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Advanced Quantum Mechanics - Problem Set 9

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Friday, 14.12.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 18.12.2023 and Wednesday 03.01.2024.

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

*1. Addition of angular momenta

3+3+1 Points

Consider two angular momenta $\hat{\boldsymbol{L}}_1$ and $\hat{\boldsymbol{L}}_2$ with $l_1 = l_2 = 1$. In this problem we will calculate the eigenvalues and eigenfunctions of $\hat{\boldsymbol{L}}^2$. The eigenfunctions are linear combinations of the 9 functions

$$Y_{1m}(\theta_1, \varphi_1)Y_{1m'}(\theta_2, \varphi_2) = u_m v_{m'}, \qquad \text{with } m, m' = 1, 0, -1.$$

(a) Construct the 9×9 matrix representation of the operator $\hat{\boldsymbol{L}}^2$ in the $u_m v_{m'}$ basis.

(b) Calculate the eigenvalues of $\hat{\boldsymbol{L}}^2$ by diagonalizing the matrix.

(c) Calculate the corresponding eigenfunctions.

Hint: It is possible to make the matrix block-diagonal, as shown in the figure, by making suitable row- and column-operations.

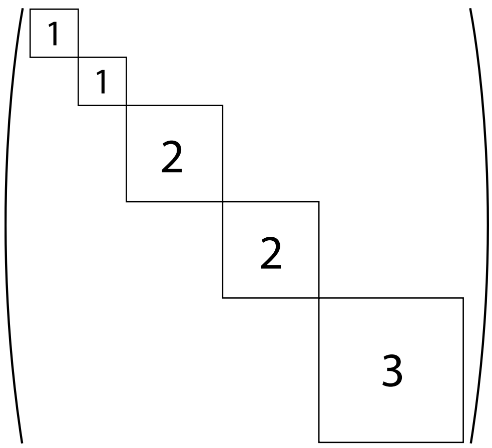

Figure 1: The matrix can be transformed into a block diagonal form.

2. Spin-orbit coupling

2+2+2 Points

Consider a particle with orbital angular momentum $\hat{\boldsymbol{L}}$ and spin angular momentum $\hat{\boldsymbol{S}}$. The total angular momentum is $\hat{\boldsymbol{J}} = \hat{\boldsymbol{L}} + \hat{\boldsymbol{S}}$.

(a) Calculate the expectation value of $\hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}$ assuming that the particle is in a state $|l, s; j, m\rangle$.

(b) An electron is moving in an electrostatic potential $\phi(r)$ with $r = |\boldsymbol{r}|$. Show that the electric field experienced by the particle is given by

$$\boldsymbol{E} = -\boldsymbol{r}\frac{1}{r}\frac{d\phi}{dr}.$$

(c) In the rest frame of the particle, the particle experiences a magnetic field $\boldsymbol{B} = -\boldsymbol{v} \times \boldsymbol{E}/c^2$. Calculate the energy $\frac{e}{m}\hat{\boldsymbol{S}} \cdot \boldsymbol{B}$, where e and m are the electron charge and mass respectively.

Remark: The result found in (c) is off by a factor of two compared to the exact result, which can be obtained using the Dirac equation. The reason is that the simple argument given above assumes a straight-line motion of the particle whereas the potential given above leads to a curved particle trajectory.

3. Addition of three angular momenta

4+2 Points

Consider three angular momenta with $l_1 = l_2 = l_3 = 1$.
First, consider adding two angular momenta $l_1 = l_2 = 1$ with m_1, m_2 to a total angular momentum l with m. As shown in problem 1

$$
\begin{aligned}
|l=1, m=1\rangle &= \frac{1}{\sqrt{2}}\Big(-|m_1=1; m_2=0\rangle + |m_1=0; m_2=1\rangle\Big) \\
|l=1, m=-1\rangle &= \frac{1}{\sqrt{2}}\Big(-|m_1=0; m_2=-1\rangle + |m_1=-1; m_2=0\rangle\Big) \\
|l=1, m=0\rangle &= \frac{1}{\sqrt{2}}\Big(-|m_1=1; m_2=-1\rangle + |m_1=-1; m_2=1\rangle\Big) \\
|l=0, m=0\rangle &= \frac{1}{\sqrt{3}}\Big(|m_1=1; m_2=-1\rangle - |m_1=0; m_2=0\rangle + |m_1=-1; m_2=1\rangle\Big)
\end{aligned}
$$

where $|m_1; m_2\rangle \equiv |l_1 = 1, m_1; l_2 = 1, m_2\rangle$.

(a) Add the three angular momenta to get a state with total angular momentum $l = 0$.

Hint: First add L_1 and L_2 and then add the resulting angular momentum with L_3. Use the same basis as in problem 1, but don't keep all 27 basis functions. Instead keep only the ones that can result to $l = 0$.

(b) Show that this state can be written as a 3×3 determinant and that it therefore is antisymmetric.

Hint: You can write $|m_1; m_2; m_3\rangle = |m_1\rangle \otimes |m_2\rangle \otimes |m_3\rangle = |m_1\rangle|m_2\rangle|m_3\rangle$.