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Advanced Quantum Mechanics - Problem Set 1

Winter Term 2023/24

Due Date: Hand in solutions to problems marked with * as a single pdf file using Moodle before the lecture on **Thursday, 19.10.2023, 15:15**. The problem set will be discussed in the tutorials on Monday 23.10.2023 and Wednesday 25.10.2023.

Website: https://home.uni-leipzig.de/stp/Quantum_Mechanics_2_WS2324.html

Moodle: https://moodle2.uni-leipzig.de/course/view.php?id=45746

Please note that starting with this problem set, solutions have to be handed in using Moodle. Please upload your solutions to the task(s) marked with * as a single pdf file.

*1. Two-level system II

2+1+1+1 Points

Consider the Hamiltonian $\hat{H}$ of a quantum mechanical system with two eigenstates $|1\rangle$ and $|2\rangle$:

$$\begin{aligned}\hat{H}|1\rangle &= E_1|1\rangle \\ \hat{H}|2\rangle &= E_2|2\rangle\ .\end{aligned}$$

Assume that $\{|1\rangle, |2\rangle\}$ is an orthonormal basis of a two-dimensional Hilbert space, such that an arbitrary operator from the Hilbert space can be written as

$$|\psi\rangle = \alpha|1\rangle + \beta|2\rangle, \quad \alpha, \beta \in \mathbb{C}\ .$$

(a) Compute the norm $\langle\psi|\psi\rangle$. Express $\hat{H}$ in spectral representation and compute the expectation value of $\hat{H}$ in state $|\psi\rangle$.

Hint: The spectral representation of an operator $\hat{A}$ with orthonormal eigenvectors $\{|e_n\rangle\}$ and eigenvalues $\{\lambda_n\}$ is defined as

$$\hat{A} = \sum_n \lambda_n |e_n\rangle\langle e_n|\ .$$

(b) Define the operators $\hat{R} = |2\rangle\langle 1|$ and $\hat{L} = |1\rangle\langle 2|$. Compute the action of these operators on the basis states and on $|\psi\rangle$.

(c) Compute $\hat{R}\hat{R}$ and $\hat{L}\hat{L}$. What are the properties of the operators $\hat{R}\hat{L}$ and $\hat{L}\hat{R}$?

(d) Express the Hamiltonian $\hat{H}$ in terms of the operators $\hat{R}$, $\hat{L}$ and the eigenvalues E_1, E_2.

2. Commutators

2+2+2+2 Points

Because operators do not commute in general, it is helpful to define the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ for the operators $\hat{A}$ and $\hat{B}$. It can also be useful to define the anti-commutator $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Let in the following $\hat{A}$,$\hat{B}$, and $\hat{C}$ be arbitrary operators and $\hat{x}$ is the position and $\hat{p}$ the momentum operator. Show that:

(a) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$.

(b) If $\hat{A}$ and $\hat{B}$ are Hermitian operators, so are $i[\hat{A}, \hat{B}]$ and $\{\hat{A}, \hat{B}\}$.

(c) For integers $n \geq 1$ holds $[\hat{A}, \hat{B}^n] = n[\hat{A}, \hat{B}]\hat{B}^{n-1}$ assuming that $[\hat{B}, [\hat{A}, \hat{B}]] = 0$.

(d) $[\hat{p}, f(\hat{x})] = -i\hbar f'(\hat{x})$. Assume here that $f(x)$ can be expressed as a power series $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)x^n$ and use the commutation relation $[\hat{x}, \hat{p}] = i\hbar$.

3. Expectation values for the harmonic oscillator

4 Points

Compute the expectation values of the operators $\hat{x}$, $\hat{p}$, $\hat{x}^2$, and $\hat{p}^2$ in the state

$$\Phi(x) = \frac{1}{\sqrt{2}} \left(\psi_0(x) + \psi_2(x)\right) ,$$

where ψ_0 and ψ_2 are wave functions for the ground state and the second excited state of the one-dimensional harmonic oscillator, respectively. Use the relations

$$\psi_{n+1}(x) = \frac{1}{\sqrt{n+1}} \hat{a}^\dagger \psi_n(x)$$
$$\psi_{n-1}(x) = \frac{1}{\sqrt{n}} \hat{a}\psi_n(x) ,$$

where $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for the harmonic oscillator.