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Quantum Field Theory of Many-Particle Systems - Problem Set 3

Summer Semester 2024

Due: The problem set will be discussed in the tutorial on **Friday, 26.04.2024, 13:30**. Problem 2 is a bonus problem for extra points and will not be counted towards the 50% threshold for exam admission.

Internet: The problem sets can be downloaded from https://home.uni-leipzig.de/stp/QFT_of_MPS_SS24.html

1. Functional Derivative *11 Punkte*

A functional $F[\varphi]$ maps the function $\varphi(x)$ to the real numbers. The functional derivative of a functional with respect to a function is defined as

$$\frac{\delta F[\varphi]}{\delta\varphi(z)} = \lim_{\epsilon\to 0} \frac{F[\varphi(x) + \epsilon\delta(x-z)] - F[\varphi(x)]}{\epsilon} \ . \tag{1}$$

This definition is in analogy to the definition of a partial derivative

$$\frac{\partial F(\boldsymbol{x})}{\partial x_j} = \lim_{\epsilon\to 0} \frac{F(\boldsymbol{x} + \epsilon\boldsymbol{e}_j) - F(\boldsymbol{x})}{\epsilon} \ .$$

When making the transition from partial to functional derivatives, the discrete index j turns into a continuous index x, and the unit vector in j direction turns into the Dirac delta distribution $\delta(x-z)$. The derivative of a functional is a function and depends on the position z.
Using this definition, compute the functional derivatives of the following functionals:

(a) $F[\varphi] = \varphi(x_0)$ for a fixed x_0.

(b) $F[\varphi] = \varphi(x_0)^2$ for a fixed x_0.

(c) For a function $f(x)$ that can be expanded in a power series, show that the functional derivative of $F[\varphi] = f(\varphi(x_0))$ is given by

$$\frac{\delta F[\varphi]}{\delta\varphi(z)} = f'(\varphi(x_0))\,\delta(z - x_0) \ .$$

(d) $F[\varphi] = \int_a^b \mathrm{d}x\, A(x)\varphi(x)$

(e) $F[\varphi] = \int \mathrm{d}^3x\, A(x)[\varphi(x)]^2$

(f) $F[\varphi] = \int \mathrm{d}^3x\, A(x)[\varphi(x)]^n$

(g) $F[\varphi] = \int \mathrm{d}^3x\, A(x) f(\varphi(x))$

(h) $F[\varphi] = \int \mathrm{d}^nx\, [\nabla\varphi(x)\cdot\nabla\varphi(x)]$

(i) $F[\varphi] = \int \mathrm{d}^nx\, g(\nabla\varphi(x))$

(j) $F[\varphi] = \int \mathrm{d}^nx\, f\big(\varphi(x), \nabla\varphi(x), \Delta\varphi(x), \nabla^3\varphi(x), ...\big)$

(k) $F[q] = \int \mathrm{d}t\, \mathcal{L}(q(t), \dot{q}(t))$

2. Jordan-Wigner transformation *Bonus 1+2+2+2 Punkte*

In this problem we will show that a spin 1/2 system can be represented in terms of fermionic operators. In particular, we represent an up spin as a particle and a down spin as the vacuum $|0\rangle$ such that $|\uparrow\rangle = |1\rangle = c^\dagger|0\rangle$ and $|\downarrow\rangle = |0\rangle = c|1\rangle$, with c denoting a fermionic operator. Naively we might try to construct a representation by writing $\sigma^+ = c^\dagger$ and $\sigma^- = c$, with $\sigma^z = 2c^\dagger c - 1$.

(a) Show that, in this representation, the spins indeed satisfy the spin algebra $[\sigma^x, \sigma^y] = 2i\sigma^z$.

(b) There is however a problem with this representation since spins on different sites commute whilst fermionic operators anti-commute. This sign somehow has to be fixed. In one dimension this can be done by the following transformation, known as a Jordan-Wigner transformation:

$$\sigma_n^+ = c_n^\dagger e^{-i\pi\sum_{j<n} n_j}, \qquad \sigma_n^- = c_n e^{i\pi\sum_{j<n} n_j}, \qquad \sigma_n^z = 2c_n^\dagger c_n - 1.$$

Here $n_j = c_j^\dagger c_j$ is the number operator. In one dimension the interpretation of this representation is as follows: We order the particles on a line. By then counting the number of particles to the left of a given site we pick up a phase of $+1$ or -1. This allows us to consider the particles as fermions. Using the Jordan-Wigner transformation, show that $\sigma_n^+\sigma_{n+1}^- = c_n^\dagger c_{n+1}$.

(c) Consider now the so-called spin-1/2 XY model in a magnetic field h pointing in the z-direction

$$H = -\sum_{n=1}^{N-1} \left[J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y\right] - \sum_{n=1}^{N} h\sigma_n^z.$$

Here $J_x + J_y > 0$ and we will assume periodic boundary conditions. Using the Jordan-Wigner transformation, show that the Hamiltonian can be written, up to unimportant constants, as

$$H = \sum_{n=1}^{N-1}\left[-(J_x+J_y)(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + (J_x - J_y)(c_{n+1}^\dagger c_n^\dagger + c_n c_{n+1})\right] - \sum_{n=1}^{N} 2hc_n^\dagger c_n.$$

(d) For $J_y = 0$ the Hamiltonian in the previous part reduces to the transverse Ising model. In this case diagonalize the Hamiltonian and thus calculate the spectrum. For which values of J_x does the system become gapless?