Institut für Theoretische Physik
Universität Leipzig

Prof. Dr. B. Rosenow
M. Thamm

Advanced Quantum Mechanics - Problem Set 3

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 02.11.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 06.11.2023 and Wednesday 08.11.2023.

1. Parity operator

1+2+2 Points

Consider a one-dimensional real-space wave-function $\psi(x)$ and let $\hat{P}$ denote the parity operator such that $\hat{P}\psi(x) = \psi(-x)$.

(a) Show that $\hat{P}$ commutes with the Hamiltonian $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\hat{x})$ as long as $V(x)$ is an even function in x, i.e. $V(x) = V(-x)$.

(b) Starting from the Rodrigues formula for Hermitian polynomials, $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$ with $n \in \mathbb{N}$, show that the eigenfunctions $\psi_n(x)$ of the one-dimensional harmonic oscillator, with mass m and frequency ω, are also eigenfunctions of the parity operator. What are the eigenvalues?

(c) Define the operator

$$\hat{\Pi} = \exp\left[i\pi\left(\frac{1}{2\alpha}\hat{p}^2 + \frac{\alpha}{2\hbar^2}\hat{x}^2 - \frac{1}{2}\right)\right], \quad \alpha \in \mathbb{R}^+,$$

where $\hat{x}$ and $\hat{p}$ denote the position and momentum operators. Show that $\hat{\Pi}$ is a parity operator.

Hint: Consider $\hat{\Pi}\psi(x)$ and expand $\psi(x)$ with respect to a suitable basis.

*2. Landau levels

3+3+2 Points

A spinless particle of charge q is confined to the x-y plane and subjected to a magnetic field in the z-direction, $\boldsymbol{B} = (0, 0, B)$.

(a) Using the Landau gauge $\boldsymbol{A} = (0, Bx, 0)$ show that the Schrödinger equation can be written as

$$\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \left(\frac{\partial}{\partial y} - i\frac{qB}{\hbar}x\right)^2\right)\Psi(x,y) = E\Psi(x,y).$$

Hint: You can use the minimal coupling rule to obtain the canonical momentum.

(b) Show that a solution of the Schrödinger equation above can be written as $\Psi(x,y) = e^{iky}u(x-a)$, and find an expression for a in terms of k. What does $u(x-a)$ look like? Explain why the energy eigenvalues are given by

$$E = \frac{\hbar q B}{m}\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, \dots$$

(c) The particles are now confined to an area of length X in the x-direction and Y in the y-direction. Using periodic boundary conditions, $\Psi(y) = \Psi(y+Y)$ in the y-direction, calculate the maximum value of n per unit area.

Hint: Don't forget that $a \leq X$.

3. Quantum quench

2+3+1+1 Points

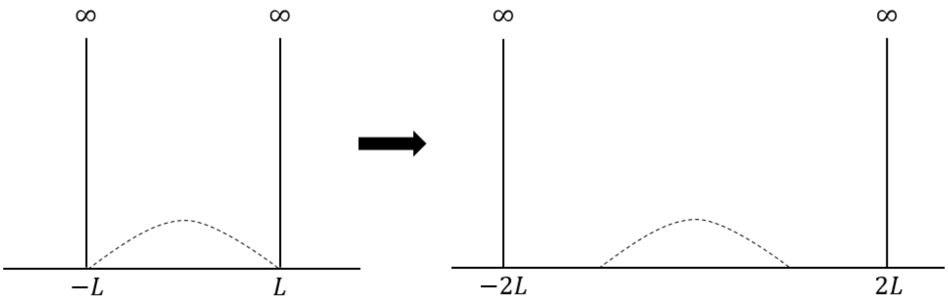

Figure 1: At $t = 0$ the infinite square well is instantaneously broadened.

Consider a particle of mass m that moves inside an infinite square well of width $2L$ ($-L < x < L$). Suppose that the particle is in the lowest energy state such that the eigenenergy and wave function of the particle are given by

$$E_1 = \frac{\hbar^2\pi^2}{8mL^2} \quad \text{and} \quad \psi_1(x) = \frac{1}{\sqrt{L}}\cos\frac{\pi x}{2L},$$

respectively. Now we assume that the walls of the well move instantaneously such that the well's width doubles ($-2L < x < 2L$) (this protocol is called a quantum quench). This change does not affect the state of the particle which still remains in the lowest energy state of the original well immediately after the change.

(a) In an expansion of the state $\psi_1 = \sum_n^\infty c_n \tilde{\psi}_n$ in terms of eigenstates $\tilde{\psi}_n$ of the wide well after the quench, which of the expansion coefficients c_n will vanish? Use the parity symmetry of the Hamiltonian to find the answer.

(b) Determine the wave function of the particle at times $t > 0$. What is the probability to find the system in an arbitrary eigenstate of the broadened well?

(c) Calculate the expectation value of the energy for $t > 0$.

(d) If we move the walls with a finite speed u instead of instantaneously, our results should still constitute a reasonable approximation, provided that u is much larger than a characteristic velocity v_0 of the system, i.e. $v_0 \ll u$. State v_0.