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Statistical Mechanics of Deep Learning - Problem set 10

Winter Term 2023/24

Hand in: Friday, 05.01 at 10:00 am, you can upload your solutions to the course webpage on Moodle platform.

20. On-line learning of soft committee machines

4+3 Points

Consider the order parameter dynamical equations of a soft committee machines, `https://arxiv.org/abs/2104.14546`

$$\frac{dR}{d\alpha} = \frac{2\eta}{\pi}\frac{1}{1+Q}\frac{1}{K}\left\{\frac{1+Q-MR^2}{\sqrt{2(1+Q)-R^2}} - \frac{MR}{\sqrt{1+2Q}}\right\} \tag{1}$$

$$\frac{dQ}{d\alpha} = \frac{4\eta}{\pi}\frac{1}{1+Q}\frac{1}{K}\left\{\frac{MR}{\sqrt{2(1+Q)-R^2}} - \frac{MQ}{\sqrt{1+2Q}}\right\} \tag{2}$$

with $M = K$, the number of hidden units of the teacher network M is equal to the student network K in the realizable case. The generalization error is given by

$$\begin{aligned}\varepsilon_g = \frac{K}{\pi}\Bigg[\frac{\pi}{6} &+ \arcsin\left(\frac{Q}{1+Q}\right) + (K-1)\arcsin\left(\frac{C}{1+Q}\right) - 2\arcsin\left(\frac{R}{\sqrt{2(1+Q)}}\right) \\ &- 2(K-1)\arcsin\left(\frac{S}{\sqrt{2(1+Q)}}\right)\Bigg]\end{aligned} \tag{3}$$

(a) Show that equations (1) and (2) have the following fixed points, which correspond to the symmetric phase solutions discussed in the lecture

$$R_{pl} = \frac{1}{\sqrt{K(2K-1)}}\ ,\ Q_{pl} = \frac{1}{2K-1}$$

(b) Use the results obtained in (a) to compute the generalization error at the platue

$$\varepsilon_{pl} = \frac{K}{\pi}\left[\frac{\pi}{6} - K\arcsin\left(\frac{1}{2K}\right)\right]$$

Hint : note that on the platue, we have $R_{pl} = S_{pl}$ and $Q_{pl} = C_{pl}$

21. Statistical mechanics of soft committee machines *3+6+3 Points*

Applying replica methode on a soft committee machine in the limit $K \to \infty$ with $K \ll N$, yields the free energy (see `https://arxiv.org/abs/cond-mat/9812197`) in the variables $\hat{p} = Kp$, $\hat{C} = (K-1)C$, and $\hat{S} = KS$

$$\boldsymbol{f} = \frac{2F}{NK} = \alpha\left[\frac{\beta(v-2w+1/3)}{1+\beta(u-v)} + \ln[1+\beta(u-v)]\right] + \frac{\delta-\Delta^2}{\delta-1} \tag{4}$$
$$- \ln(1-\delta) - \frac{\delta+\hat{p}-(\Delta+\hat{S})^2}{\tilde{C}}$$

with $u = 1/3+\hat{C}/\pi$, $v = [2\text{arcsin}(\delta/2)+\hat{p}]/\pi$, $w = [2\text{arcsin}(\Delta/2)+\hat{S}]/\pi$ and $\tilde{C} = K(1+\hat{C}-\delta-\hat{p})$, while the parameters $\delta = q-p$ and $\Delta = R-S$ measure the degree of specialization in the network.

The generalization error is computed by averaging a quadratic cost function over all examples, $\varepsilon_g = \frac{1}{2}\langle(\sigma-\tau)^2\rangle_\xi$, in terms of the order parameters R_{ij}^a, Q_{ij}^{aa}, the student-teacher and student-student overlaps respectivly, the generalization error is given by

$$\varepsilon_g = \frac{1}{6} + \frac{1}{K\pi}\sum_{i,j=1}^{K}\left[\text{arcsin}\left(\frac{Q_{ij}^{aa}}{2}\right) - 2\text{arcsin}\left(\frac{R_{ij}^{a}}{2}\right)\right] \tag{5}$$

with the replica symmetric order parameters (a, b are replica indices while i, j are the hidden units indices)

$$Q_{ij}^{aa} = \begin{cases} 1 & ,\text{If } i=j \\ C & ,\text{If } i\neq j \end{cases}, \quad R_{ij}^{a} = \begin{cases} R & ,\text{If } i=j \\ S & ,\text{If } i\neq j \end{cases}, \quad Q_{ij}^{ab} = \begin{cases} q & ,\text{If } i=j \\ p & ,\text{If } i\neq j \end{cases} \quad \text{for } a \neq b$$

The saddle point equations of the free energy yields two different types of solutions : an unspecialized committee symmetric solution with $\Delta = \delta = 0,\ \hat{C} = 0$ and specialized solution with $\Delta, \delta > 0,\ \hat{C} = 0$. We aim in this problem to compute the generalization error in the symmetric phase by calculating the order parametres R_{ij}^a and Q_{ij}^{aa} which minimize the free energy in the symmetric phase.

(a) Start by rewriting all variables $\hat{C}$ in the free energy in terms of $\tilde{C}$ (taking the limit $K \to \infty$), then linearize to first order in δ and Δ.

(b) Now compute the saddle point equations with respect to the variables $\hat{S}$, $\hat{p}$, $\tilde{C}$, δ, Δ

$$\frac{\partial \boldsymbol{f}}{\partial \hat{S}} = \frac{\partial \boldsymbol{f}}{\partial \hat{p}} = \frac{\partial \boldsymbol{f}}{\partial \tilde{C}} = 0,$$

and show that $\hat{S} = \hat{p} = 1$ in the symmetric phase with $\delta = 0$ and $\Delta = 0$.

(c) Use the results obtained in (b) and the fact that $\hat{C} = 0$ in the symmetric phase to compute the symmetric platue value

$$\varepsilon_{pl} = \frac{1}{3} - \frac{1}{\pi}$$