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Advanced Quantum Mechanics - Problem Set 8

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 07.12.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 11.12.2023 and Wednesday 13.12.2023.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

*1. Chiral Symmetry

1+1+2+1+2+3 Points

Define the fifth γ-matrix as $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and consider the Dirac Hamiltonian

$$H_D = \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta m,$$

with

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix},$$
$$\beta = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}.$$

(a) Show that $\{\gamma^\mu \partial_\mu, \gamma^5\} = 0$. The first term in the anti-commutator is known as the Dirac operator. Since the Dirac Hamiltonian can be constructed using $\gamma^0\gamma^i = \alpha^i$ and $\gamma^0 = \beta$, the Hamiltonian anti-commutes with the operator $i\gamma^1\gamma^2\gamma^3$.

(b) Show that $\left(\gamma^5\right)^2 = \mathbb{1}_4$.

(c) Using that $\sigma^l\sigma^m = i\varepsilon_{lmk}\sigma^k + \delta_{lm}\mathbb{1}_2$ show that $\gamma^l\gamma^m = -i\varepsilon_{lmk}\Sigma^k - \delta_{lm}\mathbb{1}_4$ for $l, m = 1, 2, 3$ and explicitly compute $\gamma^1\gamma^3$. Here $\Sigma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$.

(d) Consider now an operator $\hat{C}$ with the property that $\hat{C}^2 = \mathbb{1}$ and $\{\hat{H}, \hat{C}\} = 0$. Show that if $|E_n\rangle$ is an eigenstate of the Hamiltonian H with eigenvalue E_n, then $|-E_n\rangle = C|E_n\rangle$ is also an eigenstate of the Hamiltonian with eigenvalue $-E_n$.

(e) Consider now a two-level system with energy eigenvalues $\pm E_n$. Write down the matrix representations of $\hat{C}$ and $\hat{H}$, and show that H is anti-diagonal in the basis where C is diagonal.

(f) Generalize your result in (e) to N non-degenerate levels. That is show that it is possible to diagonalize C in such a way that H becomes anti-diagonal. What happens qualitatively when there are degenerate eigenstates?

Hint: Diagonalize C (you know its eigenvalues). You can construct H using your result in (e). Write down a suitable basis. Think about how to rearrange the rows and columns of your matrices such that the diagonal elements in C are sorted with the positive eigenvalues coming before the negative eigenvalues.

2. Klein Tunneling in graphene *1+3+6 Points*

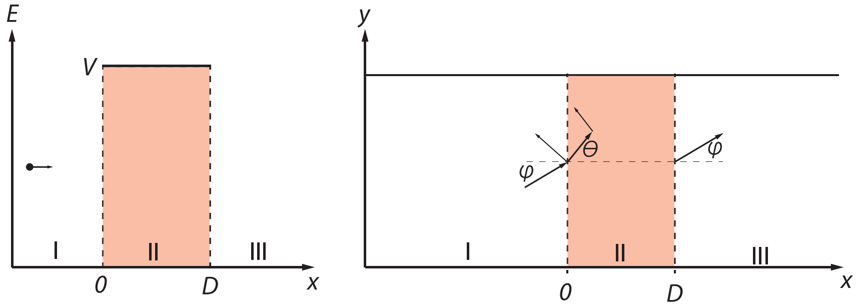

Figure 1: Left: Schematic drawing of a Dirac electron incident on a potential barrier. Right: Definition of the angles used in the problem. Assume that the sample is infinite in the y-direction.

Consider a Dirac electron with energy E incident on a potential barrier of size V as shown in the figure.

(a) Why is it sufficient to only require continuity of the wave-function and not its derivative?

(b) Assume the electron is incident at some angle ϕ in regions I and III and θ in region II, such that $k_x = k\cos\phi$, $k_y = k\sin\phi$ in regions I and III, while $\theta = \arctan(k_y/q_x)$ with $q_x = \sqrt{(V-E)^2/v^2 - k_y^2}$ and $v = |\boldsymbol{k}|/m$ in region II. Explain why the wave-functions in the different regions can be written as

$$\begin{aligned}
\psi_{\mathrm{I}}(x) &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i\phi} \end{pmatrix} e^{i(k_x x + k_y y)} + \frac{r}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i(\pi-\phi)} \end{pmatrix} e^{i(k_y y - k_x x)}, \\
\psi_{\mathrm{II}}(x) &= \frac{a}{\sqrt{2}}\begin{pmatrix} 1 \\ s'e^{i\theta} \end{pmatrix} e^{i(q_x x + k_y y)} + \frac{b}{\sqrt{2}}\begin{pmatrix} 1 \\ s'e^{i(\pi-\theta)} \end{pmatrix} e^{i(k_y y - q_x x)}, \\
\psi_{\mathrm{III}}(x) &= \frac{t}{\sqrt{2}}\begin{pmatrix} 1 \\ se^{i\phi} \end{pmatrix} e^{i(k_x x + k_y y)}.
\end{aligned}$$

Here $s = \mathrm{sgn}(E)$ and $s' = \mathrm{sgn}(E - V)$. What is the physical significance of r, a, b, and t?

(c) Use the continuity of the wave-function to calculate the transmission through the barrier $T(\theta, \phi, Dq_x) = |t|^2$. What do you get for $Dq_x = n\pi$ with n integer? For general values of Dq_x, investigate what happens when $\phi, \theta \to 0$.

Hint: You might want to use a computer algebra system to solve the resulting linear equation system for t, and to compute $|t|^2$.