense of [33] on the double linear dynamical system of $M^+ \cup M^-$. This is a one-particle structure (p, \mathcal{H}) on $M^+ \cup M^-$ such that p has a dense range on each of M^+ and M^{-} ; the Killing time evolution is implemented by a strongly continuous unitary group e^{itH} , where H has no zero eigenvalue; there is a complex conjugation C on \mathcal{H} such that $Cp(f) = -p(I^*\bar{f})$ for all $f \in C_0^{\infty}(M^+ \cup M^-)$; and $p(C_0^{\infty}(M^{\pm}))$ is in the domain of $e^{\pm \frac{\beta}{2}H}$ with

$$e^{\mp \frac{\rho}{2}H}p(f) = -p(I^*f), \quad f \in C_0^{\infty}(M^{\pm}).$$

To obtain this double β -KMS structure we take $(p^d_{(\beta)}, \mathcal{H}_e)$ with

$$p^{d}_{(\beta)}(f^{+} + f^{-}) := p_{(\beta)}(f^{+}) - e^{-\frac{\beta}{2}H}p_{(\beta)}(I^{*}f^{-}),$$

where $f^{\pm} \in C_0^{\infty}(M^{\pm})$. It can be verified that this is well-defined and it has all the desired properties, where the Killing time evolution is again implemented by $H = -H_e$. The complex conjugation on \mathcal{H}_e is the given conjugation C, which satisfies CH = -HC, so it exchanges the negative and positive frequency subspaces of H_e . We denote by $\omega_2^{(\beta),d}$ the two-point distribution on $M^+ \cup M^-$ determined by $p^d_{(\beta)}$.

Kay has shown that this double β -KMS one-particle structure is unique [33] and he considered corresponding quasi-free double β -KMS states on double wedge algebras in [34, 35]. In our case, we may obtain the following result:

Theorem 3.5 Let M be a globally hyperbolic spacetime with a stationary bifurcate Killing horizon and assume that there is an isometric, involutive diffeomorphism I of $M^+ \cup M^-$ onto itself which reverses the time orientation and satisfies $I^*\xi^a = \xi^a$. We consider the Klein-Gordon equation with a stationary potential V such that V > 0 and $I^*V = V$. For each $\beta > 0$ there exists a unique double β -KMS state $\omega^{(\beta),d}$ on $M^+ \cup M^-$ whose restriction to M^+ is $\omega^{(\beta)}$. This state is pure, quasi-free, Hadamard, it has the Reeh-Schlieder property and its two-point distribution is given by $\omega_2^{(\beta),d}$.

Proof: By Equation (16) there is at most one double β -KMS state on $M^+ \cup M^-$ which restricts to a given β -KMS state on M^+ . It is clear that the quasi-free state $\omega^{(\beta),d}$ with two-point distribution $\omega_2^{(\beta),d}$ restricts to $\omega^{(\beta)}$ on M^+ and we will show this is a double β -KMS state and prove its properties.

Using the complex conjugation C and the properties of $p^d_{(\beta)}$ we find

$$\omega_{2}^{(\beta),d}(I^{*}f,I^{*}f') = \langle p_{(\beta)}^{d}(I^{*}\overline{f}), p_{(\beta)}^{d}(I^{*}f') \rangle = \langle -Cp_{(\beta)}^{d}(f), -Cp_{(\beta)}^{d}(\overline{f'}) \rangle$$

$$= \langle p_{(\beta)}^{d}(\overline{f'}), p_{(\beta)}^{d}(f) \rangle = \omega_{2}^{(\beta),d}(f',f)$$
(18)

Because $\omega^{(\beta),d}$ restricts to $\omega^{(\beta)}$ on M^+ it is Hadamard there. The symmetry property above proves that $\omega^{(\beta),d}$ is Hadamard on M^- as well, because I reverses the time orientation and it interchanges the arguments of $\omega_2^{(\beta),d}$. Furthermore, we may use $Cp_{(\beta)}^d(f^{\pm}) = -p_{(\beta)}^d(I^*f^{\pm}) = e^{\pm \frac{\beta}{2}H}p_{(\beta)}^d(f^{\pm})$ for any $f^{\pm} \in C_0^{\infty}(M^{\pm}, \mathbb{R})$ to see that

$$\begin{split} \omega_{2}^{(\beta),d}(f^{+},f^{-}) &= \langle p_{(\beta)}^{d}(f^{+}), p_{(\beta)}^{d}(f^{-}) \rangle = \langle Cp_{(\beta)}^{d}(f^{-}), Cp_{(\beta)}^{d}(f^{+}) \rangle \\ &= \langle e^{\frac{\beta}{2}H} p_{(\beta)}^{d}(f^{-}), e^{-\frac{\beta}{2}H} p_{(\beta)}^{d}(f^{+}) \rangle = \omega_{2}^{(\beta),d}(f^{-},f^{+}) \\ &= -\langle e^{-\frac{\beta}{2}H} p_{(\beta)}(I^{*}f^{-}), p_{(\beta)}(f^{+}) \rangle. \end{split}$$

It follows that

$$\omega^{(\beta),d}(W(f^-)W(f^+)) = e^{-\frac{1}{2}\omega_2^{(\beta),d}(f^+ + f^-, f^+ + f^-)} = \omega(W(I^*f^-))\omega(W(f^+))e^{-\omega_2^{(\beta),d}(f^-, f^+)}.$$

On the other hand, we can use the β -KMS condition to find $F_{W(I^*f^-),W(f^+)}$, which may be written as

$$F_{W(I^*f^-),W(f^+)}(z) = \omega(W(I^*f^-))\omega(W(f^+))e^{-F_{I^*f^-,f^+}(z)}$$

where the function $F_{I^*f^-,f^+}(z)$ on $\overline{\mathcal{S}}_{\beta}$ is holomorphic on the interior and is given by

$$F_{I^*f^-,f^+}(z) = \langle p_{(\beta)}(I^*f^-), e^{izH} p_{(\beta)}(f^+) \rangle.$$

Evaluating this function at $z = \frac{i\beta}{2}$ we see that $\omega^{(\beta),d}$ is a double β -KMS state. We now prove that $p_{(\beta)}^d$ has a dense range on $C_0^{\infty}(M^+ \cup M^-, \mathbb{R})$. We use the complex conjugation C to write \mathcal{H}_e as a direct sum of the real Hilbert spaces of real vectors, with $C\psi = \psi$, and imaginary ones, with $C\psi = -\psi$. Taking the time derivative of $Ce^{itH_e}C = e^{itH_e}$ at t = 0we find $CH_eC = -H_e$ and therefore $C(P_+ - P_-)C = -(P_+ - P_-)$. This means that the linear involution $Q := P_+ - P_-$ maps real vectors to imaginary ones and vice versa. The operators $X^{\pm} := |H_e|^{-\frac{1}{2}} \tanh(\frac{\beta}{4}|H_e|)^{\pm \frac{1}{2}}$ are strictly positive for each choice of the sign and the range of p_{cl} on $C_0^{\infty}(M^+)$ is a core for both of these operators (cf. Thm. 4.2 in [50]). The range of $X^{\pm}p_{cl}$ on $C_0^{\infty}(M^+)$ is therefore dense. Furthermore, the complex conjugation C commutes with $|H_e|$ and $Cp_{cl}(f) = p_{cl}(\bar{f})$, which means that $X^{\pm}p_{cl}$ has a dense range in the real subspace of \mathcal{H}_e if f ranges over the real-valued test-functions $C_0^{\infty}(M^+,\mathbb{R})$. A straightforward computation shows that

$$p^{d}_{(\beta)}(f - I^{*}f) = \sqrt{2}X^{-}p_{cl}(f),$$

$$p^{d}_{(\beta)}(f + I^{*}f) = -\sqrt{2}QX^{+}p_{cl}(f)$$

with $f \in C_0^{\infty}(M^+, \mathbb{R})$. By varying f, the arguments on the left remain in $C_0^{\infty}(M^+ \cup M^-, \mathbb{R})$, and the ranges on the right are dense in the space of real and imaginary vectors, respectively. Therefore $p_{(\beta)}^d$ has a dense range on $C_0^{\infty}(M^+ \cup M^-, \mathbb{R})$. The fact that $p_{(\beta)}^d$ has a dense range already on $C_0^{\infty}(M^+ \cup M^-, \mathbb{R})$ entails that $\omega^{(\beta),d}$ is pure [37] and that it is the unique state with this two-point distribution [36].

Finally, note that the GNS-representation space of $\omega^{(\beta),d}$ is the same as for $\omega^{(\beta)}$. Since the latter already has the Reeh-Schlieder property (cf. [53]), the same is true for the former, at least on M^+ . That this also holds on M^- follows from the symmetry with respect to the wedge reflection I at the one-particle level.

Remark 3.6 In analogy to Remark 3.4 one may obtain additional, pure, double β -KMS states by applying an automorphism of the Weyl algebra determined by $W(f) \mapsto e^{i\phi(f)}W(f)$, where $\phi(x)$ is now a real-valued (weak) solution to the Klein-Gordon equation on $M^+ \cup M^-$ which is independent of the Killing time and satisfies $I^*\phi = \phi$. Subsequently one may obtain mixed double β -KMS states by taking mixtures of these pure ones. It is straightforward to verify that the double β -KMS condition is invariant under these automorphisms and under taking mixtures. Note that any double β -KMS state is uniquely determined by its β -KMS restriction to M^+ and Equation

(16). Conversely, any β -KMS state ω on M^+ has a double β -KMS extension. To see this we note that ω can be obtained from $\omega^{(\beta)}$ by applying suitable automorphisms and mixing. These operations can be extended to $\omega^{(\beta),d}$ on $M^+ \cup M^-$, by requiring each $\phi(x)$ to be symmetric under the wedge reflection I. This yields a double β -KMS state ω^d with the prescribed restriction ω .

3.3 Double β -KMS states in the static case and Wick rotation

Let us now consider the Klein-Gordon equation on a spacetime M with a static bifurcate Killing horizon. Because the right wedge M^+ is a (possibly disconnected) standard static spacetime we can obtain the two-point distributions of its ground and β -KMS states from a Wick rotation. We will briefly review this procedure and show how it can be extended to the disconnected spacetime $M^+ \cup M^-$.

In the static case, we have D = 0 in Equation (14) and we quote the following properties of C from [50], Proposition 4.3:

Proposition 3.7 Consider the partial differential operator C of Equation (14), defined on the dense domain $C_0^{\infty}(\Sigma^+)$ of $L^2(\Sigma^+)$ (in the metric volume form $dvol_h$). C preserves its domain and all integer powers of C are essentially self-adjoint on this domain. The self-adjoint operator \overline{C} is strictly positive (i.e. positive and injective) and it satisfies $C \geq Vv^2$. Finally, $C_0^{\infty}(\Sigma^+)$ is in the domain of $\overline{C}^{\pm \frac{1}{2}}$ for both signs.

From now on, we shall use C to denote the unique self-adjoint extension.

In analogy to the Lorentzian theory on M^+ one considers a Euclidean theory on M_R^+ for any given R > 0. This theory is defined by the Euclidean version

$$K_R := -\Box_{g_R} + V$$

of the Klein-Gordon operator, which satisfies

$$v^{\frac{3}{2}}K_R v^{\frac{1}{2}} = -\partial_{\tau}^2 + C.$$

Here, τ is the imaginary Killing time, compactified to a circle of radius R, and the function v and the operator C depend only on the spatial coordinates on Σ^+ . The operator K_R is symmetric and positive on the dense domain $C_0^{\infty}(M_R^+)$ of $L^2(M_R^+)$. We let \hat{K}_R be the self-adjoint Friedrichs extension of K_R , which satisfies $\hat{K}_R \geq V$ on the domain of $\hat{K}_R^{\frac{1}{2}}$, so that \hat{K}_R is strictly positive and the domain of $\hat{K}_R^{-\frac{1}{2}}$ contains $C_0^{\infty}(M_R^+)$ (cf. [50] Lemma A.6). Hence, the operator

$$G_R := \hat{K}_R^{-1}$$

defines a distribution density on $(M_R^+)^{\times 2}$ (loc.cit. Theorem A.1), which is the Euclidean Green's function.

Remark 3.8 In [50] we used a different, but equivalent, definition of the Euclidean Green's function. There we noted that vK_Rv is essentially self-adjoint on the domain of test-functions, that the closure $\overline{vK_Rv} \ge Vv^2$ is strictly positive and that the domain of $(\overline{vK_Rv})^{-\frac{1}{2}}$ contains the space of all test-functions $C_0^{\infty}(M_R^+)$, which is a core. We then set $G_R := v(\overline{vK_Rv})^{-1}v$, which again defines a distribution density on $(M_R^+)^{\times 2}$. To see that both definitions are equivalent we argue as follows. Define the operator $X := (\overline{vK_Rv})^{\frac{1}{2}}v^{-1}$ on the domain of test-functions. X^* extends $v^{-1}(\overline{vK_Rv})^{\frac{1}{2}}$, which is defined on the range of $(\overline{vK_Rv})^{-\frac{1}{2}}$, acting on the domain of test-functions. This means that X^* is densely defined, so X is closable. Now $|\overline{X}|^2 = X^*\overline{X}$ extends K_R . Note that the domain of |X| equals the form domain of \hat{K}_R , which implies that $\hat{K}_R = X^*\overline{X} = v^{-1}(\overline{vK_Rv})v^{-1}$. Taking the inverses, we see that both definitions of the Euclidean Green's function are equivalent.

The dependence of G_R on the imaginary time τ can be determined explicitly, leading to a continuous function from $\mathbb{S}_R^{\times 2}$ into the distribution densities on $(\Sigma^+)^{\times 2}$ which is given by

$$G_R(\tau, \tau'; \overline{f}, f') = \langle C^{-\frac{1}{2}} v f, \frac{\cosh((\tau - \tau' + \pi R)\sqrt{C})}{2\sinh(\pi R\sqrt{C})} v f' \rangle$$
(19)

when $\tau - \tau' \in [-2\pi R, 0]$. We now note the following result, which is familiar from Wick rotations in Minkowski spacetime [44, 45].

Proposition 3.9 (Reflection positivity) Consider the open region $V' \subset M_R^+$ defined by $V' := \{(\tau, x) \in M_R^+ | \tau \in (0, \pi R)\}$ and let $R_\tau : M_R^+ \to M_R^+$ be the imaginary time reflection $R_\tau(\tau, x) := (-\tau, x)$. For every $\phi \in C_0^\infty(V')$ we then have

$$G_R(\overline{R^*_\tau\phi},\phi) \ge 0.$$

Proof: Without the imaginary time reflection R_{τ} this formula would be clear from the positivity of $G_R = \hat{K}_R^{-1}$. To see that the positivity remains valid in the presence of the reflection we note that it suffices to consider test-functions of the form $\phi(\tau, x) = \chi(\tau)f(x)$, by Schwartz Kernels Theorem. In that case, we may use Proposition 3.7 to introduce the vector $\psi \in L^2(\Sigma^+)$ defined by

$$\psi := (I - e^{-2\pi R\sqrt{C}})^{-\frac{1}{2}} C^{-\frac{1}{4}} v f$$

and we note that

$$G_R(\overline{R^*_{\tau}\phi},\phi) = \frac{1}{2} \int_{\mathbb{S}_R^{\times 2}} (\langle e^{-\tau\sqrt{C}}\psi, e^{-\tau'\sqrt{C}}\psi \rangle + \langle e^{(\tau-\pi R)\sqrt{C}}\psi, e^{(\tau'-\pi R)\sqrt{C}}\psi \rangle)\overline{\chi}(\tau)\chi(\tau')d\tau \ d\tau',$$

where we used the support properties of χ and Equation (19). Performing the integrations we end up with a sum of squared norms of vectors in $L^2(\Sigma^+)$, which is clearly non-negative.

 G_R can be analytically continued to $z = t + i\tau$ ([50] Theorem 6.4). In this way we find a holomorphic function $G_R^c(z, z')$ from $(z, z') \in \mathcal{C}_R^{\times 2}$ with $\operatorname{Im}(z-z') \in (-2\pi R, 0)$ into the distribution densities on $(\Sigma^+)^{\times 2}$:

$$G_R^c(z, z'; \overline{f}, f') = \langle C^{-\frac{1}{2}} v f, \frac{\cos((z - z' + i\pi R)\sqrt{C})}{2\sinh(\pi R\sqrt{C})} v f' \rangle$$

$$\tag{20}$$

for $f, f' \in C_0^{\infty}(\Sigma^+)$. This function has continuous boundary values at $\operatorname{Im}(z - z') \in [-2\pi R, 0]$. Restricting G_R^c to real times t, t' with $\operatorname{Im}(z - z') \to 0^-$ yields the two-point distribution $\omega_2^{(\beta)}$ of the quasi-free β -KMS state ω^{β} on M^+ with $\beta = 2\pi R$. A similar result holds for the ground state, in the degenerate case $R = \infty$. That ω^{β} is Hadamard follows from the fact that it is a boundary value of a holomorphic function and positivity can be shown using the initial data formulation and reflection positivity.

After this review of the Wick rotation for β -KMS states on M^+ it is now easy to describe a corresponding result for the disconnected spacetime $M^+ \cup M^-$, if the spacetime M has a (weak) wedge reflection.¹² Indeed, due to the wedge reflection there is an embedding $\chi: M^+ \cup M^- \rightarrow (M^+)_R^c$ (cf. Equation (3)), such that the complement of its range in $(M^+)_R^c$ is the union of the two regions where $\operatorname{Im}(z) \in (-\pi R, 0)$, respectively $\operatorname{Im}(z) \in (-2\pi R, -\pi R)$. Analogously, the image of $\chi^{\times 2}$ in $(M_R^c)^{\times 2}$ is the boundary of the region

$$\{(z, x; z', x') | \operatorname{Im}(z) \in (-\pi R, 0), \operatorname{Im}(z') \in (0, \pi R)\}$$

where $\text{Im}(z - z') \in (-2\pi R, 0)$. Taking the continuous extension of G_R^c to this boundary (cf. Equation (20)) defines a distribution density $\omega_2^{(\beta),d}$ on $M^+ \cup M^-$, which extends the β -KMS two-point distribution on M^+ .

To see what this boundary value looks like we proceed as follows. For any test-function $f \in C_0^{\infty}(\Sigma \setminus \mathcal{B})$ we may use the wedge reflection ι to write $f = f^+ + \iota^* f^-$ with unique $f^+, f^- \in C_0^{\infty}(\Sigma^+)$.

¹²As in Section 3.2 it suffices to assume that there is an isometric, involutive diffeomorphism I of $M^+ \cup M^-$ which reverses the time orientation and which satisfies $I^*\xi^a = \xi^a$, because we are not yet investigating the behaviour near the Killing horizon.

It is then easy to see that $\omega_2^{(\beta),d}$ takes the form

$$\begin{split} \omega_2^{(\beta),d}(t,f;t',f') &= \langle C^{-\frac{1}{2}}|v|\overline{f^+}, \frac{\cos((t-t'+i\pi R)\sqrt{C})}{2\sinh(\pi R\sqrt{C})}|v|(f')^+\rangle \\ &+ \langle C^{-\frac{1}{2}}|v|\overline{f^-}, \frac{\cos((t-t')\sqrt{C})}{2\sinh(\pi R\sqrt{C})}|v|(f')^+\rangle \\ &+ \langle C^{-\frac{1}{2}}|v|\overline{f^+}, \frac{\cos((t-t')\sqrt{C})}{2\sinh(\pi R\sqrt{C})}|v|(f')^-\rangle \\ &+ \langle C^{-\frac{1}{2}}|v|\overline{f^-}, \frac{\cos((t-t'-i\pi R)\sqrt{C})}{2\sinh(\pi R\sqrt{C})}|v|(f')^-\rangle \end{split}$$

for any test-functions $f, f' \in C_0^{\infty}(\Sigma \setminus \mathcal{B})$. $\omega_2^{(\beta),d}$ is a bi-solution to the Klein-Gordon equation on $M^+ \cup M^-$, given by the Klein-Gordon operator

$$\begin{split} K &= \partial_t^2 + A \\ A &= -|v|^{\frac{1}{2}} \nabla_i^{(h)} |v| h^{ij} \nabla_j^{(h)} |v|^{\frac{1}{2}} + V v^2 \end{split}$$

with the potential function V extended from M^+ to $M^+ \cup M^-$ such that

$$\iota^* V = V.$$

Because V is stationary this implies that $I^*V = V$, $A|_{M^+} = C$, $A|_{M^-} = \iota^*C$ and $I^*K = K$. Note that $\omega_2^{(\beta),d}$ is again Hadamard, because on M^- the reversed Killing time orientation is compensated for by taking the boundary value of a holomorphic function from the opposite imaginary direction when compared to M^+ .

To close this section, we wish to show that $\omega_2^{(\beta),d}$ is indeed the double β -KMS state on $M^+ \cup M^$ as defined in Section 3.2. For this purpose, we first compute the initial data of $\omega_2^{(\beta),d}$ on Σ . Note that $L^2(\Sigma \setminus \mathcal{B}) \simeq L^2(\Sigma^+) \oplus L^2(\Sigma^-)$. The weak wedge reflection ι gives rise to a unitary involution T_ι of $L^2(\Sigma \setminus \mathcal{B})$ defined by $T_\iota(\psi^+ \oplus \psi^-) := \iota^*\psi^- \oplus \iota^*\psi^+$, which shows in particular that $L^2(\Sigma^-) \simeq L^2(\Sigma^+)$. From Proposition 3.7 and the definition of C we immediately conclude the following:

Corollary 3.10 Consider the partial differential operator

$$A := -|v|^{\frac{1}{2}} \nabla_i^{(h)} |v| h^{ij} \nabla_j^{(h)} |v|^{\frac{1}{2}} + V v^2$$

on $\Sigma \setminus \mathcal{B}$, where V satisfies $\iota^* V = V$ and A is defined on the dense domain $C_0^{\infty}(\Sigma \setminus \mathcal{B})$ of $L^2(\Sigma \setminus \mathcal{B})$ (in the metric volume form $dvol_h$). A preserves this dense domain and all integer powers of A are essentially self-adjoint on it. The self-adjoint operator \overline{A} is strictly positive and it satisfies $A \geq Vv^2$. Finally, $C_0^{\infty}(\Sigma \setminus \mathcal{B})$ is in the domain of $\overline{A}^{\pm \frac{1}{2}}$ for both signs.

From now on we will use A to denote the unique self-adjoint extension.

The initial data of $\omega_2^{(\beta),d}$ can be conveniently expressed in terms of A and T_{ι} as:

$$\begin{split} \omega_{2,00}^{(\beta),d}(f,f') &= \frac{1}{2} \langle A^{-\frac{1}{2}} | v |^{\frac{1}{2}} \overline{f}, \coth(\pi R \sqrt{A}) | v |^{\frac{1}{2}} f' \rangle + \frac{1}{2} \langle A^{-\frac{1}{2}} | v |^{\frac{1}{2}} T_{\iota} \overline{f}, \sinh(\pi R \sqrt{A})^{-1} | v |^{\frac{1}{2}} f' \rangle \\ \omega_{2,10}^{(\beta),d}(f,f') &= \frac{-i}{2} \langle \overline{f}, f' \rangle \\ \omega_{2,01}^{(\beta),d}(f,f') &= \frac{i}{2} \langle \overline{f}, f' \rangle \\ \omega_{2,11}^{(\beta),d}(f,f') &= \frac{1}{2} \langle A^{\frac{1}{2}} | v |^{-\frac{1}{2}} \overline{f}, \coth(\pi R \sqrt{A}) | v |^{-\frac{1}{2}} f' \rangle - \frac{1}{2} \langle A^{\frac{1}{2}} | v |^{-\frac{1}{2}} T_{\iota} \overline{f}, \sinh(\pi R \sqrt{A})^{-1} | v |^{-\frac{1}{2}} f' \rangle \end{split}$$

where we used the fact that $\partial_t = v \partial_{x^0} = \pm |v| \partial_{x^0}$ on Σ^{\pm} , where x^0 is the Gaussian normal coordinate, and the restriction of a distribution density from $M^+ \cup M^-$ to $\Sigma \setminus \mathcal{B}$ involves a change of measure, which yields a factor $|v|^{-\frac{1}{2}}$ for every test-function on $\Sigma \setminus \mathcal{B}$. The distributions $\omega_{2,10}^{(\beta),d}$ and $\omega_{2,01}^{(\beta),d}$ show that the anti-symmetric part of $\omega_2^{(\beta),d}$ is indeed the canonical commutator. We may use the reflection positivity of Proposition 3.9 to show that $\omega_2^{(\beta),d}$ is of positive type, so it defines a quasi-free state.

To see that the two-point distribution $\omega_2^{(\beta),d}$ determined by the initial data above corresponds to the distribution of Section 3.2 we proceed as follows. In the static case, the one-particle structure $(p_{(\beta)}, \mathcal{H}_{(\beta)})$ on M^+ can be given explicitly in terms of initial data (cf. [50] Proposition 4.3¹³), namely $\mathcal{H}_{(\beta)} = L^2(\Sigma^+)^{\oplus 2}$ and

$$p_{(\beta)}(f) = \frac{1}{\sqrt{2}} \left(I - e^{-\beta\sqrt{C}} \right)^{-\frac{1}{2}} \left(C^{\frac{1}{4}} v^{-\frac{1}{2}} f_0 + i C^{-\frac{1}{4}} v^{\frac{1}{2}} f_1 \right)$$

$$\oplus \frac{1}{\sqrt{2}} e^{-\frac{\beta}{2}\sqrt{C}} \left(I - e^{-\beta\sqrt{C}} \right)^{-\frac{1}{2}} \left(C^{\frac{1}{4}} v^{-\frac{1}{2}} f_0 - i C^{-\frac{1}{4}} v^{\frac{1}{2}} f_1 \right),$$

where $f_0 := Ef|_{\Sigma}$ and $f_1 := n^a \nabla_a Ef|_{\Sigma}$. The Killing time evolution is implemented by $H = \sqrt{C} \oplus -\sqrt{C}$. This expression can be rewritten in a nicer way by using ι^* on the second summand of $\mathcal{H}_{(\beta)}$ to identify it with $L^2(\Sigma \setminus \mathcal{B})$ and by exploiting the fact that $A = C \oplus 0 + T_\iota(C \oplus 0)T_\iota$. After some straightforward computations one finds

$$p_{(\beta)}^{d}(f) = \frac{1}{\sqrt{2}} (I - e^{-\beta\sqrt{A}})^{-\frac{1}{2}} \left((I + T_{\iota}e^{-\frac{\beta}{2}\sqrt{A}})A^{\frac{1}{4}}|v|^{-\frac{1}{2}}(f_{0} \oplus 0) + i(I - T_{\iota}e^{-\frac{\beta}{2}\sqrt{A}})A^{-\frac{1}{4}}|v|^{\frac{1}{2}}(f_{1} \oplus 0) \right).$$

We may now obtain the double β -KMS one-particle structure from Section 3.2. Keeping in mind that the wedge reflection I reverses the time orientation, so that $(I^*f^-)_1 = -\iota^*(f_1^-)$, we find that

$$p_{(\beta)}^{d}(f) = \frac{1}{\sqrt{2}} (I - e^{-\beta\sqrt{A}})^{-\frac{1}{2}} \left((I - T_{\iota}e^{-\frac{\beta}{2}\sqrt{A}})A^{\frac{1}{4}}|v|^{-\frac{1}{2}}f_{0} + i(I + T_{\iota}e^{-\frac{\beta}{2}\sqrt{A}})A^{-\frac{1}{4}}|v|^{\frac{1}{2}}f_{1} \right)$$
(21)

(up to unitary equivalence). It is a straightforward exercise to verify the initial data of $\omega_2^{(\beta),d}$ from this expression, using Equations (11) and (12).

4 Hadamard's parametrix construction

The definition of the HHI state, and the verification that it is a Hadamard state, will involve a detailed comparison of the Euclidean Green's function G_R and its Wick rotation to the Lorentzian spacetime M. In this section, we will focus on the local singularity structures in this comparison. The local singularities of a fundamental solution of a second-order operator can nicely be characterised using Hadamard's parametrix construction. Here we will describe this construction in some detail for both the Euclidean and the Lorentzian setting. Our presentation is essentially an expanded version of Section 17.4 of [30] (see also [2, 1] for a more detailed description in the case of the advanced and retarded fundamental solutions in a Lorentzian setting).

There is no harm in considering the more general situation of a pseudo-Riemannian manifold $N = (\mathcal{N}, g_{ab})$ on which we consider a partial differential operator P given in local coordinates as

$$P = -\partial_{\mu}g^{\mu\nu}\partial_{\nu} + b^{\mu}\partial_{\mu} + c,$$

¹³Note that the statement of the proposition has a sign error, which can be corrected by changing the sign of each f_1 . The error enters in the proof of loc.cit. via erroneous expressions for $V^*P_{\pm}V$. Here we use the corrected expression.

where $g^{\mu\nu}$ is the inverse of the pseudo-Riemannian metric $g_{\mu\nu}$ and b^{μ} , c are a smooth vector field and function, respectively, on \mathcal{N} . Consider any $y \in \mathcal{N}$ and choose coordinates x^{μ} near ysuch that $g_{\mu\nu}(y) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ is a real-valued diagonal matrix with eigenvalues contained in $\{+1, -1\}$. We will denote the inverse of $\eta_{\mu\nu}$ by $\eta^{\mu\nu}$. The basic idea of Hadamard's construction is to approximate the operator P near y by $P_0 := -\partial_{\mu}\eta^{\mu\nu}\partial_{\nu}$ and to make sense of the formal geometric series $P^{-1} = (P_0 + (P - P_0))^{-1} = \sum_{k=0}^{\infty} (P_0 - P)^k P_0^{-k-1}$.

4.1 The Hadamard coefficients

To see how this approximation works, we first consider the operator $P_0 = -\partial_\mu \eta^{\mu\nu} \partial_\nu$ on \mathbb{R}^d . The formal geometric series for P^{-1} motivates us to consider fundamental solutions F_k of $\frac{1}{k!}P_0^{k+1}$, $k \geq 0$, which can be studied using Fourier analysis. If we define the principal symbol $p(\xi) := \eta^{\mu\nu}\xi_\mu\xi_\nu$ of P_0 with characteristic set $C := p^{-1}(0)$, then the Fourier transforms \hat{F}_k of F_k are given by $k!p(\xi)^{-k-1}$ on $\mathbb{R}^d \setminus C$. Furthermore, they are homogeneous of degree -2(k+1) and they satisfy

$$p(\xi)\vec{F}_k(\xi) = k\vec{F}_{k-1}(\xi)$$
$$-2\eta^{\mu\nu}\xi_{\nu}\hat{F}_k(\xi) = \partial_{\xi_{\mu}}\hat{F}_{k-1}(\xi)$$

outside C for $k \ge 1$, where the bottom equality essentially expresses the invariance of \widehat{F}_k under the symmetry group of $(\mathbb{R}^d, \eta_{\mu\nu})$. Similarly, in the case k = 0:

$$p(\xi)\overline{F}_0(\xi) = 1.$$

We now assume that we can find tempered distributions $\widehat{F}_k(\xi)$, $k \ge 0$, on all of \mathbb{R}^d which extend the distributions $p(\xi)^{-k-1}$, which have inverse Fourier transforms¹⁴ $F_k \in C^{k+1-d}(\mathbb{R}^d)$ if $2k \ge d-1$ and which still satisfy the two relations above for $k \ge 0$, modulo an additional term with a smooth inverse Fourier transform. To find such extensions is a non-trivial issue, which depends on the signature of the $\eta_{\mu\nu}$. In Subsection 4.2 below we will comment on the existence and uniqueness aspects for the Euclidean and Lorentzian case, but for now we will simply assume that distributions with these properties are given. This means that the tempered distributions F_k satisfy

$$P_0 F_k \sim \begin{cases} kF_{k-1} & k \ge 1\\ \delta_0 & k = 0 \end{cases},$$

$$\partial_\mu F_k \sim \frac{-1}{2} (\partial_\mu s) F_{k-1}, \ k \ge 1,$$
(22)

where $s(x) := \frac{1}{2} \eta_{\mu\nu} x^{\mu} x^{\nu}$ and ~ means equality modulo a smooth function.

In order to fully exploit the properties of the F_k on the pseudo-Riemannian manifold N, we need to choose Riemannian normal coordinates on $U^{\times 2}$, where $U \subset N$ is a convex normal neighbourhood. More precisely, we will use arbitrary coordinates \tilde{y}^{μ} on U and an orthonormal frame $(e_{\mu})^a$ of TU in order to describe the Riemannian normal coordinates in terms of an embedding $\rho: U^{\times 2} \to \mathbb{R}^{\times d} \times U$ as in Section 2.3 (cf. Equation (7)). We then define the pull-backs $\tilde{F}_k := \rho^*(F_k \otimes 1)$ as distributions on $U^{\otimes 2}$, i.e. $\tilde{F}_k(v, y) = F_k(v)$. In the following discussion, we continue to work in the coordinates (v, y) and all derivatives will be taken with respect to v. From Equation (10) and the last line of (22) we then find

$$(\exp_y^* g)^{\mu\nu}(v)\partial_\nu \tilde{F}_k(v,y) \sim g^{\mu\nu}(y)\partial_\nu \tilde{F}_k(v,y)$$

for $k \geq 1$. Equation (22) then leads to

$$-\partial_{\mu}g^{\mu\nu}(v)\partial_{\nu}\tilde{F}_{k}(v,y) \sim P_{0}\tilde{F}_{k}(v,y) \sim k\tilde{F}_{k-1}(v,y), \qquad (23)$$
$$g^{\mu\nu}(v)\partial_{\nu}\tilde{F}_{k}(v,y) \sim \frac{-1}{2}v^{\mu}\tilde{F}_{k-1}(v,y)$$

¹⁴Because $\hat{F}_k(\xi)$ falls off like $\|\xi\|^{-2k-2}$ in the Euclidean setting, it seems plausible that we can require the distributions F_k to be even more regular, e.g. $F_k \in C^{k+1-d}(\mathbb{R}^d)$. As this extra regularity only occurs in the Euclidean setting it will not be essential to our arguments and we will not pursue it.

for $k \geq 1$. Note that $-\partial_{\mu}g^{\mu\nu}(x)\partial_{\nu}$ is the principal part of P. It is the special virtue of the Riemmanian normal coordinates v^{μ} , centred on y, which allowed us to replace $g^{\mu\nu}(0)$ by $g^{\mu\nu}(v)$, leading to agreement between the highest-order part of P and P_0 at any y. In addition to these properties for \tilde{F}_k , $k \geq 1$, we will assume that \tilde{F}_0 satisfies

$$-\partial_{\mu}g^{\mu\nu}(v)\partial_{\nu}\tilde{F}_{0}(v,y) \sim \delta_{0}(v)$$

$$g^{\mu\nu}(v)\partial_{\nu}\tilde{F}_{0}(v,y) = \frac{-1}{2}v^{\mu}\tilde{F}_{-1}(v,y)$$

$$(24)$$

for some distribution \tilde{F}_{-1} .

Returning to the formal geometric series for P^{-1} , the idea is now to approximate a right parametrix for P on U by

$$\mathcal{R}^{(N)} := \sum_{k=0}^{N} u_k \tilde{F}_k \tag{25}$$

for some smooth coefficients $u_k \in C^{\infty}(U^{\times 2})$. A straightforward computation using Equation (23) shows that for $k \ge 1$

$$P(u_k \tilde{F}_k) \sim (Pu_k) \tilde{F}_k + k u_k \tilde{F}_{k-1} + u_k b^{\mu} \partial_{\mu} \tilde{F}_k - 2g^{\mu\nu} (\partial_{\nu} u_k) \partial_{\mu} \tilde{F}_k$$

$$= (Pu_k) \tilde{F}_k + k u_k \tilde{F}_{k-1} - \frac{1}{2} (g_{\mu\nu} v^{\nu} b^{\mu} u_k - 2v^{\nu} \partial_{\nu} u_k) \tilde{F}_{k-1},$$

where b^{μ} and $g^{\mu\nu}$ are evaluated at v. For k = 0 we have

$$P(u_0\tilde{F}_0) \sim (Pu_0)\tilde{F}_0 + u_0\rho^*(\delta_0 \otimes 1) + u_0b^{\mu}\partial_{\mu}\tilde{F}_0 - 2g^{\mu\nu}(\partial_{\nu}u_0)\partial_{\mu}\tilde{F}_0 = (Pu_0)\tilde{F}_0 + u_0\rho^*(\delta_0 \otimes 1) - \frac{1}{2}(g_{\mu\nu}v^{\nu}b^{\mu}u_0 - 2v^{\mu}\partial_{\mu}u_0)\tilde{F}_{-1}.$$

Adding these equations together we find

$$P\mathcal{R}^{(N)} \sim u_0 \rho^* (\delta_0 \otimes 1) + (Pu_N)\tilde{F}_N + \sum_{k=0}^N (Pu_{k-1} + ku_k - \frac{1}{2} (g_{\mu\nu} v^\nu b^\mu u_k - 2v^\nu \partial_\nu u_k))\tilde{F}_{k-1}$$
(26)

where we set $u_{-1} \equiv 0$. The factor Pu_N is smooth by assumption and $\tilde{F}_N \in C^{N+1-d}(U^{\times 2})$ if $2N \geq d-1$, so the second term becomes more regular as N increases. Moreover, the terms in the sum can be made to vanish by choosing the coefficients u_k appropriately (cf. [30] Lemma 17.4.1):

Lemma 4.1 There are unique functions $u_k \in C^{\infty}(U^{\times 2})$, $k \ge 0$, such that $u_0(0, y) = 1$ and

$$2ku_k - g_{\mu\nu}v^{\nu}b^{\mu}u_k + 2v^{\nu}\partial_{\nu}u_k = -2Pu_{k-1}$$

in the coordinates (v, y). In coordinates $x, y \in U$ these functions are given recursively by

$$u_0(x,y) := \exp\left(\frac{1}{2}\int_0^1 g_{\mu\nu}(x(t))b^{\mu}(x(t))\dot{x}^{\nu}(t)dt\right),$$

$$u_k(x,y) := -u_0(x,y)\int_0^1 t^{k-1}\frac{(Pu_{k-1})(x(t),y)}{u_0(x(t),y)}dt,$$

where $t \mapsto x(t)$ is the unique geodesic in U with x(0) = y and x(1) = x.

In the special case that $P = -\Box + c$ we have $b^{\mu} = -\frac{1}{2}g^{\mu\nu}\partial_{\nu}\log|g|$, where $g := \det g_{\alpha\beta}$, and one may show that

$$u_0(x,y)^2 = (-1)^d \frac{\det g(x)}{|\det g(x)|} \frac{1}{\sqrt{g(x)g(y)}} \det \left(\partial_{x^{\mu}} \partial_{y^{\nu}} \sigma(x,y)\right)$$

$$= \Delta_{VVM}(x,y),$$
(27)

the Van Vleck-Morette determinant, because both sides satisfy the same differential equation in the coordinate x, which may be integrated along geodesics:

$$\nabla^{\mu}\sigma\cdot\nabla_{\mu}\log(\Delta_{VVM}) = d - \Box\sigma$$

(cf. [46, 8, 39]). In the case of u_0^2 this equation may be verified by using Riemannian normal coordinates around y and using Lemma 4.1. We therefore find:

Definition 4.2 Let U be a convex normal neighbourhood in a pseudo-Riemannian manifold $N = (\mathcal{N}, g_{ab})$. The Hadamard coefficients $u_k, k \geq 0$, on U for the operator $P = -\Box + c$ are the smooth functions defined by

$$u_0(x,y) := \sqrt{\Delta_{VVM}(x,y)},$$

$$u_{k+1}(x,y) := -u_0(x,y) \int_0^1 t^k \frac{(Pu_k)(x(t),y)}{u_0(x(t),y)} dt,$$

where $t \mapsto x(t)$ is the unique geodesic segment in U from x(0) = y to x(1) = x.

Our Hadamard coefficients u_k equal Moretti's heat kernel coefficients a_k [39, 40] (at least when c is real-valued) and they differ from the Hadamard coefficients V^k in [2] by $V^k = k!u_k$.

The approximate parametrices $\tilde{\mathcal{R}}^{(N)}$ can be used to construct an exact local right parametrix using Borel's Lemma and from that one can construct a local right fundamental solution for the operator P. Different choices of extensions \hat{F}_k may give rise to different approximate fundamental solutions, but the Hadamard coefficients only depend on the operator P and the geometry of the pseudo-Riemannian manifold. We refer the reader to [2] for an elaboration of this procedure in the case of advanced and retarded fundamental solutions for wave equations.

Under suitable circumstances a right parametrix is also a left parametrix. This follows from

Theorem 4.3 (Moretti's Theorem) If $M = (\mathcal{M}, g_{ab})$ is a Riemannian or Lorentzian manifold and $P = -\Box + c$ with real-valued c, then the Hadamard coefficients are symmetric: $u_k(x, y) = u_k(y, x)$ for all $k \ge 0$.

See [39, 40] for a proof.¹⁵

4.2 The distributions F_k

To complete our discussion of the Hadamard parametrix construction we return to the issue of finding suitable distributions F_k such that $F_k \in C^{k+1-d}(\mathbb{R}^d)$ if $2k \ge d-1$ and such that Equations (22) and (24) hold. We do this first for the Euclidean and then for the Lorentzian case.

4.2.1 The Euclidean case

In the Euclidean case, the characteristic set C reduces to the origin, so we need to extend the homogeneous distributions $k!p(\xi)^{-k-1}$ and $-\log |p(\xi)|$ (for k = -1) from $\mathbb{R}^d \setminus \{0\}$ to \mathbb{R}^d . For each k such an extension always exists (cf. [29] Section 3.2 and 7.1). The difference of two extensions is supported at the origin, so its inverse Fourier transform is a polynomial, which does not contribute to the singularities appearing in Equation (25). In particular, Equation (22) is automatically satisfied.

A convenient explicit expression for such extensions is given in the following result:

Proposition 4.4 For a fixed $d \ge 2$ and any $k \in \mathbb{N}_0$ we define the distributions

$$F_k^e(x) := s(x)^{k+1-\frac{d}{2}} (c_k + d_k \log(s(x))), \quad k \ge 0,$$
(28)

 $^{^{15}}$ Presumably this result also holds in other pseudo-Riemannian manifolds, because the procedure of [40] can be used inductively to change the signature of the metric by changing the sign of one eigenvalue at a time [41].

viewed as locally integrable functions, where the constants $c_k, d_k, k \ge 0$, are given by

$$c_{k} = \begin{cases} 0 & k+1-\frac{d}{2} \in \mathbb{N}_{0} \\ (4\pi)^{-\frac{d}{2}} 2^{\frac{d}{2}-1-k} \Gamma(\frac{d}{2}-1-k) & \text{otherwise} \end{cases}$$

$$d_{k} = \begin{cases} -(4\pi)^{-\frac{d}{2}} (-2)^{\frac{d}{2}-1-k} \Gamma(k+2-\frac{d}{2})^{-1} & k+1-\frac{d}{2} \in \mathbb{N}_{0} \\ 0 & \text{otherwise} \end{cases}$$

$$(29)$$

Then $F_k^e \in C^{k+1-d}(\mathbb{R}^d)$ if $2k \ge d-1$ and the F_k^e satisfy Equation (22).

Proof: The constants are chosen such that $(2k+2-d)c_k = -c_{k-1}$ and $(2k+2-d)d_k = -d_{k-1}$ for $k \ge 1$ and $k \ne \frac{d}{2} - 1$. For $k = \frac{d}{2} - 1 \ge 1$ we have $2d_k = -c_{k-1}$ while $c_k = d_{k-1} = 0$. Outside x = 0 we may then compute

$$\partial_{\mu}F_{k}^{e} = \begin{cases} \frac{-1}{2}(\partial_{\mu}s)F_{k-1}^{e} + d_{k}(\partial_{\mu}s)s^{k-\frac{d}{2}} & k \ge 1, \ k \ne \frac{d}{2} - 1\\ \frac{-1}{2}(\partial_{\mu}s)F_{k-1}^{e} & k = \frac{d}{2} - 1 \ge 1 \end{cases}$$
(30)

These equations extend to x = 0 in the distributional sense, because all functions and distributions involved are locally integrable. Applying this result twice, or using similar computations, we find

$$P_0 F_k^e = \begin{cases} kF_{k-1}^e + (d-2-4k)d_k s^{k-\frac{d}{2}} & k \ge 1, \ k \ne \frac{d}{2} - 1 \\ kF_{k-1}^e & k = \frac{d}{2} - 1 \ge 1 \end{cases}$$

For k = 0 we have $P_0 F_0^e = \delta_0$, by [29] Theorem 3.3.2 and the fact that the volume of the unit sphere in \mathbb{R}^d is given by $\operatorname{vol}(\mathbb{S}^{d-1}) = 2\pi^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)^{-1}$. (This fixes c_0 and d_0 . The other constants are determined by the recursion relations above.)

Equation (22) follows once it is realised that the extra terms with $s^{k-\frac{d}{2}}$ vanish unless $k-\frac{d}{2} \in \mathbb{N}_0$, in which case they are polynomials and hence smooth. F_k^e is continuous for $k \geq \frac{d-1}{2}$, both for even and odd dimensions d. The claimed regularity for $2k \geq d-1$ then follows by induction from Equation (30).

The corresponding distribution \tilde{F}_0^e does not satisfy the bottom line of Equation (24) as stated, but

$$\partial_{\mu}F_0^e = \frac{c_{-1}}{2}(\partial_{\mu}s)s^{-\frac{d}{2}}$$

as a product is a locally integrable function (cf. [29] Theorem 3.3.2). From this it follows that

$$g^{\mu\nu}(v)\partial_{\nu}\tilde{F}^{e}_{0}(v,y) = g^{\mu\nu}(0)\partial_{\nu}\tilde{F}^{e}_{0}(v,y) = \frac{c_{-1}}{2}v^{\mu}s(v)^{-\frac{d}{2}} -\partial_{\mu}g^{\mu\nu}(v)\partial_{\nu}\tilde{F}^{e}_{0}(v,y) = P_{0}\tilde{F}^{e}_{0}(v,y) = \delta_{0}(v),$$

which suffices to show that the k = 0 term in the summation of Equation (26) vanishes when u_0 solves the transport equation of Lemma 4.1.

4.2.2 The Lorentzian case

In the Lorentzian case different choices of extensions can lead to different fundamental solutions. The easiest choice is to take $F_k = E_k^{\pm}$ with

$$\widehat{E}_k^{\pm}(\xi) := \lim_{V^+ \ni \epsilon \to 0^+} k! (\eta^{\mu\nu} (\xi_{\mu} \mp i\epsilon_{\mu}) (\xi_{\nu} \mp i\epsilon_{\nu}))^{-k-1},$$

where we use a fixed choice of the sign for all k. These distributions are well-defined by [29] Theorem 3.1.15 and they are tempered, because they are homogeneous (loc.cit. Theorem 7.1.18). Also note that they are invariant under the proper, orthochronous Lorentz group. It is not hard to show that the inverse Fourier transforms $E_k^{\pm}(x)$ are supported in the future (+) or past (-) light cone (cf. [15] Ch. 4) and that they satisfy Equation (22) with equalities. In fact, one may prove by induction over k that $E_k^{\pm}(x)$ is uniquely determined by the top line of (22) with equality and

by its support property. Indeed, the difference of two such distributions must be a solution to the wave equation with either past or future compact support and therefore it must vanish.

To deduce the regularity of $E_k^{\pm}(x)$ for sufficiently large k one may proceed as follows. Note that each $E_k^{\pm}(x)$ is again Lorentz invariant and hence so is the difference $E_k := E_k^- - E_k^+$. Furthermore, $E_k(-x^0, x^i) = E_k(x^0, x^i)$ and a detailed analysis of Lorentz invariant distributions (as in [17] Section 8) shows that we may write

$$E_k(x) = \operatorname{sign}(x^0) \rho(-\sigma(x))$$

for a unique distribution ρ on \mathbb{R} whose support lies in $\mathbb{R}_{\geq 0}$. Next we note that both E_k^{\pm} are homogeneous of degree 2k + 2 - d (cf. [29] Thm. 7.1.16) and hence ρ is homogeneous of degree $k+1-\frac{d}{2}$. This means that $\rho(s) = e_k s^{k+1-\frac{d}{2}}$ on s > 0 and $\rho(s) = 0$ on s < 0. Here e_k is a constant, which must be real because E_k is real. When $2k \geq d-3$ then $k+1-\frac{d}{2}$ is not an integer ≤ -1 , so ρ is uniquely determined by homogeneity and its restriction to $\mathbb{R} \setminus \{0\}$ ([29] Thm. 3.2.3). By inspecting the supports we therefore find

$$E_k^{\pm}(x) = e_k \theta\left(\pm x^0\right) \theta(-\sigma(x)) |\sigma(x)|^{k+1-\frac{d}{2}}$$

when $2k \ge d-3$. The fact that $E_k^{\pm} \in C^{k+1-d}$ when $2k \ge d-1$ can now be shown by induction from the bottom line of Equation (22).

A different approach to the distributions $E_k^{\pm}(x)$ is given in full detail in [2], which also proves that the Hadamard parametrix based on these distributions gives rise to the unique advanced (-)and retarded (+) fundamental solutions, according to the choice of sign. Comparing our formulae with those of this reference¹⁶ allows us to determine the constants e_k as

$$e_k = \left(2^{k+\frac{d}{2}} \pi^{\frac{d}{2}-1} \Gamma\left(k+2-\frac{d}{2}\right)\right)^{-1}.$$
(31)

(These may also be found by a direct Fourier transformation.)

Feynman and anti-Feynman fundamental solutions are only unique up to a smooth function. To obtain them from the Hadamard parametrix construction one first defines the distributions $v_k^{\pm}(s) := \lim_{\epsilon \to 0^+} k! (s \mp i\epsilon)^{-k-1}$ on \mathbb{R} , which are homogeneous of degree -k-1 and whose wave front sets are given by

$$WF(v_k^{\pm}) = \left\{ (0,\varsigma) \in \mathbb{R}^{\times 2} | \mp \varsigma > 0 \right\}.$$

These distributions have well-defined pull-backs $w_k^{\pm}(\xi)$ to $\mathbb{R}^d \setminus \{0\}$ under the map $\xi \mapsto p(\xi)$ (cf. [29] Thm. 8.2.4). The pull-backs are Lorentz invariant, homogeneous of degree -2k - 2 and they have

$$WF(w_k^{\pm}) \subset \left\{ (\xi, x) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}^d | \ p(\xi) = 0, \ x^{\mu} = a\eta^{\mu\nu}\xi_{\nu}, \ \mp a > 0 \right\}$$

If we let $\hat{E}_k^{F,\pm}$ be any extensions of w_k^{\pm} to \mathbb{R}^d , then they are automatically tempered (by homogeneity outside the origin) and it is straightforward to verify that the distributions $E_k^{F,\pm}$ satisfy Equations (22) and (24). Moreover, we have

$$WF(E_k^{F,\pm}) \subset T_0^* \mathbb{R}^d \cup \left\{ (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d | \sigma(x) = 0, \xi_\mu = a\eta_{\mu\nu} x^\nu, \ \pm a > 0 \right\}$$
(32)

(cf. [29] Thm. 8.1.8). Note that different choices of extension of w_k^{\pm} differ by a distribution supported at $\xi = 0$, which has a smooth inverse Fourier transform. In fact, the distributions $E_k^{F,\pm}$ are uniquely determined up to smooth functions by the first line of Equation (22) and the condition that their wave front set is contained in the right-hand side of Equation (32). This can be shown by induction, using the fact that the difference of two solutions $E_k^{F,\pm}$ of Equation (22) solves the wave equation with a smooth source term. It then follows from the Propagation of Singularities Theorem [12] that the wave front set estimates can only hold if the difference is a

¹⁶The notations of reference [2] relate to ours as follows: $\gamma(x) = 2\sigma(x)$ and $R_{\pm}(\alpha) = \frac{1}{k!}E_k^{\pm}$ when $\alpha = 2k+2$ and n = d.

smooth function. The regularity of $E_k^{F,\pm}$ for sufficiently large k is a bit harder to see directly, but we will return to this momentarily.

The Hadamard series that is used to characterise the singularities of Hadamard states for a real scalar QFT arises as follows. We consider the differences $F_k := -i(E_k^{F,+} - E_k^-), k \ge 0$, which satisfy

$$P_0 F_k \sim \begin{cases} kF_{k-1} & k \ge 1\\ 0 & k = 0 \end{cases}$$

and

$$WF(F_k) \subset \left\{ (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d | \sigma(x) = 0, \ \xi_\mu = a\eta_{\mu\nu}x^\nu, \ \xi_0 < 0 \right\}.$$

The wave front set estimate can be proved by induction, using the Propagation of Singularities Theorem to propagate singularities in the past light cone to singularities in the future light cone, where E_k^- vanishes and only the singularities of $E_k^{F,+}$ can occur. It can also be shown by induction that the distributions F_k are uniquely determined up to smooth functions by their wave front set estimate and the condition that $F_k(x) - F_k(-x) = iE_k(x)$.

A convenient expression for the F_k can be obtained from a Wick rotation of the Euclidean distributions F_k^e . For this purpose we consider the holomorphic function $s^c(z, x^i) := -z^2 + \sum_{i=1}^{d-1} (x^i)^2$ of $z := x^0 + i\tau$ and we define

$$F_k^l(x) = \lim_{\tau \to 0^+} s^c(z, x')^{k+1-\frac{d}{2}} (c_k + d_k \log s^c(z, x')),$$
(33)

with c_k, d_k as in Equation (29) (cf. [39]). In this formula all logarithms and all fractional powers are defined as holomorphic functions with the principal branch cut along the non-positive real axis. Note that the range of s^c does not intersect the branch cut of the logarithm as long as $\tau \neq 0$ and that taking instead the limit $x^0 \to 0$ would yield $F_k^e(\tau, x^i)$ (as long as $\tau \neq 0$).

Performing derivatives before we take the limit $\tau \to 0^+$ and proceeding as in the proof of Proposition 4.4 one may verify by direct computation that

$$P_0 F_k^l = \begin{cases} k F_{k-1}^l + (d-2-4k) d_k \sigma^{k-\frac{d}{2}} & k \ge 1, \ k \neq \frac{d}{2} - 1 \\ k F_{k-1}^l & k = \frac{d}{2} - 1 \ge 1 \\ 0 & k = 0 \end{cases}$$
(34)

It is apparent from these equations that any singularities must lie on the light cone and the wave front sets can only contain lightlike vectors, which must be future pointing by standard arguments ([29] Thm. 8.1.6). This proves the desired wave front set estimate. The fact that $F_k^l \in C^{k+1-d}$ when $2k \ge d-1$ can again be shown by induction, as in the proof of

The fact that $F_k^l \in C^{k+1-d}$ when $2k \ge d-1$ can again be shown by induction, as in the proof of Proposition 4.4, or it can be verified by direct computation. Furthermore, when k is large enough one can easily see that $F_k^l(x) - F_k^l(-x)$ is Lorentz invariant, odd under spacetime reflection in x = 0 and homogeneous of degree 2k + 2 - d. Since there is only one such distribution, up to a multiplicative factor, it follows that $F_k^l(x) - F_k^l(-x)$ is a multiple of E_k when k is large enough. A direct computation shows that

$$\begin{split} F_k^l(x) - F_k^l(-x) &= e_k' \theta(\pm x^0) \theta(-\sigma(x)) |\sigma(x)|^{k+1-\frac{d}{2}}, \\ e_k' &= \begin{cases} 2\pi i^{2k+1-d} d_k & d \text{ even}, \\ 2i^{2k-d} c_k & d \text{ odd}, \end{cases} \end{split}$$

and a comparison of the coefficients c_k, d_k in Equation (29) with the coefficients e_k in Equation (31) shows that¹⁷ $e'_k = ie_k$ and hence $F^l_k(x) - F^l_k(-x) = iE_k(x)$ when k is large enough. The same equality then holds for all k, by Equations (22, 34). Taken altogether this proves that the expressions for F^l_k above satisfy $F^l_k \sim F_k = -i(E^{F,+}_k - E^-_k)$. Note that the regularity of F^l_k and of E^{\pm}_k for sufficiently large k also implies the desired regularity for $E^{F,+}_k$ and hence for $E^{F,-}_k$, by complex conjugation.

¹⁷Here we rewrite c_k using $\Gamma\left(\frac{1}{2}-n\right) = (-1)^n \pi \Gamma\left(\frac{1}{2}+n\right)^{-1}$.

We now consider the Hadamard series based on the F_k^l :

$$H^{(N)} := -i\sum_{k=0}^{N} u_k \tilde{F}_k, \quad \tilde{F}_k := \rho^*(F_k \otimes 1),$$

where again $\rho(x, y) := (\exp_y^{-1}(x), y)$. By Moretti's Theorem $H^{(N)}$ approximates a bi-solution to the Klein-Gordon equation. A two-point distribution ω_2 is called Hadamard if and only if $\omega_2 - H^{(N)}$ is $C^{(N+2-d)}$ for all $N \ge \frac{d-3}{2}$. The factor -i is needed to ensure that $H^{(N)}$ can be made into a distribution of positive type by adding a suitable $C^{(N+3-d)}$ function.

4.3 Infinitesimal analytic continuation of the Hadamard series

In Subsection 2.3 we have made a detailed comparison of the geometry near the Cauchy surface Σ of a spacetime M with a static bifurcate Killing horizon and a (weak) wedge reflection (Σ, ι) , with the geometry near $\overline{\mu}(\Sigma)$ in $M' := \overline{M_{R_H}^+}$ when $R_H \equiv \kappa^{-1}$. The purpose of this subsection is to establish a comparison between the Hadamard series in these two manifolds. We consider the same geometric situation as in Section 2.3, with a Cauchy surface $\Sigma \subset M$ satisfying the properties of Definition 2.2 and a coordinate neighbourhood $U \subset \Sigma$. By shrinking U if necessary, we may assume that there are convex normal neighbourhoods $V \subset M$ and $V' \subset M'$ such that $U \subset V$ and $U' := \overline{\mu}(U) \subset V'$. We let x^{μ} and $(x')^{\mu}$ be Gaussian normal coordinates near U and U', which are related as in Section 2.3 and we let $y^{\mu}, (y')^{\mu}$ be copies of the same coordinates.

Whereas the Hadamard coefficients u_k depend on the choice of a second-order partial differential operator P, the distributional factors \tilde{F}_k in the Hadamard series for a fundamental solution or two-point distribution only depend on the local geometry. We will first consider the Hadamard coefficients:

Proposition 4.5 Consider the operators $K := -\Box_g + V$ on M and $K' := -\Box_{g'} + V'$ on M', where V and V' are smooth functions which are stationary,

$$\xi^{\mu}\partial_{\mu}V = 0, \quad \overline{\xi}^{\mu}_{R}\partial'_{\mu}V' = 0,$$

and such that $\iota^* V = V = \overline{\mu}^* V'$ on Σ . Let u_k be the Hadamard coefficients for K on M and u'_k those for K' on M'. Then the following equality holds on $U^{\times 2}$:

$$\partial_{x^0}^l \partial_{y^0}^m u_k = i^{l+m} (\overline{\mu}^{\times 2})^* \left(\partial_{(x')^0}^l \partial_{(y')^0}^m u_k' \right).$$

Proof: First we will show that

$$\partial_{x^0}^l V = i^n \overline{\mu}^* \left(\partial_{(x')^0}^l V' \right)$$

on U. For l = 0 this is true by assumption. We proceed by induction over l, exploiting the fact that V and V' are stationary. Indeed,

$$0 = \partial_{x^0}^l (\xi^\mu \partial_\mu) V - i^l \overline{\mu}^* \left(\partial_{(x')^0}^l (\overline{\xi_R}^\mu \partial'_\mu) V' \right)$$
$$= \xi^0 \partial_{x^0}^{l+1} V - i^l \overline{\mu}^* \left(\overline{\xi_R}^0 \partial_{(x')^0}^{l+1} V' \right)$$

on U, where the last equality follows from the induction hypothesis for $l' \in \{0, \ldots, l\}$ and Corollary 2.12. Since $\xi^0 = \overline{\mu}^*(\overline{\xi_R})^0 = v \neq 0$ on the dense subset $\Sigma \setminus \mathcal{B}$ of Σ , the claim for l + 1 follows and the proof by induction is complete.

From Proposition 2.11 we find that

$$-\partial_{x^0}^l \det g_{\mu\nu} = i^l \overline{\mu}^* \left(\partial_{(x')^0}^l \det(\overline{g}_R)_{\mu\nu} \right)$$

on U, and similarly for det $(\partial_{x^{\mu}}\partial_{y^{\nu}}\sigma(x,y))$, due to Corollary 2.17. From Equation (27) we have

$$u_0(x,y) = \frac{1}{\sqrt[4]{g(x)g(y)}} \sqrt{(-1)^{d+1} \det\left(\partial_{x^{\mu}} \partial_{y^{\nu}} \sigma(x,y)\right)}$$

and similarly for $u'_0(x',y')$, but with $(-1)^d$ instead of $(-1)^{d+1}$. Combining these results yields the statement of the Proposition for u_0 and u'_0 , i.e. for k = 0. We will now proceed by induction over k, so we may assume that the claim has been shown for all $k' \in \{0, \ldots, k\}$ for some $k \ge 0$ and we aim to prove it for k+1.

From the induction hypothesis, Corollary 2.12 and our results on V and V' we may conclude that on $U^{\times 2}$:

$$\partial_{x^{0}}^{l} \partial_{y^{0}}^{m} K u_{k} = i^{l+m} (\overline{\mu}^{\times 2})^{*} \left(\partial_{(x')^{0}}^{l} \partial_{(y')^{0}}^{m} K' u_{k}' \right).$$

To proceed we need to recall some notations from the proof of Proposition 2.14. Given $x^{\mu} = (0, x^i)$ and $y^{\mu} = (0, y^{i})$ in U we define the geodesics $r \mapsto \gamma_{0}^{\mu}(r, y^{i})$ and $s \mapsto \gamma_{0}^{\mu}(s, x^{i})$ in V. For some $\epsilon > 0$ we may define the map $\gamma^{\mu} : (-\epsilon, \epsilon)^{\times 2} \times [0, 1] \to V$ such that $t \mapsto \gamma^{\mu}(r, s, t)$ is the unique geodesic in V between $\gamma_0(r)$ and $\gamma_1(s)$. We define a map $(\gamma')^{\mu}: (-\epsilon, \epsilon)^{\times 2} \times [0, 1] \to V'$ in complete analogy, using the points $x' := \overline{\mu}(x)$ and $y' := \overline{\mu}(y)$. We have shown in the proof of Proposition 2.14 that

$$\partial_s^l \partial_r^m \gamma^\mu(0,0,t) = i^{l+m+c} \partial_s^l \partial_r^m (\gamma')^\mu(0,0,t),$$

where c = -1 when $\mu = 0$ and c = 0 otherwise. In particular, we have for t = 0:

$$\partial_r^m \gamma_0^\mu(0) = i^{m+c} \partial_r^m (\gamma')^\mu(0).$$

Combining this with our previous results we find

$$\partial_{x^{0}}^{l} \partial_{y^{0}}^{m}(Ku_{k})(x(t), y) = \partial_{s}^{l} \partial_{r}^{m}(Ku_{k})(\gamma(r, s, t), \gamma_{0}(r))|_{r=s=0}$$

= $\partial_{(x')^{0}}^{l} \partial_{(y')^{0}}^{m}(K'u'_{k})(x'(t), y'),$

where we have written x(t) for the unique geodesic in U from y to x, x'(t) for the corresponding geodesic in U'. The last equality uses the chain rule and a matching up of factors i on both sides, where we note that a γ^0 or γ_0^0 comes with an extra derivative ∂_{x^0} or ∂_{y^0} , so the additional factors of *i* cancel out. Note that the derivatives with respect to x^0 and y^0 also force use to vary the curve x(t), for which purpose we needed to use γ^{μ} .

The same argument actually works for all $u_{k'}$ with $k' \leq k$ and in particular for k = 0. All these results can then be inserted into the formula for u_{k+1} and u'_{k+1} given in Lemma 4.1. Differentiating under the integral sign then proves the claim for k+1, so the proof is complete. \square

Now we turn to the distributional parts of the Hadamard series. We choose orthonormal frames $(e_{\alpha})^{\mu}$ and $(e'_{\alpha})^{\mu}$ on V and V' as in Lemma 2.15, and we use these to define Riemannian normal coordinates on $V^{\times 2}$ and $(V')^{\times 2}$, respectively, denoting the coordinate changes by ρ and ρ' . We then consider the distributions F_k^l of Equation (33) and F_k^e of Equation (28) and the corresponding distributions $\tilde{F}_k^l := \rho^*(F_k^l \otimes 1)$ and $\tilde{F}_k^e := (\rho')^*(F_k^e \otimes 1)$. Away from the diagonal of $U^{\times 2}$ one may easily show that in Gaussian normal coordinates

$$\partial_{x^{0}}^{l} \partial_{y^{0}}^{m} \tilde{F}_{k}^{l}(x, y) = i^{l+m} (\overline{\mu}^{\times 2})^{*} (\partial_{(x')^{0}}^{l} \partial_{(y')^{0}}^{m} \tilde{F}_{k}^{e})(x, y)$$

because of Corollary 2.17 and the fact that \tilde{F}_k^l and \tilde{F}_k^e are locally given by the same expression in terms of σ and $\bar{\sigma}_R$, respectively. However, it is necessary to have a more detailed understanding of this infinitesimal analytic continuation also at the diagonal. For this purpose, we will first consider the Lorentzian case, which is better behaved regarding restrictions to the Cauchy surface Σ . We will use the distribution δ on $U^{\times 2}$, defined by $\delta(f) := \int_U f|_{\Delta}$, where $f|_{\Delta}(x) := f(x, x)$.

Theorem 4.6 For \tilde{F}_k^l the following equalities hold:

$$\begin{split} \tilde{F}_{k}^{l}|_{U^{\times 2}} &= \sigma_{U}^{k+1-\frac{d}{2}}(c_{k}+d_{k}\log(\sigma_{U}))\\ \partial_{x^{0}}\tilde{F}_{k}^{l}|_{U^{\times 2}} &= -\partial_{y^{0}}\tilde{F}_{k}^{l}|_{U^{\times 2}} = \begin{cases} 0 & k \ge 1\\ \frac{-i}{2}\delta & k = 0 \end{cases}\\ \partial_{x^{0}}\partial_{y^{0}}\tilde{F}_{k}^{l}|_{U^{\times 2}} &= \begin{cases} \frac{-1}{2}(\partial_{x^{0}}\partial_{y^{0}}\sigma)\tilde{F}_{k-1}^{l}|_{U^{\times 2}} + (\partial_{x^{0}}\partial_{y^{0}}\sigma)|_{U^{\times 2}}d_{k}\sigma_{U}^{k-\frac{d}{2}} & k \ge 1, \ k \neq \frac{d}{2} - 1\\ \frac{-1}{2}(\partial_{x^{0}}\partial_{y^{0}}\sigma)\tilde{F}_{k-1}^{l}|_{U^{\times 2}} & k \ge 1, \ k = \frac{d}{2} - 1 \end{split}$$

where the distributional restrictions are well-defined in the sense of microlocal analysis and σ_U is the squared geodesic distance of (Σ, h_{ij}) on U. Furthermore,

$$(\partial_{x^0} + \partial_{y^0})^2 \tilde{F}_0^l|_{U^{\times 2}} = C(d) \sigma_U^{\epsilon + \frac{1}{2} - \frac{d}{2}} \frac{-(\partial_{x^0} + \partial_{y^0})^2 \sigma|_{U^{\times 2}}}{(\sigma_U)^{\epsilon + \frac{1}{2}}}$$

for any $\epsilon \in \left(0, \frac{1}{2}\right)$, where $C(d) = \left(1 - \frac{d}{2}\right)c_0 + d_0$.

The reason why we treat $\partial_{x^0} \partial_{y^0} \tilde{F}_0^l$ differently is to facilitate the comparison with the Euclidean case later on.

Proof: The distributions \tilde{F}_k^l have wave front sets which are contained in the lightlike vectors. In particular, they do not intersect the conormal bundle $N^*U^{\times 2}$, because U is spacelike. By a standard result in microlocal analysis ([29] Theorem 8.2.4) the restrictions of all \tilde{F}_k^l and all of their derivatives are well-defined, so it remains to compute them. For this, we use the limit in Equation (33), which holds in the sense of the Hörmander pseudo-topology so that it commutes with restrictions. (This can be seen most easily by a slight adaptation of the proof of Theorem 8.1.6 in [29]. See also [9] for a discussion of the Hörmander topology.)

Note that

$$\sigma_{\tau}(x,y) := (\rho^*(s^z \otimes 1))(x,y) = \sigma(x,y) + \frac{1}{2}\tau^2 + i\tau v^0(x,y),$$

so we may write

$$\tilde{F}_{k}^{l}(x,y) = \lim_{\tau \to 0^{+}} \sigma_{\tau}(x,y)^{k+1-\frac{d}{2}} (c_{k} + d_{k} \log \sigma_{\tau}(x,y)).$$

This limit again commutes with restrictions, because we have only tensored in a constant function 1 and applied a smooth change of coordinates ρ to obtain $\tilde{F}_k^l = \rho^*(F_k^l \otimes 1)$. Furthermore, for $x, y \in U$ we have $\sigma_\tau(x, y) = \sigma_U(x, y) + \frac{1}{2}\tau^2$ (cf. the proof of Corollary 2.17). As $(\sigma_U)^r$ is locally integrable for all $r > \frac{1-d}{2}$ the formula for $\tilde{F}_k^l|_{U\times 2}$ immediately follows from the dominated convergence theorem. The convergence of the limit even works for all continuous test-functions $f \in C_0^0(U)$, i.e. in the sense of measure theory. The same is true when k is greater than or equal to the number of normal derivatives, where we use Corollary 2.17 to obtain the correct formulae.

For the remaining cases, we may take the normal derivatives before taking the limit $\tau \to 0^+$. Note that $v^0|_{U^{\times 2}} \equiv 0$, by Corollary 2.13 and $\partial_{x^0}\sigma|_{U^{\times 2}} = \partial_{y^0}\sigma|_{U^{\times 2}} = 0$ by Corollary 2.17. Furthermore, if $\gamma(t)$ is the unique geodesic through $y \in U$ with tangent vector $n^{\mu}(y)$, then $v^0(\gamma(s), \gamma(t)) = s - t$, from which it follows that

$$\begin{array}{rcl} \partial_{x^{0}}v^{0}(x,y)|_{x=y} &=& \partial_{s}v^{0}(\gamma(s),y)|_{s=0} = 1,\\ \partial_{y^{0}}v^{0}(x,y)|_{x=y} &=& \partial_{t}v^{0}(y,\gamma(t))|_{t=0} = -1,\\ \partial_{x^{0}}\partial_{y^{0}}v^{0}(x,y)|_{x=y} &=& \partial_{s}\partial_{t}v^{0}(\gamma(s),\gamma(t))|_{s=t=0} = 0. \end{array}$$

Taking the derivatives and restrictions, we now find

$$\begin{aligned} \partial_{x^{0}} \tilde{F}_{0}^{l}|_{U^{\times 2}} &= \lim_{\tau \to 0^{+}} C(d) \sigma_{\tau}^{-\frac{d}{2}} i \tau \partial_{x^{0}} v^{0}|_{U^{\times 2}} \\ \partial_{y^{0}} \tilde{F}_{0}^{l}|_{U^{\times 2}} &= \lim_{\tau \to 0^{+}} C(d) \sigma_{\tau}^{-\frac{d}{2}} i \tau \partial_{y^{0}} v^{0}|_{U^{\times 2}} \\ \partial_{x^{0}} \partial_{y^{0}} \tilde{F}_{1}^{l}|_{U^{\times 2}} &= \lim_{\tau \to 0^{+}} \sigma_{\tau}^{-\frac{d}{2}} \left\{ (a + b \log \sigma_{\tau}) i \tau \partial_{x^{0}} v^{0} i \tau \partial_{y^{0}} v^{0} - (\frac{c_{0}}{2} + d_{1} \delta_{d,2} + \frac{d_{0}}{2} \log \sigma_{\tau}) \right\} |_{U^{\times 2}} \end{aligned}$$

for some constants a, b and with $\delta_{d,2}$ denoting the Kronecker delta. The limits can be computed using the technical Lemma A.1 in the appendix, leading to the stated results.

It remains to prove the final equation for the case k = 0. By a straightforward computation, using the fact that $(2 - d)d_0 = 0$ for all d, we find that

$$(\partial_{x^0} + \partial_{y^0})^2 \tilde{F}_0^l|_{U^{\times 2}} = C(d) \lim_{\tau \to 0^+} (\sigma_U + \frac{1}{2}\tau^2)^{\epsilon + \frac{1}{2} - \frac{d}{2}} \frac{\Phi}{(\sigma_U + \frac{1}{2}\tau^2)^{\epsilon + \frac{3}{2}}}$$

$$\Phi := -\frac{d}{2}\tau^2((\partial_{x^0} + \partial_{y^0})v^0|_{U^{\times 2}})^2 - \left(\sigma_U + \frac{1}{2}\tau^2\right)(\partial_{x^0} + \partial_{y^0})^2\sigma_\tau|_{U^{\times 2}}.$$

The function Φ has a Taylor series in x^i, y^i, τ which vanishes up to (and including) third order at any point with $x^i = y^i$ and $\tau = 0$, because σ and its first-order derivatives vanish on the diagonal Δ and we have $((\partial_{x^0} + \partial_{y^0})P\sigma)|_{\Delta} = \partial_0((P\sigma)|_{\Delta})$ for any partial differential operator P. It follows that the quotient

$$\frac{\Phi}{(\sigma_U + \frac{1}{2}\tau^2)^{\epsilon + \frac{3}{2}}}$$

is continuous at points where $x^i = y^i$ and $\tau = 0$. As $\tau \to 0^+$ it converges uniformly on compact sets to $\frac{-(\partial_{x^0} + \partial_{y^0})^2 \sigma|_{U \times 2}}{\sigma_U^{\epsilon + \frac{1}{2}}}$. The power $(\sigma_U + \frac{1}{2}\tau^2)^{\epsilon + \frac{1}{2} - \frac{d}{2}}$ converges to $\sigma_U^{\epsilon + \frac{1}{2} - \frac{d}{2}}$ in the sense of measure theory, by the dominant convergence theorem. It follows that the product also converges to the expression claimed in the theorem.

We now turn to the Euclidean case and obtain a similar result:

Theorem 4.7 Let $V' \subset M'$ be a causally convex normal neighbourhood and let $U' \subset V' \cap \overline{\mu}(\Sigma)$ be a relatively compact open set. Then there is a $\delta > 0$ such that the Gaussian normal coordinate $(x')^0$ is a well-defined coordinate near $\overline{U'}$ for all $|(x')^0| \leq \delta$. For F_k^e as in Proposition 4.4 and $k \in \mathbb{N}_0$ we have:

$$\begin{split} \tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+} &= \sigma_{U'}^{k+1-\frac{d}{2}}(c_{k}+d_{k}\log(\sigma_{U'}))\\ \partial_{(x')^{0}}\tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+} &= -\partial_{(y')^{0}}\tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+} = \begin{cases} 0 & k \ge 1\\ \frac{1}{2}\delta_{0} & k = 0 \end{cases}\\ \partial_{(x')^{0}}\partial_{(y')^{0}}\tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+} &= \begin{cases} \frac{-1}{2}(\partial_{(x')^{0}}\partial_{(y')^{0}}\bar{\sigma}_{R})\tilde{F}_{k-1}^{e}|_{(U')^{\times 2}}^{+}\\ +(\partial_{(x')^{0}}\partial_{(y')^{0}}\bar{\sigma}_{R})\tilde{F}_{k-1}^{e}|_{(U')^{\times 2}}^{+} & k \ge 1, \ k \neq \frac{d}{2} - 1\\ \frac{-1}{2}(\partial_{(x')^{0}}\partial_{(y')^{0}}\bar{\sigma}_{R})\tilde{F}_{k-1}^{e}|_{(U')^{\times 2}}^{+} & k \ge 1, \ k = \frac{d}{2} - 1 \end{split}$$

where $\sigma_{U'}$ is Synge's world function restricted to U' and $|_{(U')^{\times 2}}^+$ denotes the distributional limit as $(y')^0 = -(x')^0 \to 0^+$. The expressions on the right-hand side of the first line are locally integrable functions. Furthermore,

$$(\partial_{(x')^0} + \partial_{(y')^0})^2 \tilde{F}_0^e|_{(U')^{\times 2}} = C(d)\sigma_{U'}^{\epsilon + \frac{1}{2} - \frac{d}{2}} \frac{(\partial_{(x')^0} + \partial_{(y')^0})^2 \bar{\sigma}_R|_{(U')^{\times 2}}}{(\sigma_{U'})^{\epsilon + \frac{1}{2}}}$$

for any $\epsilon \in \left(0, \frac{1}{2}\right)$, where $C(d) = \left(1 - \frac{d}{2}\right)c_0 + d_0$.

Proof: Because $\overline{U'}$ is compact it is obvious that an $\epsilon > 0$ exists such that the Gaussian normal coordinate is well-defined. We will first show that

$$\lim_{(x')^0, (y')^0 \to 0} \frac{\sigma_{U'}((x')^i, (y')^i) + \frac{1}{2} |(x')^0 - (y')^0|^2}{\bar{\sigma}_R(x', y')} \to 1$$
(35)

uniformly on compact sets. Both numerator and denominator converge uniformly on compact sets to $\sigma_{U'}((x')^i, (y')^i)$. On compact subsets of $(U')^{\times 2}$ which do not intersect the diagonal the claim

then follows. Near the diagonal we require an additional argument. We may choose Riemannian local coordinates $(x')^i$ on U' centred at a given point $p \in U'$ and extend these to Gaussian normal coordinates on $\overline{U'} \times [-\delta, \delta]$. In these coordinates, the metric takes the form $g = (d(x')^0)^{\times 2} + h'_{ij}d(x')^i d(x')^j$, where $h'_{ij}(x')$ is a real, symmetric matrix that defines a bounded, positive operator on \mathbb{R}^{d-1} . There are constants M > 1 > m > 0 such that $m\delta_{ij} \leq h'_{ij}(x') \leq M\delta_{ij}$. For any $x', y' \in \overline{U'} \times [-\delta, \delta]$ we then find that $h'_{ij}(x') \geq c h'_{ij}(y')$ with $c := \frac{m}{M} \in (0, 1]$. By shrinking the region W to a smaller neighbourhood of p and by shrinking δ if necessary we can get the constants m, M and c to be arbitrarily close to 1.

If $\gamma:[0,1] \to V'$ is the unique geodesic in V' between $x' = ((x')^0, (x')^i)$ and $y' = ((y')^0, (y')^i)$, we may write it as $\gamma = (\gamma^0, \gamma^i)$. If $\tilde{x'} = ((\tilde{x'})^0, (x')^i)$ and $\tilde{y'} = ((\tilde{y'})^0, (y')^i)$ we can use the curve $\tilde{\gamma}(s) := ((1-s)(\tilde{x'})^0 + s(\tilde{y'})^0, \gamma^i(s))$ (which may not be a geodesic) to derive the following estimate:

$$\begin{split} \bar{\sigma}_R(x',y') &= \frac{1}{2} \int_0^1 dt \; g_{\mu\nu}(\gamma) \dot{\gamma}^{\mu} \dot{\gamma}^{\nu} = \frac{1}{2} \int_0^1 dt \; g_{\mu\nu}(\gamma) \dot{\tilde{\gamma}}^{\mu} \dot{\tilde{\gamma}}^{\nu} + (\dot{\gamma}^0)^2 - (\dot{\tilde{\gamma}}^0)^2 \\ &\geq \frac{1}{2} \int_0^1 dt \; cg_{\mu\nu}(\tilde{\gamma}) \dot{\tilde{\gamma}}^{\mu} \dot{\tilde{\gamma}}^{\nu} + (\dot{\gamma}^0)^2 - (\dot{\tilde{\gamma}}^0)^2 \\ &\geq c \bar{\sigma}_R(\tilde{x'}, \tilde{y'}) + \frac{1}{2} (|(x')^0 - (y')^0|^2 - |(\tilde{x'})^0 - (\tilde{y'})^0|^2), \end{split}$$

where we used the fact that the geodesic in $\Sigma \cap V'$ between $\tilde{x'}$ and $\tilde{y'}$ is the unique curve of minimal length and similarly for the geodesic between $(x')^0$ and $(y')^0$ in $(-\delta, \delta)$ with the Riemannian metric $(d(x')^0)^2$ (cf. [43] Proposition 5.16). In particular, it follows that

$$c\sigma_{U'}((x')^{i},(y')^{i}) + \frac{c}{2}|(x')^{0} - (y')^{0}|^{2} \le \bar{\sigma}_{R}(x',y') \le c^{-1}\sigma_{U'}((x')^{i},(y')^{i}) + \frac{1}{2c}|(x')^{0} - (y')^{0}|^{2}.$$

For any compact $K \subset U'$ we can cover a neighbourhood of the compact intersection of $K^{\times 2}$ with the diagonal by a finite number of sufficiently small compact sets to see that the last estimate still holds and that the constant can be chosen arbitrarily close to 1 if $(x')^0$ and $(y')^0$ are small enough. Therefore, the limit in Equation (35) converges uniformly on compact sets.

If $r > \frac{1-d}{2}$ and $(x')^0, (y')^0 \to 0$, then $\bar{\sigma}_R(x, y)^r$ converges as a measure to the locally integrable function $\sigma_{U'}(x, y)^r$. This follows from the dominated convergence theorem, the uniform convergence above and the estimate $\sigma_{U'}((x')^i, (y')^i) + \frac{1}{2}|(x')^0 - (y')^0|^2 \ge \sigma_{U'}((x')^i, (y')^i)$. Hence, for kgreater than or equal to the number of normal derivatives, the distributions

$$\tilde{F}_k^e, \quad \partial_{(x')^0} \tilde{F}_k^e, \quad \partial_{(y')^0} \tilde{F}_k^e, \quad -\partial_{(x')^0} \partial_{(y')^0} \tilde{F}_k^e,$$

are continuous functions of $(x')^0 \neq (y')^0$ with values in the distributions on $(U')^{\times 2}$. Moreover, the limits $(x')^0 \to 0^-$ and $(y')^0 \to 0^+$ exist and are given by the indicated locally integrable functions, where we use Corollary 2.17 to treat the normal derivatives of $\bar{\sigma}_R$.

For $k = 1, k \neq \frac{d}{2} - 1$, we have

$$\begin{aligned} \partial_{(x')^{0}}\partial_{(y')^{0}}\tilde{F}_{1}^{e} &= \left(\partial_{(x')^{0}}\partial_{(y')^{0}}\bar{\sigma}_{R}\right)\left(-\frac{1}{2}\tilde{F}_{0}^{e}+d_{1}\bar{\sigma}_{R}^{1-\frac{d}{2}}\right) \\ &+ \left(\partial_{(x')^{0}}\bar{\sigma}_{R}\right)\left(\partial_{(y')^{0}}\bar{\sigma}_{R}\right)\bar{\sigma}_{R}^{-\frac{d}{2}}\left(\left(1-\frac{d}{2}\right)\left(d_{1}-\frac{1}{2}c_{0}\right)-\frac{1}{2}d_{1}\right) \end{aligned}$$

as a locally integrable function on $(U' \times (-\epsilon, \epsilon))^{\times 2}$, whereas for $k = 1 = \frac{d}{2} - 1$ we find

$$\partial_{(x')^0} \partial_{(y')^0} \tilde{F}_1^e = (\partial_{(x')^0} \partial_{(y')^0} \bar{\sigma}_R) \left(-\frac{1}{2} \tilde{F}_0^e \right) + (\partial_{(x')^0} \bar{\sigma}_R) (\partial_{(y')^0} \bar{\sigma}_R) \bar{\sigma}_R^{-\frac{d}{2}} \frac{c_0}{2}.$$

In both cases we can use the Euclidean version of Equation (10) to deduce that $|\partial_{(x')^0}\bar{\sigma}_R| \leq \sqrt{2\bar{\sigma}_R}$ and $|\partial_{(y')^0}\bar{\sigma}_R| \leq \sqrt{2\bar{\sigma}_R}$. We may use this to estimate the factors in the second term as

$$|(\partial_{(x')^0}\bar{\sigma}_R)(\partial_{(y')^0}\bar{\sigma}_R)| \le |\partial_{(x')^0}\bar{\sigma}_R|^{\frac{1}{2}} (2\bar{\sigma}_R)^{\frac{3}{4}}.$$

Arguing as before, we see that $\partial_{(x')^0} \partial_{(y')^0} \tilde{F}_1^e$ has a limit as $(x')^0 \to 0^-$ and $(y')^0 \to 0^+$, where the second term vanishes and the first term yields the expression stated in the theorem.

Now we turn to the case k = 0 with one normal derivative. With the constant C(d) defined in Theorem 4.6 we then have

$$\begin{aligned} \partial_{(x')^0} \tilde{F}_0^e(x',y') &= C(d) (\partial_{(x')^0} \bar{\sigma}_R(x',y')) \bar{\sigma}_R(x',y')^{-\frac{d}{2}} \\ &= C(d) \frac{(\partial_{(x')^0} \bar{\sigma}_R(x',y'))}{\tau} \left(\frac{\sigma_{U'}((x')^i,(y')^i) + \frac{1}{2} |(x')^0 - (y')^0|^2}{\bar{\sigma}_R(x',y')} \right)^{\frac{d}{2}} \\ &\quad \tau \left(\sigma_{U'}((x')^i,(y')^i) + \frac{1}{2} \tau^2 \right)^{-\frac{d}{2}} \end{aligned}$$

where we take $\frac{1}{2}\tau = (y')^0 = -(x')^0 > 0$. The quotient to the power $\frac{d}{2}$ converges to 1 uniformly on compact sets as $\tau \to 0^+$. Expanding $\partial_{(x')^0} \bar{\sigma}_R(x', y')$ in a Taylor series around $(x')^0 = (y')^0 = 0$ we see that the first quotient converges uniformly on compact sets to

$$\frac{1}{2}\partial_{(x')^0}\partial_{(y')^0}\bar{\sigma}_R(x',y')|_{(U')^{\times 2}} - \frac{1}{2}\partial_{(x')^0}^2\bar{\sigma}_R(x',y')|_{(U')^{\times 2}}$$

which is constantly -1 on the diagonal. Since the last factors converge to $\frac{-1}{2C(d)}\delta$ by Lemma A.1 in Appendix A we find that

$$\partial_{(x')^0} \tilde{F}_0^e |_{(U')^{\times 2}}^+ = \frac{1}{2} \delta.$$

The argument for $\partial_{(y')^0} F_0^e|_{(U')^{\times 2}}^+$ is entirely analogous, but incurs an additional sign from the first quotient.

For the final equality, we use the fact that $(2-d)d_0 = 0$ for all d to compute for all $(x')^0 \neq (y')^0$:

$$(\partial_{(x')^0} + \partial_{(y')^0})^2 \tilde{F}_0^l = C(d) \bar{\sigma}_R^{\epsilon + \frac{1}{2} - \frac{d}{2}} \frac{\Phi'}{\bar{\sigma}_R^{\epsilon + \frac{3}{2}}}$$
$$\Phi' := \left\{ -\frac{d}{2} ((\partial_{(x')^0} + \partial_{(y')^0}) \bar{\sigma}_R)^2 + \bar{\sigma}_R (\partial_{(x')^0} + \partial_{(y')^0})^2 \bar{\sigma}_R \right\}$$

with $\epsilon \in (0, \frac{1}{2})$. The function Φ' has a Taylor series in x^i, y^i and $\tau = \frac{1}{2}(y')^0 = -\frac{1}{2}(x')^0$ which vanishes up to (and including) third order at any point with $x^i = y^i$ and $\tau = 0$. It follows that the quotient

$$\frac{\Phi'}{\bar{\sigma}_B^{\epsilon+\frac{3}{2}}}$$

is continuous at such points. As $\tau \to 0^+$ it converges to $\frac{(\partial_{(x')^0} + \partial_{(y')^0})^2 \bar{\sigma}_R|_{(U') \times 2}}{\bar{\sigma}_R^{\epsilon + \frac{1}{2}}}$ uniformly on com-

pact sets. The power $\bar{\sigma}_R^{\epsilon+\frac{1}{2}-\frac{d}{2}}$ converges to $\sigma_{U'}^{\epsilon+\frac{1}{2}-\frac{d}{2}}$ in the sense of measure theory, by the dominant convergence theorem. It therefore follows that the product also converges to the expression claimed in the theorem.

Remark 4.8 To define the restrictions of distributions appearing in Theorem 4.7 it does not seem to suffice to appeal to general results in microlocal analysis with finite Sobolev regularity. For example, $\hat{F}_k^e(\xi) = k! |\xi|^{-2-2k}$ outside $\xi = 0$, where $|\xi|$ denotes the Euclidean norm. It follows that F_k^e is in the Sobolev space $H^{(s)}(\mathbb{R}^d)$ for all $s < 2 + 2k - \frac{d}{2}$. In order to define its restriction to a (time zero) hyperplane using [28] Lemma 11.6.1 we need to require that $s > \frac{1}{2}$. This is possible only when $k > \frac{d-3}{4}$, so in particular it fails for k = 0 unless d = 2.

We can now compare the initial data of the distributions appearing in the Lorentzian and the Euclidean version of the Hadamard series:

Proposition 4.9 For all $k \ge 0$ we have

$$\begin{split} \tilde{F}_{k}^{l}|_{U^{\times 2}} &= (\overline{\mu}^{\times 2})^{*} (\tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+}) \\ \partial_{x^{0}} \tilde{F}_{k}^{l}|_{U^{\times 2}} &= i(\overline{\mu}^{\times 2})^{*} (\partial_{(x')^{0}} \tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+}) \\ \partial_{y^{0}} \tilde{F}_{k}^{l}|_{U^{\times 2}} &= i(\overline{\mu}^{\times 2})^{*} (\partial_{(y')^{0}} \tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+}) \\ \partial_{x^{0}} \partial_{y^{0}} \tilde{F}_{k}^{l}|_{U^{\times 2}} &= -(\overline{\mu}^{\times 2})^{*} (\partial_{(x')^{0}} \partial_{(y')^{0}} \tilde{F}_{k}^{e}|_{(U')^{\times 2}}^{+}). \end{split}$$

Proof: With the exception of the case k = 0 with two normal derivatives, this follows immediately from the results of Theorems 4.6 and 4.7, where the sign in the case of two normal derivatives is due to Corollary 2.17. It remains to show the result for k = 0 and two normal derivatives.

We consider the Hadamard series on M' for the operator $P' = -\Box_{\bar{g}_R}$. If we choose u'_0 to be the Hadamard coefficient for this operator, then the last term in Equation (26) vanishes and we have

$$P'(u'_0 \tilde{F}^e_0) \sim (P'u'_0) \tilde{F}^e_0 + u'_0 \rho^*(\delta_0 \otimes 1).$$
(36)

Note that $u'_0\rho^*(\delta_0 \otimes 1) = \rho^*(\delta_0 \otimes 1) = \delta$, by a change of coordinates and the fact that $u'_0 = 1$ on the diagonal. In Equation (36) we write P' in terms of Gaussian normal coordinates (acting on the first argument):

$$P' = -\partial_{(x')^0}^2 - \frac{1}{2} (\det h'_{ij})^{-1} (x') (\partial_{(x')^0} \det h'_{ij} (x')) \partial_{(x')^0} - \Box_{h'}.$$

Since $u'_0 \neq 0$ is smooth we can then rewrite the equivalence as

$$\partial_{(x')^0}^2 \tilde{F}_0^e \sim \frac{2}{u'_0} (g')^{\mu\nu} (\partial'_\mu u'_0) \partial'_\nu \tilde{F}_0^e - \frac{1}{2 \det h'_{ij}} (\partial'_0 \det h'_{ij}) \partial'_0 \tilde{F}_0^e - \Box_{h'} \tilde{F}_0^e + \delta.$$

It is then clear from Theorem 4.7 that all terms on the right-hand side of the equivalence have a limit in Gaussian normal coordinates as $-(x')^0 = (y')^0 \rightarrow 0^+$. Since both sides of the correspondence differ by a smooth function, the same must be true for the left-hand side and

$$\partial_{(x')^0}^2 \tilde{F}_0^e|_{(U')^{\times 2}}^+ \sim \left(\frac{2}{u'_0} (h')^{ij} (\partial'_i u'_0) \partial'_j - \Box_{h'}\right) \tilde{F}_0^e|_{(U')^{\times 2}}^+$$

by Propositions 2.11 and 4.5, where all operators still act on x' and we divided out $u'_0 \neq 0$.

From Theorem 4.7 we see that $\tilde{F}_0^e(y',x') = \tilde{F}_0^e(x',y')$, whereas u'_0 is symmetric by Moretti's Theorem 4.3. We can therefore apply the same argument as above to the case where P' acts on the argument y' to find that the limit of $-\partial^2_{(y')^0}\tilde{F}_0^e$ exists as $-(x')^0 = (y')^0 \to 0^+$. The limit is obtained (up to equivalence \sim) from the expression for $\partial^2_{(x')^0}\tilde{F}_0^e|_{(U')\times 2}^+$ by letting all operators act on y' instead of x'. Appealing to the last statement of Theorem 4.7 and taking a linear combination one finds that also $\partial_{(x')^0}\tilde{F}_0^e$ has such a limit and that

$$\begin{aligned} \partial_{(x')^{0}}\partial_{(y')^{0}}\tilde{F}_{0}^{e}|_{(U')^{\times 2}}^{+} &\sim \quad \frac{C(d)}{2}\sigma_{U'}^{\epsilon+\frac{1}{2}-\frac{d}{2}}\frac{(\partial_{(x')^{0}}+\partial_{(y')^{0}})^{2}\bar{\sigma}_{R}|_{(U')^{\times 2}}}{\sigma_{U'}^{\epsilon+\frac{1}{2}}} \\ &+ \left(\frac{1}{2}\Box_{h',(x')}+\frac{1}{2}\Box_{h',(y')}-\frac{1}{u'_{0}}(h')^{ij}(x')(\partial'_{(x')^{i}}u'_{0})\partial'_{(x')^{j}}\right) \\ &- \frac{1}{u'_{0}}(h')^{ij}(y')(\partial'_{(y')^{i}}u'_{0})\partial'_{(y')^{j}}\right)\tilde{F}_{0}^{e}|_{(U')^{\times 2}}^{+}.\end{aligned}$$

Similar arguments apply in the Lorentzian setting with the operator $P = -\Box_g$. In that case, we use Theorem 4.6 instead of 4.7, the restrictions to $U^{\times 2}$ are less problematic and the term $\rho^*(\delta_0 \otimes 1)$ is absent from the beginning, because we use the Hadamard series for a solution instead

of a fundamental solution. Using Proposition 4.5, Corollary 2.17 and the earlier results of this proposition one finds

$$\partial_{x^{0}}\partial_{y^{0}}\tilde{F}_{0}^{l}|_{U^{\times 2}}\sim -(\overline{\mu}^{\times 2})^{*}(\partial_{(x')^{0}}\partial_{(y')^{0}}\tilde{F}_{0}^{e}|_{(U')^{\times 2}}^{+}).$$

To show that we even have equality it suffices to compute both sides away from the diagonal, where they are smooth functions. This computation is straightforward and the equality follows from Corollary 2.17. This completes the proof. $\hfill \Box$

5 The Hartle-Hawking-Israel state in static black holes

We now turn to the rigorous construction of the HHI state in spacetimes with a static bifurcate Killing horizon and a wedge reflection. Assuming that κ is constant singles out a particular radius $R_H = \kappa^{-1}$ and hence a particular inverse temperature

$$\beta_H = 2\pi R_H = \frac{2\pi}{\kappa},$$

the inverse Hawking temperature. The Riemannian metric $(\bar{g}_{R_H})_{ab}$ on $M' := \overline{M_{R_H}^+}$ is smooth and we consider the elliptic operator

$$\bar{K} := -\Box_{\bar{g}_{R_H}} + V$$

on M', where V denotes the unique stationary potential on M' that extends the one on $M_{R_H}^+$ (cf. Lemma 2.9). Just like K_{R_H} in Section 3.2, \bar{K} is a symmetric and positive operator on the dense domain $C_0^{\infty}(M')$ in $L^2(M')$ and it has a self-adjoint Friedrichs extension \hat{K} which is strictly positive, because $\hat{K} \geq V$. We may therefore consider the Euclidean fundamental solution $\bar{G} := \hat{K}^{-1}$, which defines a distribution density on $(M')^{\times 2}$.

Because $\mathcal{B} = M' \setminus M_{R_H}^+$ is a submanifold of codimension two we may identify $L^2(M_{R_H}^+) = L^2(M')$. It is clear that \tilde{K} extends K_{R_H} , which is defined on $C_0^{\infty}(M_{R_H}^+)$, and hence the form domain of \hat{K} extends that of \hat{K}_{R_H} . However, K_{R_H} is in general not essentially self-adjoint, so it is not obvious if $\hat{K} = \hat{K}_{R_H}$. We will now prove that this is in fact the case, starting with a lemma:

Lemma 5.1 For every $\epsilon > 0$ there is a $\chi_{\epsilon} \in C^{\infty}(0, \infty)$ such that $\chi \equiv 1$ near 0, $\chi \equiv 0$ near $[\epsilon, \infty)$, $\chi'_{\epsilon} \leq 0$ everywhere and

$$\int_0^\infty r\chi_\epsilon'(r)^2 dr \leq \epsilon.$$

Proof: For any $\delta \in (0, \frac{1}{2})$ we may choose a $\chi_{\delta} \in C_0^{\infty}(0, \infty)$ with support in (0, 1) and such that $0 \leq \chi_{\delta} \leq 1$ everywhere and $\chi_{\delta} \equiv 1$ on $[\delta, 1 - \delta]$. We set $\tilde{\chi}_{\delta}(r) := \sqrt{2} \left(\frac{r}{\epsilon}\right)^{\epsilon-1} \chi_{\delta}\left(\frac{r}{\epsilon}\right)$, so $\tilde{\chi}_{\delta} \in C_0^{\infty}(0, \infty)$ has support in $(0, \epsilon)$ and $\tilde{\chi}_{\delta} \geq 0$ everywhere. We now use a simple substitution to estimate:

$$\int_{0}^{\infty} r \tilde{\chi}_{\delta}(r)^{2} dr = \int_{0}^{\infty} 2\epsilon^{2} r^{2\epsilon-1} \chi_{\delta}(r)^{2} dr \leq \int_{0}^{1} 2\epsilon^{2} r^{2\epsilon-1} dr = \epsilon,$$

$$c(\chi_{\delta}) := \int_{0}^{\infty} \tilde{\chi}_{\delta}(r) dr = \int_{0}^{\infty} \sqrt{2\epsilon} r^{\epsilon-1} \chi_{\delta}(r) dr$$

$$\geq \int_{\delta}^{1-\delta} \sqrt{2\epsilon} r^{\epsilon-1} dr = \sqrt{2} \left((1-\delta)^{\epsilon} - \delta^{\epsilon} \right).$$

As $\lim_{\delta \to 0^+} \sqrt{2} \left((1-\delta)^{\epsilon} - \delta^{\epsilon} \right) = \sqrt{2}$ we can choose $\delta > 0$ and χ_{δ} such that $c(\chi_{\delta}) \ge 1$. Then $\chi_{\epsilon}(r) := \frac{1}{c(\chi_{\delta})} \int_{r}^{\infty} \tilde{\chi}_{\delta}(s) ds$ has all the desired properties. \Box

Proposition 5.2 $\hat{K} = \hat{K}_{R_H}$ and consequently $\bar{G} = G_{R_H}$.

Proof: We need to show that \hat{K}_{R_H} extends $\hat{\bar{K}}$ as a quadratic form, since the converse is clear. In particular, we need to show that for every $f \in C_0^{\infty}(M')$, which is a form core of $\hat{\bar{K}}$, there is a sequence $f_n \in C_0^{\infty}(M' \setminus \mathcal{B})$ such that $f = \lim_{n \to \infty} f_n$ and $\lim_{m,n \to \infty} \langle f_m - f_n, K_{R_H}(f_m - f_n) \rangle = 0$. This entails that

$$\langle f, \hat{K}_{R_H} f \rangle = \lim_{m, n \to \infty} \langle f_m, K_{R_H} f_n \rangle = \lim_{m, n \to \infty} \langle f_m, \bar{K} f_n \rangle = \langle f, \bar{K} f \rangle,$$

so $\hat{K}_{R_H} = \hat{K}$ on the form core of the latter and the desired extension property follows. Equivalently, we need to show that $f = \lim_{n \to \infty} f_n$ and $\hat{K}_{R_H}^{\frac{1}{2}} f = \lim_{n \to \infty} \hat{K}_{R_H}^{\frac{1}{2}} f_n$. The linearity of this problem allows us to use a partition of unity argument, so it suffices to consider f supported in a region U, where U ranges over a set of coordinate neighbourhoods which cover M'. We may assume that U contains some point on \mathcal{B} , otherwise the claim is trivial.

Near any point $p \in \mathcal{B}$ we can find a coordinate neighbourhood U on which we can choose local coordinates (X, Y, x^i) as in the proof of Lemma 2.8, and we denote the corresponding polar coordinates by (τ, r, x^i) . For any $n \in \mathbb{N}$ we fix a function $\chi_{n^{-1}}$ as in Lemma 5.1 with $\epsilon = n^{-1}$ and we note that the function $\chi_{n^{-1}}(r)$ on U is smooth even at r = 0, because $\chi_{n^{-1}} \equiv 1$ near r = 0. For any $f \in C_0^{\infty}(U)$ we may now define

$$f_n(X, Y, x^i) := (1 - \chi_{n^{-1}}(r))f(X, Y, x^i).$$

We note that $f_n \in C_0^{\infty}(U \setminus \mathcal{B})$, because the first factor vanishes near \mathcal{B} and the second has compact support in U. As n increases, the support of $\chi_{n^{-1}}(r)$ shrinks in the radial direction, but its values remain uniformly bounded by 1. From this it follows that

$$\lim_{n \to \infty} f - f_n = 0, \qquad \lim_{n \to \infty} \sqrt{V}(f - f_n) = 0, \qquad \lim_{n \to \infty} \partial_{x^i}(f - f_n) = 0.$$

Using the polar coordinates we have

$$|v|^{-1}\partial_{\tau}f = \frac{1}{R_H} \left(-\frac{Y}{|v|}\partial_X f + \frac{X}{|v|}\partial_Y f\right)$$

away from \mathcal{B} . Since $\lim_{r\to 0^+} \frac{r}{|v|} = \kappa^{-1}$ and $r^{-1}X$ and $r^{-1}Y$ are bounded we see that $|v|^{-1}\partial_{\tau}f$ remains bounded on U and hence

$$\lim_{n \to \infty} \partial_\tau (f - f_n) = 0$$

as before. Also

$$\partial_r f = \frac{X}{r} \partial_X f + \frac{Y}{r} \partial_Y f$$

remains bounded on U, but now the derivative of $\chi_{n^{-1}}$ enters in the limit

$$\lim_{n \to \infty} \partial_r (f - f_n) = \lim_{n \to \infty} \chi'_{n^{-1}} f + \chi_{n^{-1}} \partial_r f.$$

The second term vanishes in the limit, by the boundedness of $\partial_r f$ and the shrinking support of $\chi_{n^{-1}}$. The first term also vanishes in the limit, due to the fact that the integration measure satisfies $\sqrt{\det(\bar{g}_{R_H})_{\mu\nu}} \leq Cr$ for some C > 0 on the compact support of f and $\int r \chi'_{n^{-1}}(r)^2 \leq n^{-1}$ by Lemma 5.1.

Putting all this into the definition of \bar{K} and \bar{K} we see that $\lim_{n\to\infty} (f - f_n) = 0$ and also $\lim_{n\to\infty} \hat{K}^{\frac{1}{2}}(f - f_n) = 0$. Using the positivity of \bar{K} we can then estimate

$$\begin{aligned} \|\hat{K}_{R_{H}}^{\frac{1}{2}}(f_{n}-f_{m})\|^{2} &= \langle (f_{n}-f)-(f_{m}-f), \bar{K}((f_{n}-f)-(f_{m}-f)) \rangle \\ &\leq 2\|\hat{K}^{\frac{1}{2}}(f_{n}-f)\|^{2}+2\|\hat{K}^{\frac{1}{2}}(f_{m}-f)\|^{2}. \end{aligned}$$

The right-hand side vanishes in the limit $m, n \to \infty$, showing that f is indeed in the domain of $\hat{K}_{R_{H}}^{\frac{1}{2}}$. It follows that $\hat{K}_{H_{R}} = \hat{K}$ and $\bar{G} = G_{R_{H}}$.

Recall that the imaginary time reflection $R_{\tau} : (\tau, x) \mapsto (-\tau, x)$ is a diffeomorphism of $M_{R_H}^+$, which is isometric. Near \mathcal{B} we can express R_{τ} in terms of the coordinates (X, Y, x^i) introduced in the proof of Lemma 2.8, where it is given by a reflection of Y. It is then easy to see that R_{τ} extends in a unique way to a diffeomorphism of M', which remains isometric. Furthermore, it leaves the hypersurface $\overline{\mu}(\Sigma)$ pointwise fixed, whereas it sends the normal derivative n^{μ} of this hypersurface to $-n^{\mu}$. By definition of Gaussian normal coordinates near $\overline{\mu}(\Sigma)$ it follows that R_{τ} is given locally by a reflection in the Gaussian normal coordinate.

Using \bar{G} on M' we will now prove that it gives rise to a pure Hadamard state ω^{HHI} on the Lorentzian side, which restricts to the double β -KMS state $\omega^{(\beta),d}$ of Theorem 3.5 in the exterior regions. This is our main result:

Theorem 5.3 Consider a spacetime M with a static bifurcate Killing horizon, a wedge reflection and a globally constant surface gravity κ and let Σ be a Cauchy surface as in Definition 2.2. There exists a unique state ω^{HHI} on the Weyl algebra of M with a Hadamard two-point distribution ω_2^{HHI} extending $\omega_2^{(\beta),d}$. This state is pure, quasi-free, Hadamard, invariant under the Killing flow, it extends $\omega^{(\beta),d}$ and ω_2^{HHI} is determined by the initial data

$$\begin{split} &\omega_{2,00}^{HHI}(f,f') &= \lim_{\tau \to 0^+} \bar{G}(-\tau,\overline{\mu}_*f;\tau,\overline{\mu}_*f') \\ &\omega_{2,10}^{HHI}(f,f') &= i \lim_{\tau \to 0^+} \partial_{(x')^0} \bar{G}(-\tau,\overline{\mu}_*f;\tau,\overline{\mu}_*f') \\ &\omega_{2,01}^{HHI}(f,f') &= i \lim_{\tau \to 0^+} \partial_{(y')^0} \bar{G}(-\tau,\overline{\mu}_*f;\tau,\overline{\mu}_*f') \\ &\omega_{2,11}^{HHI}(f,f') &= -\lim_{\tau \to 0^+} \partial_{(x')^0} \partial_{(y')^0} \bar{G}(-\tau,\overline{\mu}_*f;\tau,\overline{\mu}_*f') \end{split}$$

on the Cauchy surface Σ , for all $f, f' \in C_0^{\infty}(\Sigma)$.

Proof: Let $f, f' \in C_0^{\infty}(\Sigma)$, both supported in a compact set $K \subset \Sigma$. Let $\epsilon > 0$ be such that the Gaussian normal coordinate x^0 is well-defined for $|x^0| < \epsilon$ on a neighbourhood of K and such that the Gaussian normal coordinate $(x')^0$ is well-defined for $|(x')^0| < \epsilon$ on a neighbourhood of $\overline{\mu}(K) \subset \overline{\mu}(\Sigma)$. We may then consider the Euclidean Green's function, smeared with f, f', in Gaussian normal coordinates, $\overline{G}((x')^0, f; (y')^0, f')$. This defines a distribution in the variables $(x')^0, (y')^0$. To see that the limits, which give the initial data of ω_2^{HHI} , are well-defined, we note that \overline{G} is smooth away from the diagonal of $(M')^{\times 2}$, whereas its singularities on the diagonal are characterised by the Hadamard construction. Hence, $\overline{G}((x')^0, f; (y')^0, f')$ is smooth on $(x')^0 \neq$ $(y')^0$ and the limits are well-defined by Theorem 4.7.

 $(y')^0$ and the limits are well-defined by Theorem 4.7. The data $\omega_{2,ij}^{HHI}$ define a unique distributional bi-solution to the Klein-Gordon equation by Equation (12). These data are smooth away from the diagonal of $\Sigma^{\times 2}$, whereas the singularities on the diagonal coincide with those of a Hadamard state, due to Propositions 4.5 and 4.9. This shows that ω_2^{HHI} has the correct singularity structure. Note that Theorem 4.7 also implies that the anti-symmetric part of ω_2^{HHI} is the canonical commutator $\frac{i}{2}E$. To prove that ω_2^{HHI} is of positive type we use the reflection positivity of \bar{G} (Proposition 3.9),

To prove that ω_2^{HHI} is of positive type we use the reflection positivity of \bar{G} (Proposition 3.9), now with the reflection in the Gaussian normal coordinates (see the comments above this theorem). For any test-function $\chi \in C_0^{\infty}((0,\epsilon),\mathbb{R})$ we may set $F(x') := \chi((x')^0)f_1((x')^j) - i\chi'((x')^0)f_0((x')^j)$ and use the reflection positivity of \bar{G} to deduce that

$$\overline{G}(\overline{F}(-(x')^0, (x')^j), F(x')) \ge 0.$$

Letting χ approach a δ -distribution at some $\tau \in (0, \epsilon)$ leads to

$$\bar{G}(\delta((x')^0 + \tau)\overline{f_1} - i\partial_{(x')^0}\delta((x')^0 + \tau)\overline{f_0}, \delta((y')^0 - \tau)f_1 - i\partial_{(y')^0}\delta((y')^0 - \tau)f_0) \ge 0,$$

where the change of sign in the factor *i* due to complex conjugation is cancelled by the change of sign due to the derivative in the reflected Gaussian normal coordinate. Taking the limit of the final estimate as $\tau \to 0^+$ and using the definition of the initial data of ω_2^{HHI} in Equation (12) yields the desired positivity.

Since ω_2^{HHI} is a Hadamard two-point distribution, it defines a unique quasi-free Hadamard state ω^{HHI} on M. Its restriction to the union of the exterior wedges is $\omega^{(\beta),d}$, because $\bar{G} = G_{R_H}$ yield the same initial data on $(\Sigma \setminus \mathcal{B})^{\times 2}$. To see why the infinitesimal Wick rotation and the actual Wick rotation yield the same result, it suffices to note that the normal and the Killing time derivatives on Σ are related by the smooth factor v, which is non-zero away from the bifurcation surface.

Note that ω_2^{HHI} is the only Hadamard extension to M of $\omega_2^{(\beta),d}$. Because $\omega_2^{(\beta),d}$ is invariant under the Killing flow, the same must be true for ω_2^{HHI} and hence also for ω^{HHI} . The fact that ω^{HHI} is pure follows from the fact that $\omega^{(\beta),d}$ is pure (Theorem 3.5) and Propo-sition 5.4 below. The uniqueness of ω^{HHI} follows from its purity and the uniqueness of the Hadamard extension of $\omega_2^{(\beta),d}$, by a result of Kay [36].

Proposition 5.4 Let ω be a quasi-free Hadamard state on a globally hyperbolic spacetime M and let (p, \mathcal{H}) be the one-particle structure of its two-point distribution ω_2 . Let $\Sigma \subset M$ be a Cauchy surface and let $\mathcal{B} \subset \Sigma$ be a submanifold of codimension at least 1. Then p already has a dense range on $C_0^{\infty}(D(\Sigma \setminus \mathcal{B}))$, where D denotes the domain of dependence. Consequently, if the restriction of ω to $D(\Sigma \setminus \mathcal{B})$ is pure, then ω itself is pure.

Proof: We can consider the initial value formulation and replace p by the continuous linear map $q: C_0^{\infty}(\Sigma)^{\oplus 2} \to \mathcal{H}$. We denote by \mathcal{H}' the closed range of q on $C_0^{\infty}(\Sigma \setminus \mathcal{B})^{\oplus 2}$ and by \mathcal{H}^0 the closed range of q on $C_0^{\infty}(\Sigma) \oplus \{0\}$. We let P denote the orthogonal projection of \mathcal{H} onto \mathcal{H}' and we introduce a real-linear isometric involution C on \mathcal{H} which acts as the identity on $(\mathcal{H}^0)^{\perp}$ and which is defined on \mathcal{H}^0 by continuous linear extension of $Cq(f_0, 0) := q(\overline{f_0}, 0)$. Note that this is indeed isometric, because $\omega_{2,11}$ is symmetric. For any $\psi \in \mathcal{H}$ and $\chi \in \mathcal{H}'$ we have $\|\psi - \chi\| = \|C\psi - C\chi\|$. Since $\chi = P\psi$ minimises this norm we must have $CP\psi = PC\psi$, i.e. CP = PC.

We claim that the range of q on $\{0\} \oplus C_0^{\infty}(\Sigma)$ is entirely contained in the subspace \mathcal{H}' . To see why this is so we choose $f_1 \in C_0^{\infty}(\Sigma)$ and we let $\chi_n \in C_0^{\infty}(\Sigma, \mathbb{R})$ be a sequence of test-functions which remain uniformly bounded and such that each $\chi_n \equiv 1$ on a neighbourhood of supp $(f_1) \cap \mathcal{B}$, but such that the support of χ_n shrinks towards \mathcal{B} as $n \to \infty$. Then the sequence $(1 - \chi_n)f \in$ $C_0^{\infty}(\Sigma \setminus \mathcal{B})$ is also uniformly bounded and it converges pointwise to f almost everywhere. We now use the fact that $||q(0,\chi_n f_1)||^2 = \omega_{2,00}(\chi_n f_1,\chi_n f_1)$. Because ω_2 is Hadamard, $\omega_{2,00}$ is given by a locally integrable function on $\Sigma^{\times 2}$ (cf. Theorem 4.6). It therefore follows from the Dominated Convergence Theorem that $q(0, \chi_n f_1) \to 0$ as $n \to \infty$ and hence $q(0, (1 - \chi_n) f_1) \to q(0, f_1)$. Because the latter sequence remains in \mathcal{H}' , the limit must also lie in this subspace, which proves the claim.

Now we define an \mathcal{H}' -valued distribution on M by p'(f) := Pp(f). We will show that (p', \mathcal{H}') is a one-particle structure on M. The distribution p' satisfies the Klein-Gordon equation, just like p, and it has a dense range. To establish the commutator property we use the initial value formulation $q': C_0^{\infty}(\Sigma)^{\oplus 2} \to \mathcal{H}'$ of p'. For any $f, f' \in C_0^{\infty}(M)$ let $(f_0, f_1, f'_0, f'_1) \in C_0^{\infty}(\Sigma)$ denote the corresponding initial data. Then,

$$q'(f_0, f_1) = Pq(f_0, 0) + q(0, f_1)$$

since $Pq(0, f_1) = q(0, f_1)$ by the previous paragraph. By the same token and the commutator property of p:

$$\begin{array}{l} \langle q'(\overline{f_0},\overline{f_1}),q'(f_0',f_1')\rangle - \langle q'(\overline{f_0'},\overline{f_1'}),q'(f_0,f_1)\rangle \\ = & iE(f,f') + \langle q(\overline{f_0'},0),(I-P)q(f_0,0)\rangle - \langle q(\overline{f_0},0),(I-P)q(f_0',0)\rangle \\ = & iE(f,f') + \langle q(\overline{f_0'},0),(I-P)q(f_0,0)\rangle - \langle C(I-P)q(f_0',0),Cq(\overline{f_0},0)\rangle \\ = & iE(f,f') + \langle q(\overline{f_0'},0),(I-P)q(f_0,0)\rangle - \langle q(\overline{f_0'},0),(I-P)q(f_0,0)\rangle \\ = & iE(f,f'). \end{array}$$

Hence, p' has the desired commutator property and (p', \mathcal{H}') is a one-particle structure on M. Note that the two-point distribution ω'_2 corresponding to p' coincides with ω_2 on $D(\Sigma \setminus \mathcal{B})$ and that it is Hadamard, because the estimate $\omega'_2(\overline{f}, f) \leq \omega_2(\overline{f}, f)$ allows us to estimate the wave front set of the Hilbert space-valued distribution p' as in [54]. It follows that the initial data of $\omega_2 - \omega'_2$ are smooth and that they vanish on the dense subset $(\Sigma \setminus \mathcal{B})^{\times 2}$ of $\Sigma^{\times 2}$. They must therefore vanish everywhere and $\omega_2 = \omega'_2$. This implies that P = I and in particular $\mathcal{H} = \mathcal{H}'$.

Due to Lemma A.2 of [37], a quasi-free state ω is pure if and only if the one-particle structure (p, \mathcal{H}) of its two-point distribution is such that p already has a dense range on the real-valued test-functions. Combining this criterion with the results we have just shown and the assumption that the restriction of ω to $D(\Sigma \setminus \mathcal{B})$ is pure, we see that p already has a dense range on $C_0^{\infty}(D(\Sigma \setminus \mathcal{B}), \mathbb{R})$, so that ω is pure.

In the case where M is Minkowski spacetime and M^+ the Rindler wedge, the Hartle-Hawking-Israel state is well-known to be the Minkowski vacuum (cf. [55]).

We have already seen in Theorem 5.3 that ω^{HHI} is Hadamard and invariant under the Killing flow. It is also invariant under the wedge reflection, in the following sense. Just as in Section 3.2 one may use the wedge reflection I on M to define a complex anti-linear involution τ_I of the Weyl algebra of the entire spacetime by setting $\tau_I(zW(f)) = \overline{z}W(I^*f)$. One may then show that ω^{HHI} is invariant under τ_I in the conjugate linear sense of Equation (17). This is because the restriction of ω^{HHI} to $M^+ \cup M^-$ is a double β_H -KMS state, which is necessarily τ_I invariant. That this invariance extends across the bifurcation surface \mathcal{B} follows from the Hadamard property, because the Hadamard two-point distributions $\omega_2^{HHI}(x, y)$ and $\omega_2^{HHI}(I(y), I(x))$ differ by a smooth function and their initial data are equal on the dense set $(\Sigma \setminus \mathcal{B})^{\times 2}$, so they must be equal everywhere.

Any other Hadamard two-point distribution ω_2 is of the form $\omega_2 = w_2 + \omega_2^{HHI}$, where w_2 is a smooth, real-valued bi-solution to the Klein-Gordon equation. If ω_2 is invariant under the Killing flow, then so is w_2 . In particular, we can consider two points x, y on one of the Killing horizons \mathfrak{h}_A of M, as was done in [37]. These points are most conveniently expressed in terms of local coordinates x^i, y^i on \mathcal{B} and an affine coordinate x^U along the lightlike geodesics, starting at \mathcal{B} , that generate \mathfrak{h}_A . Using the invariance of w_2 and these coordinates we find

$$w_2|_{\mathfrak{h}_{A}^{\times 2}}(x^U, x^i; y^U, y^i) = w_2|_{\mathcal{B}^{\times 2}}(x^i, y^i), \tag{37}$$

because we can exploit the Killing flow to simultaneously transport $x = (x^U, x^i)$ and $y = (y^U, y^i)$ to $\tilde{x} = (0, x^i)$ and $\tilde{y} = (0, y^i)$, respectively.

The Equality (37) is directly related to the uniqueness result found by Kay and Wald [37]. Because they restricted attention to observables that are generated by U-derivatives of compactly supported data on the horizon \mathfrak{h}_A , the term involving w_2 does not contribute to their expectation value in the two-point distribution ω_2 . On this subalgebra, the state ω^{HHI} can therefore be characterised uniquely by its Killing field invariance, dropping the assumption that it restricts to a double β_H -KMS state. This uniqueness claim is interesting for physical investigations involving phenomena near the horizon, but it is not clear under what circumstances it extends to a similar uniqueness claim on the entire Weyl algebra \mathcal{A} . That question would involve a more detailed analysis of w_2 on the entire manifold $M^{\times 2}$ and of the circumstances under which w_2 must vanish.

6 Discussion

In this final section, we comment on some aspects of our result and on the possibilities of generalising it.

Let us first note that we have made rather few assumptions about the future and past regions of the spacetime. Indeed, the Wick rotation and wedge reflection only require information about the left and right wedge regions and an arbitrarily small neighbourhood of the bifurcation surface. The future and past regions are not part of any of the complexified or Riemannian manifolds that we considered. For this reason, the Wick rotation does not provide any direct information about the behaviour of the state in the future or past regions. Instead, we have obtained this information indirectly, using the Cauchy problem and causal propagation. The only assumptions that we have made about the future and past regions is the existence of the Killing field ξ^a . This assumption was only necessary to formulate the Killing field invariance of ω^{HHI} on the entire spacetime M. For the main part of the construction it seems sufficient if ξ^a is only defined on the exterior wedge regions and a neighbourhood of the bifurcation surface.

The determination of ω^{HHI} from initial data works very well for a free field, but it is doubtful that it extends to interacting fields, whose restriction to a Cauchy surface may not be well-defined. It is therefore unlikely that our proof can be extended to such interacting fields and a proof of the conjecture of [32] would presumably require different (or additional) methods. As a first step one might investigate whether our results can be generalised to perturbatively interacting fields, e.g. using the ideas of [19].

We have shown the existence of a Hadamard extension ω^{HHI} of a double β_H -KMS state in the static case using a Wick rotation. In the more general, stationary case this method of proof is no longer available, but the result could still be true. Under what circumstances, if any, a Hadamard extension of a double β_H -KMS state exists is at present unclear. In the even more general case of Kerr spacetime the non-existence of a state which is invariant under the flow of the Killing field that generates the horizon is known to follow from a certain superradiance property, which is expected to hold [37].

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A A technical lemma

Lemma A.1 Let U be a convex normal neighbourhood in a Riemannian manifold (Σ, h_{ij}) of dimension d-1 and let σ_U be half the squared geodesic distance on U. For $d \geq 2$ we let

$$C(d) := -(4\pi)^{-\frac{d}{2}} 2^{\frac{d}{2}-1} \Gamma\left(\frac{d}{2}\right)$$

and we use the distribution $\delta(f) := \int_U f|_{\Delta}$ on $U^{\times 2}$, where $f|_{\Delta}(x) := f(x, x)$ is the restriction to the diagonal. Then

$$\lim_{\tau \to 0^+} \tau \left(\sigma_U + \frac{1}{2} \tau^2 \right)^{-\frac{a}{2}} = \frac{-1}{2C(d)} \delta$$

for all continuous $f \in C_0^0(U^{\times 2})$.

Note that $C(d) = (1 - \frac{d}{2})c_0 + d_0$, with c_0 and d_0 as in Proposition 4.4. *Proof:* For $d \ge 2$ we will need the following identities:

$$\operatorname{vol}(\mathbb{S}^{d-2}) = 2\pi^{\frac{d-1}{2}} \Gamma\left(\frac{d-1}{2}\right)^{-1}$$
$$X(d) := \int_0^\infty r^{d-2} (r^2+1)^{-\frac{d}{2}} dr = \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{d-1}{2}\right) \Gamma\left(\frac{d}{2}\right)^{-1}$$

The first identity is a standard result, which is proved by expressing $\pi^{\frac{d-1}{2}}$ as a Gaussian integral and changing to polar coordinates. To determine X(d) one may show by partial integration that

 $X(d+2) = \frac{d-1}{d}X(d)$. From a direct computation one finds $X(2) = \frac{\pi}{2}$ and X(3) = 1. The result then follows from a proof by induction.

Let $h(y) := \det h_{ij}(y)$ in Gaussian normal coordinates and $\tilde{h}(v, y) := \det h_{ij}(v)$ in Riemannian normal coordinates centred on y, so that $\tilde{h}(0, y) = 1$. Then we may compute

$$\begin{split} \lim_{\tau \to 0^+} \tau \left(\sigma_U + \frac{1}{2} \tau^2 \right)^{-\frac{d}{2}} (f) &= \lim_{\tau \to 0^+} 2^{\frac{d}{2}} \int dy \ dv \ \sqrt{h}(y) \sqrt{\tilde{h}}(v, y) f(\exp_y(v), y) \tau(|v|^2 + \tau^2)^{-\frac{d}{2}} \\ &= \lim_{\tau \to 0^+} 2^{\frac{d}{2}} \int dy \ dv \ \sqrt{h}(y) \sqrt{\tilde{h}}(\tau v, y) f(\exp_y(\tau v), y) (|v|^2 + 1)^{-\frac{d}{2}} \\ &= 2^{\frac{d}{2}} \int dv \ (|v|^2 + 1)^{-\frac{d}{2}} \int dy \ \sqrt{h}(y) f(y, y) \\ &= 2^{\frac{d}{2}} \operatorname{vol}(\mathbb{S}^{d-2}) X(d) \delta(f). \end{split}$$

Inserting the formulae for X(d) and $vol(\mathbb{S}^{d-2})$ we find the desired result.

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