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Statistical Mechanics of Deep Learning - Problem set 12

Winter Term 2023/24

Hand in: Friday, 19.01 at 10:00 am, you can upload your solutions to the course webpage on Moodle platform.

24. Functional Derivative

11 Points

A functional $F[\varphi]$ maps the function $\varphi(x)$ to the real numbers. The functional derivative of a functional with respect to a function is defined as

$$\frac{\delta F\left[\varphi\right]}{\delta\varphi(z)} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F\left[\varphi(x)+\epsilon\,\delta(x-z)\right]-F\left[\varphi(x)\right]\right) \quad .$$

This definition is in analogy the the definition of a partial derivative

$$\frac{\partial F(\vec{x})}{\partial x_j} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(F(\vec{x}+\epsilon\,\vec{e}_j)-F(\vec{x})\right) \quad .$$

When making the transition from partial to functional derivatives, the discrete index j turns into the continuous index x, and the unit vector in j-direction turns into the Dirac delta-function $\delta(x-z)$.
The derivative of a functional is a function and depends on the position z. Using this definition, compute the functional derivatives of the following functionals:

(a) $F[\varphi] = \varphi(x_0)$ with a fixed x_0.

(b) $F[\varphi] = (\varphi(x_0))^2$ with a fixed x_0.

(c) Assume that the function $f(x)$ can be expanded in a power series, and show that under this assumption for $F[\varphi] = f\left(\varphi(x_0)\right)$

$$\frac{\delta F\left[\varphi\right]}{\delta\varphi(z)} = f'\left(\varphi(x_0)\right)\,\delta(z-x_0)$$

(d) $F[\varphi] = \int_a^b A(x)\varphi(x)dx$

(e) $F[\varphi] = \int d^3x\, A(x)\left(\varphi(x)\right)^2$

(f) $F[\varphi] = \int d^3x\, A(x)\left(\varphi(x)\right)^n$

(g) $F[\varphi] = \int d^3x\, A(x) f\left(\varphi(x)\right)$

(h) $F[\varphi] = \int d^nx\, \left[\nabla\varphi(x)\cdot\nabla\varphi(x)\right]$

(i) $F[\varphi] = \int d^nx\, g\left(\nabla\varphi(x)\right)$

(j) $F[\varphi] = \int d^nx\, f\left(\varphi(x), \nabla\varphi(x), \triangle\varphi(x), \nabla^3\varphi(x), \ldots\right)$

(k) $S[q] = \int dt\mathcal{L}\left(q(t), \dot{q}(t)\right)$